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Computing Shortest Distance in Hamiltonian Decomposition using Eigen Values

[1]Murugesan N. and [2]Vanadhi T.

Department of Mathematics,
Govt. Arts College, Coimbatore, Tamil Nadu, India.
Department of Mathematics,
Karpagam University, Coimbatore, Tamil Nadu, India.
E-mail: thiru.vanadhi@gmail.com

Abstract

Decomposing the graph into Hamiltonian cycles is called the Hamiltonian decomposition. The complete graph of (2n+1) vertices can be decomposed into n Hamiltonian cycles and the complete graph of 2n vertices can be decomposed into (n-1) Hamiltonian cycles. In this paper we discuss about, which one of these Hamiltonian cycles having the shortest distance by using the Eigen values when the edges of the cycles have the graceful labeling.

Introduction

In 1967 A. Rosa proved that if a bipartite graph G with n edges has an α - labeling, then for any positive integer p the complete graph K_{2np+1} can be cyclically decomposed into copies of G. This has become a part of graph theory. The generalization of the above work that is to show that every bipartite graph H which decomposes K_k and K_m also decomposes K_{km} is done by Dalibor Froncek with the title of “Decomposition of complete graphs into small graphs”.

The study of complete graph K_n into factors with given diameters was initiated in “On decompositions of complete graphs into factors with given diameters” by A. Rosa and S.Znam.

The problem of the existence of a decomposition of the complete graph K_n into m factors with prescribed s – Edge – connectivities is studied in the paper “On decomposition of complete graphs into factors with prescribed s – Edge – connectivities” by D.Palumbiny and L.Stacho.

The complete graph of odd vertices can be decomposed into n Hamiltonian cycles and the complete graph of even vertices can be decomposed into (n-1) Hamiltonian cycles with 1 – factorization. In these Hamiltonian cycles we discuss the cycle which one has the shortest distance using eigen values. Here the vertices are labeled as 1, 2, 3,…... and only the graphs with graceful labeling for edges is considered. It may be interesting to study the more general case.

Algorithm

- Write the vertex labeling values in the first row of a matrix for n cycles.
- Write the edge graceful labeling in the second row of the matrix for n cycles. As a result we get a 2xn matrix.
- Convert this matrix as a covariance matrix of order 2x2.

$$C = \begin{pmatrix} cov(x,x) & cov(x,y) \\ cov(y,x) & cov(y,y) \end{pmatrix}$$

- Find the Eigen value of C.
- Consider only the subtracted value of the eigen value
- Comparing the subtracted value of the eigen value we have to select the minimum value.
- This minimum value is the shortest distance among the Hamiltonian cycle.

This algorithm is used only when the edge labeling is graceful.

Need of considering subtracted Eigen value

Since the covariance matrix is a 2x2 matrix, there are 2 eigen values out of which one is obtained in general, from the expression $-B+\sqrt{B^2-4AC)}/2A$ and the other is obtained from $-B-\sqrt{B^2-4AC)}/2A$. The value obtained from the first expression is called added eigen value and the value obtained from the second expression is called subtracted eigen value. The difference between added eigen values of covariance matrices of different cycles is smaller than the difference between subtracted eigen values. Therefore the subtracted eigen values of covariance matrices of different cycles are considered to calculate the shortest distance in Hamiltonian decomposition of complete graphs.

Shortest distances measured in Hamiltonian decomposition using Eigen values of (2n+1) vertices

In [1] and [3] we decompose the complete graph with 2n+1 vertices into n Hamiltonian cycles. Now the shortest cycle from these Hamiltonian cycles are illustrated by the following examples.

If the graph has 3 vertices then it has only one cycle.

Suppose the given graph is complete graph with 5 vertices we decompose that into 2 Hamiltonian cycles.

Let the vertices of K_5 are labeled as (1 2 3 4 5).

Then the corresponding Hamiltonian cycles vertex labeling are (1 2 5 3 4) and (1

3 2 4 5).

The first cycle form the following matrix

$$\begin{matrix} V \\ E \end{matrix}\begin{pmatrix} 1 & 2 & 5 & 3 & 4 \\ 1 & 3 & 2 & 1 & 3 \end{pmatrix}$$

The first row values are vertex labeling and the second row values are graceful edge labeling of the first Hamiltonian cycle. Similarly the matrix of the second Hamiltonian cycle is

$$\begin{matrix} V \\ E \end{matrix}\begin{pmatrix} 1 & 3 & 2 & 4 & 5 \\ 2 & 1 & 2 & 1 & 4 \end{pmatrix}$$

These two matrices have 2 rows and 5 columns. We cannot find the eigen values for this 2x5 matrix. Therefore first we convert this matrix as a square matrix of 2x2 matrix using covariance of V and E. This converted matrix is called covariance matrix.

Covariance matrix and Eigen value of Hamiltonian cycles

We find the values of cov (V, V), cov(V, E), cov(E,V) and cov(E,E) given below and tabulated as follows.

V	V	$V-V_0$	$V-V_0$	$(V-V_0)x(V-V_0)$
1	1	-2	-2	4
2	2	-1	-1	1
5	5	2	2	4
3	3	0	0	0
4	4	1	1	1

V	E	$V-V_0$	$E-E_0$	$(V-V_0)x(E-E_0)$
1	1	-2	-1.4	2.8
2	4	-1	1.6	-1.6
5	3	2	0.6	1.2
3	3	0	0.6	0.0
4	1	1	-1.4	-1.4

E	E	$E-E_0$	$E-E_0$	$(E-E_0)x(E-E_0)$
1	1	-1.4	-1.4	2.0
4	4	1.6	1.6	2.6
3	3	0.6	0.6	0.4
3	3	0.6	0.6	0.4
1	1	-1.4	-1.4	2.0

$$\text{Cov}(V,V) = \frac{\sum (V_i - V_0)(V_i - V_0)}{(n-1)}$$

$$\text{Cov}(V,E) = \text{Cov}(E,V) = \frac{\sum (V_i - V_0)(E_i - E_0)}{(n-1)}$$

$$\text{Cov}(E,E) = \frac{\sum (E_i - E_0)(E_i - E_0)}{(n-1)}$$

Therefore we get cov(V,V) = 2.5, cov(V,E) = 1 = cov(V,E), cov(E,E) = 1 for the first cycle.

Using the above values the 2x2 covariance matrix is founded as

$$C_1 = \begin{pmatrix} 2.5 & 1 \\ 1 & 1 \end{pmatrix}$$

The characteristic equation of the matrix is $|C_1 - \lambda_1 I| = 0$

Therefore we get the Eigen values of the first Hamiltonian cycle as

$$\lambda_1 = (3, 0.5).$$

Similarly the covariance matrix of the second cycle is

$$C_2 = \begin{pmatrix} 2.5 & -0.75 \\ -0.75 & 1.5 \end{pmatrix}$$

and the eigen value of this matrix is $\lambda_2 = (2.901, 0.198)$

From these two set of Eigen values the minimum value of the subtracted Eigen value, indicates that corresponding cycle has the shortest distance. Here λ_2 have the shortest distance. This can be verified by simply adding the edges. But if we add the edge values directly in the Hamiltonian cycles the two cycles have the same values. In this case, we select the cycle arbitrarily.

Consider the complete graph with 7 vertices; it can be decomposed that into 3 Hamiltonian cycles, as follows

$$C_1 = \begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 2 & 7 & 3 & 6 & 4 & 5 \\ 1 & 5 & 4 & 3 & 2 & 1 & 4 \end{pmatrix}$$

$$C_2 = \begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 3 & 2 & 4 & 7 & 5 & 6 \\ 2 & 1 & 2 & 3 & 2 & 1 & 5 \end{pmatrix}$$

And $$C_3 = \begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 4 & 3 & 5 & 2 & 6 & 7 \\ 3 & 1 & 2 & 3 & 4 & 1 & 6 \end{pmatrix}$$

Using the above algorithm we get the Eigen values of the three matrices as

λ_1 = (4.8524, 2.2896), λ_2 = (4.989, 1.581) and λ_3 = (4.916, 2.892).

Clearly the smallest subtracted eigen value lies on the second Hamiltonian cycle. Thus C_2 has the shortest distance.

Preceding like this it is possible to calculate the shortest distance Hamiltonian cycle for all the complete graphs of odd vertices.

The graph K_5 has decomposed into two cycles of same distance in the direct addition of edge labels. We can choose one of the values from that tie distance using the above algorithm.

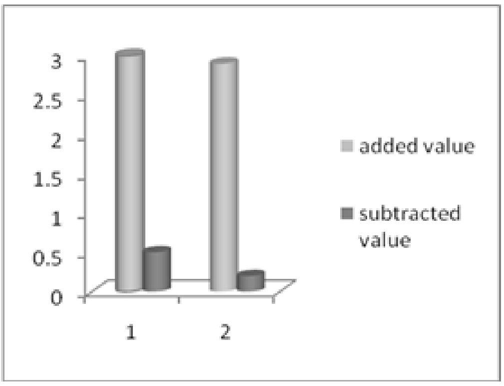
The graph K_9 decomposed into four cycles two of the cycles has the same distance in the direct addition. We can choose one of the values from that both tie distances using the above algorithm and so on.

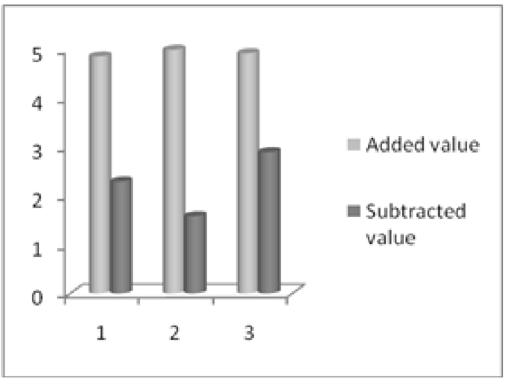
But the graphs K_7 decomposed into three cycles. Two cycles have same distance and the remaining one has distinct distance in the direct addition. By our calculation we conclude that the single distance cycle only the shortest distance of the Hamiltonian cycle.

The graphs K_{11} decomposed into five cycles. Two pairs of cycles have same distance and the remaining one has distinct distance in the direct addition. The single distance cycle has the shortest distance of the Hamiltonian cycle and so on.

Graphical representation

The graphical representations of the Eigen values of the above examples are given below.





Clearly from the subtracted eigen value we can find the shortest distance accurately. Comparing the added value that's not accurate and contradits the direct sum also. Hence we can choose the Subtracted value.

Shortest distances measured in Hamiltonian decomposition using Eigen values of (2n) vertices

Here we are finding the shortest distance of a Hamiltonian cycle from the Hamiltonian decomposition of complete graph with 2n vertices by using the above algorithm.

If the graph has 4 vertices then it has only one cycle.

Consider the complete graph with six vertices.

The matrix of two cycles using the vertex labeling and the graceful edge labeling we get

$$\begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 2 & 6 & 3 & 5 & 4 \\ 1 & 4 & 3 & 2 & 1 & 3 \end{pmatrix}$$

$$\begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 3 & 2 & 4 & 6 & 5 \\ 2 & 1 & 2 & 2 & 1 & 4 \end{pmatrix}$$

Using the above algorithm, the Eigen values of these matrices are λ_1 = (3.519, 1.450) and

λ_2 = (3.517, 1.183). Clearly λ_2 have the shortest distance.

As an example consider the complete graph with 8 vertices. Clearly this graph can be decomposed into three Hamiltonian cycles.

$$\begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 2 & 8 & 3 & 7 & 4 & 6 & 5 \\ 1 & 6 & 5 & 4 & 3 & 2 & 1 & 4 \end{pmatrix}$$

$$\begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 3 & 2 & 4 & 8 & 5 & 7 & 6 \\ 2 & 1 & 2 & 4 & 3 & 2 & 1 & 5 \end{pmatrix}$$

$$\begin{matrix} V \\ E \end{matrix} \begin{pmatrix} 1 & 4 & 3 & 5 & 2 & 6 & 8 & 7 \\ 3 & 1 & 2 & 3 & 4 & 2 & 1 & 6 \end{pmatrix}$$

The corresponding Eigen values are λ_1 = (6.068, 3.289), λ_2 = (6.175, 1.824) and

λ_3 = (6.006, 2.779). The second cycle has the shortest distance.

If the complete graph has 2n (even) vertices then the distance of all the Hamiltonian cycles are distinct. (ie) There are no two cycles which have same length.

For these cases, the algorithm written in this paper become hendy, and can be used effectively to find the shortest distance of s cycle.

Conclusion

In general, Traveling salesman problem is used to find the shortest distance cycle in the given complete graph. By using the algorithm described in this paper, the cycles of the decomposed complete graph having graceful labeling which has the shortest distance can be identified.

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