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Cone Metric Spaces and Fixed Point Theorems of Generalized Contractive Mappings

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Abstract

The purpose of this paper is to obtain sufficient conditions for the existence of a unique fixed point of generalized contractive type mappings on complete cone metric spaces depending on another function.

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Introduction

Guang and Xian [3] generalized the notion of metric spaces, replacing the set of real numbers by an ordered Banach space, defining in this way, a cone metric space. These authors also described the convergence of sequences in cone metric spaces and introduced the corresponding notion of completeness. Afterwards, they proved some fixed point theorems of contractive mappings on complete cone metric spaces. Posteriorly, some of the mentioned results were obtained by Rezapour and Hambarani [5] omitting the assumption of normality on the cone.

On the other hand, A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [1] introduced the classes of *T*-Contraction and *T*−Contractive functions, extending the Banach contraction principle and Edelstein's fixed point theorem. S. Moradi [4] introduced the notion a *T*-Kannan contractive mapping which extends the well known Kannan's fixed point theorem [2].

In this paper, we introduced the notion of generalized *T*- contractive mappin defined on a complete cone metric space (*M, d)*, and exdend the results of [3] and [4].

Definitions and preliminary results

In this section we recall the definition of cone metric spaces and some of their properties (see [3]). The following notions will be used in order to prove the main results.

Definition 2.1 ([3]):

Let *E* be a real Banach space and *P* a subset of *E*. *P* is called a cone if

i. P is closed, non empty and $P \neq \{0\}$;
ii. $ax + by \in P \, \forall \, x, y \in P$ and non negative real numbers *a, b*;
iii. $P \cap (-P) = \{0\}$.

Note also that the relations $int\, P + int\, P \subseteq int\, P$ and $\lambda\, int\, P \subseteq int\, P$ $(\lambda > 0)$ hold. For a given cone $P \subseteq E$, we can define on E a partial ordering $\leq$ with respect to P by putting $x \leq y$ if and only if $y - x \in P$. Further, $x < y$ stands for $x \leq y$ and $x \neq y$, while $x \ll y$ stands for $y - x \in int\, P$, where $int\, P$ denotes the interior of P.

Definition 2.2 ([3]):

Let E be a real Banach space and P $\subset$E be a cone. The cone P is called normal if there is a number K > 0 such that for all $x, y \in$ E, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$

The least positive number K satisfying the above inequality is called the normal constant of P.

In the following, we always suppose that E is a real Banach space, P is a cone in E with Int $P \neq 0$ and $\leq$ is the partial ordering with respect to *P.*

Definition 2.3 ([3]):

Let M be a non-empty set. Suppose that the mapping d: M x M → P satisfies:

a. $0 \leq$ d (x, y) for all x, y$\in$ M and d(x,y) = 0 if and only if $x = y$.
b. d(x, y) = d(y,x) for all $x, y \in$ M
c. d$(x, y) \leq$ d(x, z) + d(y, z) for all $x, y, z \in$ M.

Then d is called a cone metric on M and (M, d) is called a cone metric space. Notice that the notion of a cone metric space is more general than the corresponding of a metric space.. Examples of cone metric spaces can be found in [3] and [5].

Definition 2.4 ([3]):

Let (M, d) be a cone metric space, $\{x_n\}$ be a sequence in X and

$x \in X$.

i. $\{x_n\}$ converges to x if for every $c \in E$ with $0 << c$, there is an n_o such that for all $n \geq n_o$, $d(x_n, x) << c$. We denote this by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$

ii. If for any $c \in E$ with $0 << c$, there is an n_o such that for all $n, m \geq n_o$, $d(x_n, x_m) << c$, then $\{x_n\}$ is called a Cauchy sequence in M.

iii. (M, d) is called a complete cone metric space, if every Cauchy sequence in M is convergent in M.

The following result will be useful for us to prove our main results.

Lemma 2.5 ([3]):
Let (M, d) be a cone metric space, $P \in E$ a normal cone with normal constant K. Let $\{x_n\}$, $\{y_n\}$ be sequences in M and $x, y \in M$.

i. $\{x_n\}$ converges to x if and only if $\lim_{n\to\infty} d((x_n, x) = 0$.
ii. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y then x = y. That is the limit of (x_n) is unique;
iii. If $\{x_n\}$ converges to x, then $\{x_n\}$ is Cauchy sequence;
iv. $\{x_n\}$ is a Cauchy sequence if and only if $\lim_{n,m\to\infty} d((x_n, x_m) = 0$:
v. If $x_n \to x$, and $x_n \to x$, $(n\to\infty)$ then $d((x_n, y_n) \to d(x, y)$.

Definition 2.6 ([3]):
Let (M, d) be a cone metric space, P a normal cone with normal constant K and $T: M \to M$. Then

i. T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} T x_n = Tx$ for all of $\{x_n\}$ in M;
ii. T is said to be subsequentially convergent, if for every sequence $\{y_n\}$ that $\{Ty_n\}$ is convergent, implies $\{y_n\}$ has a convergent subsequence;
iii. T is said to be sequentially convergent if for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent, then of $\{y_n\}$ also is convergent.

Main result

First, we recall some definitions on cone metric space form [6].

Definition 3.1 ([6]):
Let (M, d) be a cone metric space and let $S: M \to M$ *be a* functions.

i. S is said to be a T−Kannan contraction, (TK_1−Contraction) if there is $b \in [0, 1/2)$ such that $d(TSx, TSy) \leq b\,(d(Tx, TSx) + d(Ty, TSy))\ \forall x, y \in M$.
ii. S is said to be a T- Chatterjea contraction, (TK_2−Contraction) if there is $c \in [0, 1/2)$ such that $d(TSx, TSy) \leq c\,(d(Tx, TSy) + d(Ty, TSx))\ \forall x, y \in M$.

Definition 3.2:
Let (*M, d*) be a cone metric space and let *S: M → M be a* function.

S is said to be a generalized $T-$ contraction, if there exist non negative constants *a, b, c* such that a + 2b + 2c < 1 and

$$d(TSx,TSy) \leq \ a\, d(Tx,Ty) + b(d(Tx,TSx) + \ d(Ty,TSy)) + c\,(d(Tx,TSy) + \ d(Ty,TSx))\ \forall\, x, y \in M. \ldots \quad (3.2.1)$$

Note: We observe that TK_1, TK_2 contractions are generalized T- contractions.

The following result will be useful for us to prove our main resulti.

Lemma 3.3 ([7]):
Let (M,d) be a complete cone metric space with normal cone P with normal constant K. Suppose $\lambda \in (0,1)$ and $\{x_n\}$ is a sequence in X such that

$$d\,(x_n, x_{n+1}) \leq \lambda\, d\,(x_{n-1}, x_n) \text{ for } n = 1, 2, 3, \ldots$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Theorem 3.4:
Let (M, d) be a complete cone metric space, P be a normal cone with normal constant K. In addition let T: M → M be a continuous function and S: M → M a generalised T−Contraction. Suppose S and T commute. Then,

1. For every $x_0 \in$ M, $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0$.
2. There is $v \in$ M such that $\lim_{n\to\infty} TS^n x_0 = v$
3. If T is subsequentially convergent, then
   i. $\{S^n x_0\}$ has a convergent subsequence;
   ii. There is u $\in M$ such that Su = u;

4. If T is sequentially convergent, then for each $x_0 \in$ M the iterate sequence $\{S^n x_0\}$ converges to u
5. T is contant on the fixed point set of S. If further T is one-one then S has unique fixed point.

Proof: Let $x_0 \in$ M. We define the iterate sequence $\{x_n\}$ by $x_{n+1} = Sx_n = S^{n+1}x_0$. Then

$$d(Tx_n, Tx_{n+1}) = d(TSx_{n-1}, TSx_n)$$

$$\leq \ a\, d(Tx_{n-1},Tx_n) + b(d(Tx_{n-1},TSx_{n-1}) + \ d(Tx_n,TSx_n))$$

$$+ c\,(d(Tx_{n-1},TSx_{n-1}) + \ d(Tx_n,TSx_{n-1}))$$

$$\leq \ a\, d(Tx_{n-1},Tx_n) + b(d(Tx_{n-1},Tx_n) + \ d(Tx_n,Tx_{n+1}))$$

$$+ c\,(d(Tx_{n-1},Tx_{n+1}) + \ d(Tx_n,Tx_n))$$

$$\leq (\, a + b)\big(\, d(Tx_{n-1},Tx_n)\big) + \ b(d(Tx_n,Tx_{n+1}))$$

$$+ c\,(d(Tx_{n-1},Tx_{n+1}) + 0)$$

$\leq (a+b)d(Tx_{n-1},Tx_n)+b(d(Tx_n,Tx_{n+1}))$
$+c\ (d(Tx_{n-1},Tx_n)+\ d(Tx_n,Tx_{n+1}))$
$\leq (\ a+b+c\ )\big(d(Tdx_{n-1},Tx_n)\big)+(b+c)(d(Tx_n,Tx_{n+1}))$

Hence (1- b+c)) $\big(d(Tx_n,Tx_{n+1})\big)\leq (\ a+b+c\ )\ (d(Tx_{n-1},Tx_n))$

$so\ that\ d(Tx_n,Tx_{n+1})\leq \frac{(\ a+b+c\ )}{(1-(\mathrm{b+c}))}(\ d(Tx_{n-1},Tx_n)\ )\ \ldots$ (3.4.1)

Now $\lambda = \frac{(\ a+b+c\ )}{(1-(\mathrm{b+c}))} < 1$

$\Rightarrow d(Tx_n,Tx_{n+1})\leq \lambda(d(Tx_{n-1},Tx_n))$

Hence by lemma 3.3 $\{Tx_n\}$ is a Cauchy sequence.
Since $x_n=S^nx_0$, consequently
$\lim_{n\to\infty} d(TS^nx_0,TS^{n+1}x_0)=\lim_{n\to\infty} d(Tx_n,Tx_{n+1})=0$

Thus (1) holds,
Since M is a complete there exists v ∈ M such that

$$\lim_{n\to\infty} TS^nx_0 = \lim_{n\to\infty} Tx_n = v\ \ldots \quad (3.4.2)$$

Thus (2) holds,
Now, suppose T is subsequentially convergent Then from (3.4.2) $\{S^nx_0\}$ has a convergent subsequence. Thus (3(i)) holds

So, there are u ∈ M and (x_{ni}) such that

$$\lim_{n\to\infty} S^{ni}x_0 = u\ \ldots \quad (3.4.3)$$

Since T is continuous, from (3.5.2) we obtain

$$\lim_{n\to\infty} TS^nx_0 = Tu\ \ldots \quad (3.4.4)$$

From (3.4.2) and (3.4.4) we conclude that

Tu = v. … (3.4.5)

On the other hand,
$d\big(TSu,TS^{ni}x_0\ \big)=\ d\big(TSu,TS(S^{ni-1}x_0)\ \big)$

$= d(TSu, TSx_{ni})$
$\leq \ a\, d(Tu, Tx_{ni-1},) + b(d(Tu, TSu) + \ d(Tx_{ni-1}, TSx_{ni-1}))$
$+ \ c\ (d(Tu, TSx_{ni-1}) + \ d(Tx_{ni-1}, TSu))$

On letting n → ∞, we get
$d(TSu, Tu) \leq \ ad(Tu, Tu) + b(d(Tu, TSu) + \ d(Tu, Tu))$
$+ \ c\ (d(Tu, Tu) + \ d(Tu, TSu))$
= (b+c) $d(TSu, Tu)$
∴ TSu = Tu … (3.4.6)

Since T and S commute
Tu = TSu = STu
∴ Tu is a fixed point of S. Thus (3(ii)) holds.
Now, clearly (4) holds.
Suppose x and y are fixed points of S.
Then we show that Tx =Ty
$d(TSx, TSy) \leq \ a\, d(Tx, Ty) + b(d(Tx, TSx) + \ d(Ty, TSy))$
$+ \ c\ (d(Tx, TSy) + \ d(Ty, TSx))$
$d(Tx, Ty) \leq \ a\, d(Tx, Ty) + b(d(Tx, Tx) + \ d(Ty, Ty))$
$+ \ c\ (d(Tx, Ty) + \ d(Ty, Tx))$
∴ $d(Tx, Ty)$ = (a + 2c) $d(Tx, Ty)$
∴ $d(Tx, Ty) = 0\ (since\ a + 2c < 1)$
∴ Tx = Ty … (3.4.7)
∴ T is a constant on the fixed point set of S.
Thus (5) holds.

If T is one-one,from (3.4.7) follpws that S has unique fixed point if we assume that T is one-one instead of assuming that S and T commute, then from (3.4.6) we get the following,

Corollary 3.5 ([6], Theorem 3.1)
Let (M, d) be a complete cone metric space, P be a normal cone with normal constant K.In addition let T: M → M be a continuous function and S: M → M a generalised T−Contraction. Then,

1. For every x_0 ∈ M $\lim_{n\to\infty} d(TS^n x_0, TS^{n+1} x_0) = 0$;
2. There is v ∈ M such that $\lim_{n\to\infty} TS^n x_0 = v$;

3. If T is subsequentially convergent, then $\{S^n x_0\}$ has a convergent subsequence;
4. There is u ∈ M such that Su = u;
5. If T is sequentially convergent, then for each x_0 ∈ M the iterate sequence $\{S^n x_0\}$ converges to u.

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