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Study of Approximate Value of Real Definite Integral by Mixed Quadrature Rule Obtained from Richardson Extrapolation

[1]S.R. Jena and [2]R.B. Dash

[1]Department of Mathematics, Ajay Binay Institute of Technology
C.D.A., Sector-I, Cuttack-14, Odisha, India.
E-mail: srjenamath@gmail.com
[2]Dept. of Mathematics, Ravenshaw University, Cuttack, Odisha, India.
E-mail: rajani_bdash@rediffmail.com

Abstract

In this paper, two quadrature rules of same precision are mixed up and quadrature rule of higher precision is obtained. An asymptotic error estimate of the rule has been determined and the rule has been numerically verified.

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Introduction

Here in this chapter, we mixed up quadrature rule of Gauss -Legendre 3-point rule and Weddle quadrature with a quadrature obtained from Richardson extrapolation and each of precision 7. A new rule of precision 9 is obtained and this mixed quadrature rule is used for evaluating the real integral of the form

$$I(f) = \int_{-1}^{1} f(x)dx \tag{1.1.1}$$

Mixed Quadrature Rule ofGauss-Legendre - 3 Point Rule And Weddle Quadrature Rule ($R_{WGL3}(f)$)

For the approximation evaluation of (1.1.1) the Gauss- Legendre- 3point rule is

$$R_{GL3}(f) = \frac{1}{9}\left[5f\left(-\sqrt{\frac{3}{5}}\right) + 8f(0) + 5f\left(\sqrt{\frac{3}{5}}\right)\right] \tag{2.1.1}$$

and Weddle quadrature rule is

$$R_W(f) = \frac{1}{10}\left[f(-1) + 5f\left(\frac{-2}{3}\right) + f\left(\frac{-1}{3}\right) + 6f(0) + f\left(\frac{1}{3}\right) + 5f\left(\frac{2}{3}\right) + f(1)\right] \tag{2.1.2}$$

Each of the rules of (2.1.1) and (2.1.2) is of precision 5. Now the mixed quadrature rule due to Gauss-Legendre-3point and Weddle's rule is

$$R_{WGL3}(f) = \frac{1}{511}\left[25R_{GL3}(f) + 486R_W(f)\right] \tag{2.1.3}$$

and truncation error generated by this approximation is

$$E_{WGL3}(f) = \frac{-71}{730 \times 8!} f^{8}(0) - \frac{173624}{5962275 \times 10!} f^{10}(0) - - - \tag{2.1.4}$$

In this mixed quadrature rule the error consists of at least 8^{th} order derivatives. Thus mixed quadrature theoretically is capable of computing exactly all polynomials of degree up to 7.Thus the degree of precision is 7.

Richardson Extrapolation Rule ($R_{REXT}(f)$)

$$I(f) = \int_{-1}^{1} f(x)dx = \frac{4^k I_n^{(k-1)} - I_{n/2}^{(k-1)}}{4^k - 1} \tag{2.2.1}$$

Where $n \geq 2^k$ and $k \geq 1$

For $k = 3, n = 8$ (2.2.1) becomes

$$I_8^{(3)} = \frac{64I_8^{(2)} - I_4^{(2)}}{63} \tag{2.2.2}$$

For $n \geq 4$

$$I_n^{(2)} = \frac{64I_n^{(1)} - I_{n/2}^{(1)}}{15} \tag{2.2.3}$$

Richardson extrapolation rule for n is divisible by 4

For $n = 8$ from (2.2.3)

$$I_8^{(2)} = \frac{64I_8^{(1)} - I_4^{(2)}}{15} \tag{2.2.4}$$

Where

$$I_8^{(1)} = \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8\right] \tag{2.2.5}$$

$$I_4^{(1)} = \frac{2h}{3}\left[f_0 + 4f_2 + 2f_4 + 4f_6 + f_8\right] \tag{2.2.6}$$

Substituting (2.2.5) and (2.2.6) in (2.2.4) we have

$$I_8^{(2)} = \frac{h}{45}\left[14f_0 + 64f_1 + 24f_2 + 64f_3 + 28f_4 + 64f_5 + 24f_6 + 64f_7 + 14f_8\right] \tag{2.2.7}$$

$$I_4^{(2)} = \frac{2h}{15}\left[14f_0 + 64f_2 + 24f_5 + 64f_6 + 14f_8\right] \tag{2.2.8}$$

Putting (2.2.7) and (2.2.8) in (2.2.2) and taking $h = \frac{1}{4}$, Richardson extrapolation rule is

$$I_8^{(3)} = \frac{1}{180\times 63}\left[28\times 31 f_0 + 64^2 f_1 + 64\times 22 f_2 + 64^2 f_3 + 16\times 109 f_4 + 64^2 f_5 + 64\times 22 f_6 + 64^2 f_7 + 14\times 62 f_8\right] \tag{2.2.9}$$

Each rule of $R_{WGL3}(f)$ and $I_8^{(3)}$ is of precision -7

$E_{WGL3}(f)$ and $E_{REXT}(f)$ denotes the error in approximating the integral $I(f)$ by the rule (2.1.4) and (2.2.9) respectively.

Then

$$I(f) = R_{WGL3}(f) + E_{WGL3}(f) \tag{2.2.10}$$

$$I(f) = R_{REXT}(f) + E_{REXT}(f) \tag{2.2.11}$$

Let f(x) to be differentiable in $-1 \le x \le 1$, by Taylor Series

$$f(x) = f(0) + x f^{1}(0) + \frac{x^2}{2!} f^{2}(0) + \frac{x^3}{3!} f^{3}(0) + \frac{x^4}{4!} f^{4}(0) + \frac{x^5}{5!} f^{5}(0) + \frac{x^6}{6!} f^{6}(0) + \frac{x^7}{7!} f^{7}(0) + \frac{x^8}{8!} f^{8}(0) + \frac{x^9}{9!} f^{9}(0) + \frac{x^{10}}{10!} f^{10}(0) + \text{———} \tag{2.2.12}$$

Putting (2.2.12) in (1.1.1)

$$I(f) = 2f(0) + \frac{2}{3!} f^{2}(0) + \frac{2}{5!} f^{4}(0) + \frac{2}{7!} f^{6}(0) + \frac{2}{9!} f^{8}(0) + \frac{2}{11!} f^{10}(0) + \ldots\ldots\ldots\ldots\ldots \tag{2.2.13}$$

Now taking length of the interval $h = \frac{1}{4}$, we have from (2.2.12)

$$f_0 = f(-1),\ f_1 = f\left(\frac{-3}{4}\right),\ f_2 = f\left(\frac{-2}{4}\right),\ f_3 = f\left(\frac{-1}{4}\right),\ f_4 = f(0),\ f_5 = f\left(\frac{1}{4}\right),\ f_6 = f\left(\frac{2}{4}\right),\ f_7 = f\left(\frac{3}{4}\right),\ f_8 = f(1)$$

Now putting $f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8$ in (2.2.7) we have

$$R_{REXT}(f) = 2f(0) + \frac{2}{3!}f^{2}(0) + \frac{2}{5!}f^{4}(0) + \frac{2}{7!}f^{6}(0) + \frac{163}{80\times 9!}f^{8}(0) + \frac{1639}{768\times 11!}f^{10}(0) + \ldots\ldots \qquad (2.2.14)$$

Error In Richardson Extrapolation Quadrature Rule ($E_{REXT}(f)$) And Error In Mixed Quadrature Rule Of Gauss-Legendre-3point Rule And Weddle Rule ($E_{WGL3}(f)$).

$$E_{REXT}(f) = \int_{-1}^{1} f(x)dx - R_{REXT}(f) \qquad (3.1.1)$$

Using (2.2.13) and (2.2.14) in (3.1.1)

$$E_{REXT}(f) = \frac{-1}{240\times 8!}f^{8}(0) - \frac{103}{8448\times 10!}f^{10}(0)\ldots\ldots\ldots\ldots \qquad (3.12)$$

$$E_{WGL3}(f) = \frac{-71}{730\times 8!}f^{8}(0) - \frac{173624}{5962275\times 10!}f^{10}(0) + --- \qquad (3.1.3)$$

From (3.1.2) and (3.1.3) error contains at least 8th order derivative of the integrand functions, it vanishes for all polynomials of degree is 7. That is the degree of the precision of the formula is 7.

Mixed Qadrature Rule Of Gauss-Legendre -3 Point And Weddle With A Quadrature Obtained From Richardson Extrapolation ($R_{M1}(f)$) .

Now multiplying (2.2.10) by $\frac{1}{24}$ and (2.2.11) by $\left(\frac{-71}{73}\right)$ and adding the result we obtain

$$I(f) = \frac{1704}{1631}R_{REXT}(f) - \frac{73}{1631}R_{WGL3}(f) + 0.0111435\frac{f^{10}(0)}{10!} + --- \qquad (4.1.1)$$

$$I(f) = R_{M1}(f) + E_{RM1}(f) \qquad (4.1.2)$$

Where

$$R_{M1}(f) = \frac{1}{1631}\left[1704R_{REXT}(f) - 73R_{WGL3}(f)\right] \qquad (4.1.3)$$

The notation $R_{M1}(f)$ and $E_{RM1}(f)$ are mixed quadrature rule and error in mixed quadrature rule respectively.

The truncation error generated in this approximation is given by

$$E_{RM1}(f) = \frac{0.0111435}{10!}f^{10}(0) + -- \qquad (4.1.4)$$

In (4.1.4), the error consists of at least 10th order derivatives. Thus this mixed quadrature is capable of computing exactly of all polynomials of degree up to 9. Thus the degree of persuasion is 9.The rule (4.1.3) may be called a mixed type as it is constructed from two different types of rules of the same precision.

Error Bound

An error bound of the rule (4.1.3) is given by theorem 5.1

Theorem 5.1

Statement: Let f(x) be sufficiently differentiable function in the closed interval $[-1,1]$ .The bound of the truncation error

$E_{RM1}(f) = I(f) - R_{M1}(f)$ is given by,

$$\left|E_{RM1}(f)\right| = \frac{71}{1631\times 8!}\left|\eta_2 - \eta_1\right| \qquad (5.1.1)$$

Where

$$\eta_1, \eta_2 \in [-1,1]$$

$$M = \max_{-1\le x\le 1} \left|f^9(x)\right|$$

Proof

$$E_{WGL3}(f) = \frac{-71}{730\times 8!} f^8(\eta_1) \text{ Where } \eta_1 \in [-1,1]$$

$$E_{REXT}(f) = \frac{-1}{240\times 8!} f^8(\eta_2) \text{, Where } \eta_2 \in [-1,1]$$

$$E_{RM1}(f) = \frac{1}{1631}\left[1704E_{REXT}(f) - 73E_{WGL3}(f)\right]$$

$$= \frac{71}{16310\times 8!}\int_{\eta_1}^{\eta_2} f^9(x)dx \le \frac{71M}{16310\times 8!}\int_{\eta_1}^{\eta_2} dx$$

$$\left|E_{RM1}(f)\right| \le \frac{71}{16310\times 8!}\left|\eta_2 - \eta_1\right| \text{ (Proved)}$$

Which gives a theoretical error bound as η_2, η_1 are unknown points in [-1,1].From the equation it is clear that the in approximation will be less if points η_1, η_2 are closer to each other.

Numerical Verification

Let us consider the integral quadrature rules for the approximate value

(i) $I_1(f) = \int_{-1}^{1} e^x dx$

Quadrature Rules	Approximate Value
$R_{REXT}(f)$	2.35040249
$R_{WGL3}(f)$	2.35040260
$R_{M1}(f)$	2.35040248
$I_1(f) = exact$	2.35040241

(ii) $I_2(f) = \int_{-1}^{1} e^{-x^2} dx$

Quadrature Rules	Approximate Value
$R_{REXT}(f)$	1.49353040
$R_{WGL3}(f)$	1.49365045
$R_{M1}(f)$	1.49352503
$I_2(f) = exact$	1.49365014

The mixed quadrature rule (numerically) integrated more accurately than a mixed quadrature rule of Gauss-Lgendre-3point and Weddle's rule with a quadrature obtained from Richardson extrapolation.

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