International Journal of Computational Science and Mathematics.
ISSN 0974-3189 Volume 3, Number 3 (2011), pp. 331-342

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Higher Order Methods on Layer Adapted Grids for Boundary Value Problems with Turning Points

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Abstract

A singularly perturbed boundary value problem with an exponential boundary layer is considered. The problem is discretized on the boundary layer domain with layer adapted grids using up to sixth order finite difference schemes. Numerical error is maintained at the same level for a family of extremely small values of the singular perturbation parameter. Numerical experiments support the analytical claims.

2000 Mathematics Subject Classification: 65N12; 32E25.

Keywords: Boundary Value Problems; Singular Perturbation; Differential Equations; Exponential Boundary Layers; Convergence; Stability.

In the early twentieth century, Prandtl gave the light of theoretical understanding of the singular perturbation phenomena of hydrodynamics to the Third International Congress of Mathematics. Since then, a great deal of effort has been made to conquer this anomaly. Analytically, the asymptotic expansion of O'Malley [9] and the a'priori bound theorem of Chang and Howes [1] were among the prominent approaches. With the advance of unprecedented computing power, there has been a flow of literatures on numerical solutions from the nineteen eighties. Miller, O'Riordan and Shishkin [10] constructed the Shishkin-type mesh to gain the independence of error estimation with respect to the singular perturbation parameter. Schultz and his students [2] and [6], successfully developed the stabilized high order finite difference methods.

We consider the two point boundary value problem,

$$\begin{aligned} &\varepsilon u'' = f(x,u)u' + g(x,u) \ \text{ for } \ x \in (a,b) \ \text{ and } f(x,u) \neq 0, \\ &u(a) = v_a \ \text{ and } \ u(b) = v_b, \end{aligned} \tag{1}$$

where f and g are continuous. By the improved a'priori bounds of Zhang [11], if $f(x,u) \le -k < 0$ for a positive constant k and $x \in (a,b)$, it can be analytically approximated by the two differential equations on the boundary layer and non boundary layer domain respectively,

$$f(x,u)u' + g(x,u) = 0 \text{ for } x \in (t,b), \quad u(b) = v_b, \tag{2}$$

and

$$\varepsilon u'' = f(x,u)u' + g(x,u) \text{ for } x \in (a,t), \quad u(a) = v_a \text{ and } u(t) = v_t, \tag{3}$$

where the turning point is $t = a + w\varepsilon$ and w is a constant for a family of values of ε.

If $f(x,u) \ge k > 0$ for a positive constant k and $x \in (a,b)$, the equation (1) can be analytically approximated by the following two differential equations on the boundary layer and non boundary layer domain respectively,

$$f(x,u)u' + g(x,u) = 0 \text{ for } x \in (a,t), \quad u(a) = v_a, \tag{4}$$

and

$$\varepsilon u'' = f(x,u)u' + g(x,u) \text{ for } x \in (t,b), \quad u(t) = v_t \text{ and } u(b) = v_b, \tag{5}$$

where the turning point is $t = b - w\varepsilon$ and w is a constant for a family of values of ε. Note that the boundary value v_t at the transition point t of the equations (3) and (5) is not known. To find it, we substitute $R_b(t)$ for v_t into equation (3) and substitute $R_a(t)$ for v_t into equation (5) respectively.

The reduced problems can be solved with Runge Kutta methods. On the boundary layer domain, we improve the 4th order method of Ilicasu [6], and develope the 5th and 6th methods. Through the paper, let N, N_n and N_b be the numbers of mesh points on the entire domain, on the non boundary layer domain and on the boundary layer domain repectively. Let h_n and h_b be the grid spacing on the non boundary layer domain and on the boundary layer domain respectively.

Here is the boundary value problem with an exponential boundary layer,

$$\varepsilon u'' - uu' = 0 \;\; x \in (-1,1), \quad u(-1) = 0 \text{ and } u(1) = -1. \tag{6}$$

The exact solution in [-1, 1] is given as $u(x,\varepsilon) = -\dfrac{1 - e^{\frac{-(x+1)}{\varepsilon}}}{1 + e^{\frac{-(x+1)}{\varepsilon}}}$. An important feature of the solution u is that, as a function of (x, ε), it does not behave uniformly as

(x, ε) approaches (-1, 0), that is, $\lim_{\varepsilon \to 0^+} u = -1$ for x>-1; but, $\lim_{x \to -1} u = 0$ for ε>0. For decreasing values of ε, the graphs of u are shown from right to left in the figure.

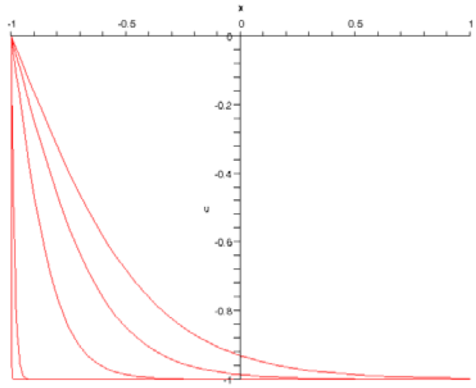

Figure 1: Graphs of the solutions of the boundary value problem (6) with ε=0.3, 0.2, 0.1, 0.01 and 0.001 from right to left

We apply the approximation schemes to get two differential equations

$$uu' = 0 \text{ for } x \in (t, 1),$$
$$u(1) = -1,$$

and

$$\varepsilon u'' = uu' \text{ for } x \in (-1, t),$$
$$u(-1) = 0 \text{ and } u(t) = R_b(\text{t}) \text{ where } \text{t} = -1 + w\varepsilon.$$

Choose the boundary layer parameter $w = 30$ for $\varepsilon \geq 10^{-12}$. The layer occurs at the left boundary. The turning point is $t = -1 + w\varepsilon$.

An improvement on the 4th order method

First, the BVP (6) was solved with the 4th order method for the singular perturbation parameter $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$. The maximum error was $1.0 * 10^{-5}$ and $1.30 * 10^{-2}$ respectively. For simplicity, let $\omega = \frac{1}{\varepsilon}$ where ε is the singular perturbation parameter. Then, we have,

$$u'' = \omega uu' \ ,$$

$$u''' = \omega (u')^2 + \omega uu'', \text{ and}$$

$$u^{(4)} = 2\omega u'u''+\omega u'u''+\omega uu'''$$
$$= 3\omega u'u''+\omega^2 u(u')^2 + \omega^2 u^2 u''$$
$$= \omega^2 u(u')^2 + (3\omega u'+\omega^2 u^2)u''.$$

Lettting $A_3 = \omega u_i^{'}$, $B_3 = \omega u_i$, and $A_4 = \omega^2 u_i u_i^{'}$, $B_4 = 3\omega u_i^{'} + \omega^2 u_i^2$, we get

$$u_i^{(3)} = A_3 u_i^{'} + B_3 u_i^{''},$$
$$u_i^{(4)} = A_4 u_i^{'} + B_4 u_i^{''}.$$

Define the following,

$$\mathrm{A}^{**} = \frac{h^4 A_4}{24},$$
$$B^{**} = h + \frac{h^3 A_3}{6},$$
$$C^{**} = \frac{h^2}{2} + \frac{h^4 B_4}{24},$$
$$D^{**} = \frac{h^3 B_3}{6}.$$

We use the 4th order finite differences to approximate the BVP (6).

$\varepsilon u_i^{''} - u_i u_i^{'} \approx c_3^{**} u_{i+1} + c_2^{**} u_i + c_1^{**} u_{i-1}$ where

$$c_3^{**} = \frac{-u_i D^{**} - \varepsilon B^{**} + \varepsilon A^{**} + u_i C^{**}}{2(A^{**}D^{**} - B^{**}C^{**})},$$
$$c_1^{**} = \frac{-u_i D^{**} - \varepsilon B^{**} - (\varepsilon A^{**} + u_i C^{**})}{2(A^{**}D^{**} - B^{**}C^{**})},$$
$$c_2^{**} = -(c_3^{**} + c_1^{**}) = \frac{u_i D^{**} + \varepsilon B^{**}}{A^{**}D^{**} - B^{**}C^{**}}.$$

The derivative $u_i^{'}$ of A_3, A_4 and B_4 is replaced with $u_i^{'} = \frac{u_{i+1} - u_{i-1}}{2h}$. We now use 4th order finite difference to replace the derivative $u_i^{'}$. For $i = 1,2,..., N-1$, let the first order derivative be given by $u_i^{'} = c_3 u_{i+1} + c_2 u_i + c_1 u_{i-1}$ where c_3, c_2 and c_1 are constants.

Using Taylor series expansion, we get

$$u_i^{'} = c_3(u_i + hu_i^{'} + \frac{h^2}{2}u_i^{''} + \frac{h^3}{6}u_i^{'''} + \frac{h^4}{24}u_i^{(4)} + ...)$$
$$+ c_2 u_i$$
$$+ c_1(u_i - hu_i^{'} + \frac{h^2}{2}u_i^{''} - \frac{h^3}{6}u_i^{'''} + \frac{h^4}{24}u_i^{(4)} + ...)$$
$$\approx$$
$$c_3\{u_i + hu_i^{'} + \frac{h^2}{2}u_i^{''} + \frac{h^3}{6}(\omega u_i^{'2} + \omega u_i u_i^{''}) + \frac{h^4}{24}[\omega^2 u_i u_i^{'2} + (3\omega u_i^{'} + \omega^2 u_i^2)u_i^{''}]\}$$
$$+ c_2 u_i$$
$$+ c_1\{u_i - hu_i^{'} + \frac{h^2}{2}u_i^{''} - \frac{h^3}{6}(\omega u_i^{'2} + \omega u_i u_i^{''}) + \frac{h^4}{24}[\omega^2 u_i u_i^{'2} + (3\omega u_i^{'} + \omega^2 u_i^2)u_i^{''}]\}$$
$$=$$
$$(c_3 + c_2 + c_1)u_i$$
$$+ [(c_3 - c_1)(h + \frac{h^3}{6}\omega u_i^{'}) + (c_3 + c_1)\frac{h^4}{24}\omega^2 u_i u_i^{'}]u_i^{'}$$
$$+ [(c_3 + c_1)(\frac{h^2}{2} + \frac{h^4}{24}3\omega u_i^{'} + \frac{h^4}{24}\omega^2 u_i^{2}) + (c_3 - c_1)\frac{h^3}{6}\omega u_i]u_i^{''}.$$

Equating the corresponding coefficients, we have

$$c_3 + c_2 + c_1 = 0,$$
$$(c_3 - c_1)(h + \frac{h^3}{6}\omega u_i^{'}) + (c_3 + c_1)\frac{h^4}{24}\omega^2 u_i u_i^{'} = 1,$$
$$(c_3 + c_1)(\frac{h^2}{2} + \frac{h^4}{24}3\omega u_i^{'} + \frac{h^4}{24}\omega^2 u_i^{'2}) + (c_3 - c_1)\frac{h^3}{6}\omega u_i = 0.$$

which is equivalent to

$$c_3 + c_2 + c_1 = 0,$$
$$(c_3 + c_1)A + (c_3 - c_1)B = 1,$$
$$(c_3 + c_1)C + (c_3 - c_1)D = 0,$$

of which, $c_3 + c_1$ and $c_3 - c_1$ are solved with Cramer's rule,

$$c_3 + c_1 = \frac{D}{AD - BC},$$
$$c_3 - c_1 = \frac{-C}{AD - BC}.$$

where

$$A = \frac{h^4}{24}\omega^2 u_i u_i^{'},$$

$$B = h + \frac{h^3}{6}\omega u_i^{'},$$

$$C = \frac{h^2}{2} + \frac{h^4}{24}3\omega u_i^{'} + \frac{h^4}{24}\omega^2 u_i^{'2},$$

$$D = \frac{h^3}{6}\omega u_i.$$

Therefore, we obtain

$$c_3 = \frac{D - C}{2(AD - BC)},$$

$$c_1 = \frac{D + C}{2(AD - BC)},$$

$$c_2 = -(c_3 + c_1) = \frac{-D}{AD - BC}.$$

The error term is

$$\frac{h^5}{120}(c_3 u^{(5)}(\eta_1) + c_1 u^{(5)}(\eta_2)) \text{ where } \eta_1, \eta_2 \in [x_i - h, x_i + h].$$

In comparison to the approach of Ilicasu and Schultz [6], c_3 and c_1 are updated to the 4th order accuracy. The improvement of the method is verified by numerical experiments and shown in Table 1 and Table 2.

The 5th and 6th Order Methods

The second improvement is to add more terms from the Taylor series to approximate the BVP (6). We expand the u_{i+1} and u_{i-1} up to the sixth order derivatives.

The following is a development of the 6th order method. The 5th order method is developed by dropping 6th order derivative terms. First, we consider the 5th order and 6th order derivatives:

$$\begin{aligned} &u^{(5)} \\ &= 3\omega u''^2 + 3\omega u' u''' + \omega^2 u' u'^2 + 2\omega^2 uu' u'' + 2\omega^2 uu' u'' + \omega^2 u^2 u''' \\ &= 3\omega u''^2 + 3\omega u'(\omega u'^2 + \omega uu'') + \omega^2 u'^3 + 4\omega^2 uu' u'' + \omega^2 u^2(\omega u'^2 + \omega uu'') \end{aligned}$$

$$= 3\omega u''^2 + 3\omega^2 u'^3 + 3\omega^2 uu'u'' + \omega^2 u'^3 + 4\omega^2 uu'u'' + \omega^3 u^2 u'^2 + \omega^3 u^3 u''$$
$$= 3\omega u''^2 + 4\omega^2 u'^3 + 7\omega^2 uu'u'' + \omega^3 u^2 u'^2 + \omega^3 u^3 u''$$
$$= (4\omega^2 u'^2 + \omega^3 u^2 u')u' + (3\omega u'' + 7\omega^2 uu' + \omega^3 u^3)u''$$

and

$$u^{(6)}$$
$$= 6\omega u''u''' + 12\omega^2 u'^2 u'' + 7\omega^2[(u'^2 + uu'')u'' + uu'u''']$$
$$+ 2\omega^3 uu'^3 + 2\omega^3 u^2 u'u'' + 3\omega^3 u^2 u'u'' + \omega^3 u^3 u'''$$
$$= 6\omega u''(\omega u'^2 + \omega uu'') + 12\omega^2 u'^2 u'' + 7\omega^2[u'^2 u'' + uu''^2 + uu'(\omega u'^2 + \omega uu'')]$$
$$+ 2\omega^3 uu'^3 + 5\omega^3 u^2 u'u'' + \omega^3 u^3(\omega u'^2 + \omega uu'')$$
$$= 6\omega^2 u'^2 u'' + 6\omega^2 uu''^2 + 12\omega^2 u'^2 u'' + 7\omega^2 u'^2 u'' + 7\omega^2 uu''^2 + 7\omega^3 uu'^3 + 7\omega^3 u^2 u'u''$$
$$+ 2\omega^3 uu'^3 + 5\omega^3 u^2 u'u'' + \omega^4 u^3 u'^2 + \omega^4 u^4 u''$$
$$= 25\omega^2 u'^2 u'' + 13\omega^2 uu''^2 + 7\omega^3 uu'^3 + 2\omega^3 uu'^3 + 12\omega^3 u^2 u'u'' + \omega^4 u^3 u'^2 + \omega^4 u^4 u''$$
$$= (9\omega^3 uu'^2 + \omega^4 u^3 u')u' + (25\omega^2 u'^2 + 13\omega^2 uu'' + 12\omega^3 u^2 u' + \omega^4 u^4)u''.$$

For simplicity, we rewrite the derivatives

$u_i^{(3)} = A_3 u_i' + B_3 u_i''$ where $A_3 = \omega u_i', B_3 = \omega u_i$,

$u_i^{(4)} = A_4 u_i' + B_4 u_i''$ where $A_4 = \omega^2 u_i u_i', B_4 = 3\omega u_i' + \omega^2 u_i^2$,

$u_i^{(5)} = A_5 u_i' + B_5 u_i''$ where $A_5 = 4\omega^2 u_i'^2 + \omega^3 u_i^2 u_i', B_5 = 3\omega u_i'' + 7\omega^2 u_i u_i' + \omega^3 u_i^3$,

and

$u_i^{(6)} = A_6 u_i' + B_6 u_i''$ where $A_6 = 9\omega^3 u_i u_i'^2 + \omega^4 u_i^3 u_i'$, and

$B_6 = 25\omega^2 u_i'^2 + 13\omega^2 u_i u_i'' + 12\omega^3 u_i^2 u_i' + \omega^4 u_i^4$.

We write

$$\varepsilon u_i'' - u_i u_i' = c_3^* u_{i+1} + c_2^* u_i + c_1^* u_{i-1},$$

where c_3^*, c_2^* and c_1^* are constants. From Taylor series expansion, we obtain

$$\varepsilon u_i'' - u_i u_i' = c_3^* u_{i+1} + c_2^* u_i + c_1^* u_{i-1}$$
$$\approx c_3^*[u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i^{(4)} + \frac{h^5}{120}u_i^{(5)} + \frac{h^6}{720}u_i^{(6)}]$$
$$+ c_2^* u_i$$
$$+ c_1^*[u_i - hu_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i^{(4)} - \frac{h^5}{120}u_i^{(5)} + \frac{h^6}{720}u_i^{(6)}]$$

$$=c_3^*[u_i+hu_i^{'}+\frac{h^2}{2}u_i^{''}+\frac{h^3}{6}(A_3u_i^{'}+B_3u_i^{''})+\frac{h^4}{24}(A_4u_i^{'}+B_4u_i^{''})+\frac{h^5}{120}(A_5u_i^{'}+B_5u_i^{''})$$

$$+\frac{h^6}{720}(A_6u_i^{'}+B_6u_i^{''})]+c_2^*u_i+c_1^*[u_i-hu_i^{'}+\frac{h^2}{2}u_i^{''}-\frac{h^3}{6}(A_3u_i^{'}+B_3u_i^{''})+\frac{h^4}{24}(A_4u_i^{'}+B_4u_i^{''})$$

$$-\frac{h^5}{120}(A_5u_i^{'}+B_5u_i^{''})+\frac{h^6}{720}(A_6u_i^{'}+B_6u_i^{''})]$$

$$=c_3^*[u_i+(h+\frac{h^3A_3}{6}+\frac{h^4A_4}{24}+\frac{h^5A_5}{120}+\frac{h^6A_6}{720})u_i^{'}+(\frac{h^2}{2}+\frac{h^3B_3}{6}+\frac{h^4B_4}{24}+\frac{h^5B_5}{120}+\frac{h^6B_6}{720})u_i^{''}]$$

$$+c_2^*u_i$$

$$+c_1^*[u_i+(-h-\frac{h^3A_3}{6}+\frac{h^4A_4}{24}-\frac{h^5A_5}{120}+\frac{h^6A_6}{720})u_i^{'}+(\frac{h^2}{2}-\frac{h^3B_3}{6}+\frac{h^4B_4}{24}-\frac{h^5B_5}{120}+\frac{h^6B_6}{720})u_i^{''}]$$

$$=(c_3^*+c_2^*+c_1^*)u_i$$

$$+[(c_3^*+c_1^*)(\frac{h^4A_4}{24}+\frac{h^6A_6}{720})+(c_3^*-c_1^*)(h+\frac{h^3A_3}{6}+\frac{h^5A_5}{120})]u_i^{'}$$

$$+[(c_3^*+c_1^*)(\frac{h^2}{2}+\frac{h^4B_4}{24}+\frac{h^6B_6}{720})+(c_3^*-c_1^*)(\frac{h^3B_3}{6}+\frac{h^5B_5}{120})]u_i^{''}.$$

Equating the corresponding coefficients of both sides, we get the following system:

$$c_3^*+c_2^*+c_1^*=0,$$

$$(c_3^*+c_1^*)A^*+(c_3^*-c_1^*)B^*=-u_i,$$

$$(c_3^*+c_1^*)C^*+(c_3^*-c_1^*)D^*=\varepsilon,$$

where

$$\mathrm{A}^*=\frac{h^4A_4}{24}+\frac{h^6A_6}{720},$$

$$B^*=h+\frac{h^3A_3}{6}+\frac{h^5A_5}{120},$$

$$C^*=\frac{h^2}{2}+\frac{h^4B_4}{24}+\frac{h^6B_6}{720},$$

$$D^*=\frac{h^3B_3}{6}+\frac{h^5B_5}{120}.$$

Note that the derivatives contained in A_3, A_4, A_5 and B_3, B_4, B_5 are replaced with the following:

$$u_i^{'}=\frac{u_{i+1}-u_{i-1}}{2h}, u_i^{''}=\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}.$$

Thus, we get

$$c_3^* + c_1^* = \frac{-u_i D^* - \varepsilon B^*}{A^* D^* - B^* C^*},$$

$$c_3^* - c_1^* = \frac{\varepsilon A^* + u_i C^*}{A^* D^* - B^* C^*}.$$

Therefore, the solution is

$$c_3^* = \frac{-u_i D^* - \varepsilon B^* + \varepsilon A^* + u_i C^*}{2(A^* D^* - B^* C^*)},$$

$$c_1^* = \frac{-u_i D^* - \varepsilon B^* - (\varepsilon A^* + u_i C^*)}{2(A^* D^* - B^* C^*)},$$

$$c_2^* = -(c_3^* + c_1^*) = \frac{u_i D^* + \varepsilon B^*}{A^* D^* - B^* C^*}.$$

The error term is

$$\frac{h^7}{5040}(c_3 u^{(7)}(\eta_3) + c_1 u^{(7)}(\eta_4)) \text{ where } \eta_3, \eta_4 \in [x_i - h, x_i + h].$$

Numerical comparison among different methods are shown in Table 1 and Table 2 for $\varepsilon=0.01$ $\varepsilon=0.001$ respectively. For the improved 4th order, 5th order and 6th order methods, the number N_n of mesh points on the non boundary layer domain is 170 and the number N_b of mesh points on the boundary layer is 300. The total number of mesh points for our approximation is $N = N_n + N_b = 470$, compared to N=2000 for that of other researchers using the uniform mesh on the entire domain. As a significant advantage, higher accuracy is obtained with less computing, which is reflected with less numbers of mesh points and iterations,

Table 1: Maximal error of different methods with ε=0.01.

Method	Number of Points	Number of Iterations	Max Error
Choudhury's Method	2000	Not known	$2.91*10^{-2}$
2nd order	2000	3201	$2.61*10^{-4}$
4th order	2000	3152	$1.00*10^{-5}$
Improved 4th order method	470	697	$8.40*10^{-5}$
5th order method	470	697	$1.34*10^{-7}$
6th order method	470	697	$7.71*10^{-8}$

Table 2: Maximal error of different methods with ε=0.001.

Method	Number of Points	Number of Iterations	Max Error
4^{th} order	2000	5102	$1.30*10^{-2}$
Improved 4^{th} order	470	697	$8.41*10^{-5}$
5^{th} order method	470	697	$1.54*10^{-7}$
6^{th} order method	470	697	$7.56*10^{-8}$

For the methods of this paper, the tolerance of iteration is set at 10^{-10}; the over relaxation factor is 1.9.

The convergence of the improved 4^{th} order method and 6^{th} order method is shown in Table 3 and Table 4 with singular perturbation parameter as small as $\varepsilon = 10^{-12}$.

Table 3: The convergence of the improved 4^{th} order method.

Number of Points	Maximal Error		
	$\varepsilon=10^{-5}$	$\varepsilon=10^{-10}$	$\varepsilon=10^{-12}$
N=350 (N_n=200, N_b=150)	$3.53*10^{-4}$	$3.53*10^{-4}$	$3.56*10^{-4}$
N=400 (N_n=200, N_b=200) 00	$1.91*10^{-4}$	$1.94*10^{-4}$	$1.94*10^{-4}$
N=450 (N_n=200, N_b=250)	$1.20*10^{-4}$	$1.22*10^{-4}$	$1.35*10^{-4}$
N=500 (N_n=200, N_b=300)	$8.39*10^{-5}$	$8.40*10^{-5}$	$9.71*10^{-5}$

Table 4: The convergence of the 6^{th} order method.

Number of Points	Maximal Error		
	$\varepsilon=10^{-5}$	$\varepsilon=10^{-10}$	$\varepsilon=10^{-12}$
N=350 (N_n=200, N_b=150)	$1.19*10^{-6}$	$1.16*10^{-6}$	$2.79*10^{-5}$
N=400 (N_n=200, N_b=200) 00	$4.51*10^{-7}$	$4.18*10^{-7}$	$2.64*10^{-5}$
N=450 (N_n=200, N_b=250)	$1.29*10^{-7}$	$2.88*10^{-7}$	$2.42*10^{-5}$
N=500 (N_n=200, N_b=300)	$7.49*10^{-8}$	$7.49*10^{-8}$	$2.39*10^{-5}$

As expected, numerical error decreases when the higher order finite difference methods are applied. We compare numerical results of different orders in Table 5.

Table 5: Maximal error comparison among the different methods with 300 points on the boundary layer.

Finite Differences	Maximal error	
	$\varepsilon=10^{-5}$	$\varepsilon=10^{-10}$
Central Difference	$3.77*10^{-4}$	$3.77*10^{-4}$
4th Order Difference	$1.02*10^{-4}$	$1.02*10^{-4}$
Improved 4th Order	$8.40*10^{-5}$	$8.40*10^{-5}$
6th Order Difference	$7.49*10^{-8}$	$7.49*10^{-8}$

The graph of the numerical solution from the 6th order method for the BVP (6), is shown in Figure 2, which reflects the existence of a boundary layer.

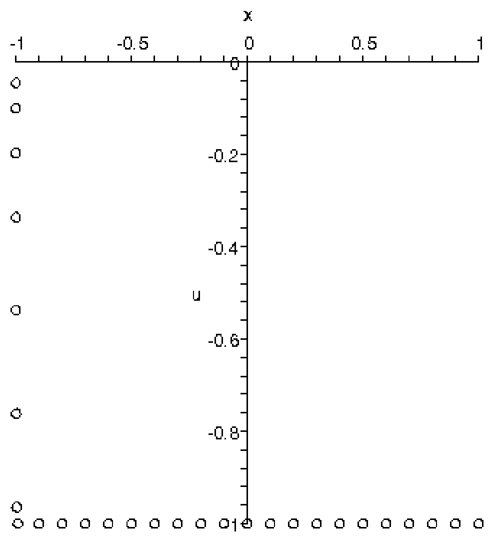

Figure 2: The graph of the numerical solution of the BVP (6).

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