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A Partial Solution to an Open Problem in Strict Menger Probabilistic Metric Spaces

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Abstract

In this paper, we give partial solution to the open problem 2.10 posed in [3].

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Introduction

In Sastry et.al [3] an open problem in a strict Menger space is given, incidentally observing a fallacy in the argument of a result of Servet Kutucku and Sushil Sharma [6].

We use the notion of strict Menger space given in [3] and make use of this to prove a fixed point theorem (Theorem 2.1) in strict Menger spaces with min t-norm. An open problem (open problem 2.2) is also given at the end of the paper.

We start with

Definition 1.1: [4] A function $F\colon \mathbb{R} \to [0,1]$ is called a distribution function if F is non-decreasing, left continuous and $\inf_{x\in\mathbb{R}} F(x) = 0$ and $\sup_{x\in\mathbb{R}} F(x) = 1$.

Definition1.2: [4] A triangular norm $*\colon [0,1]\, x\, [0,1] \to [0,1]$ is a function satisfying the following conditions:

1. $\alpha * 1 = \alpha \, \forall \, \alpha \in [0,1]$,
2. $\alpha * \beta = \beta * \alpha \, \forall \, \alpha, \beta \in [0,1]$,
3. $\gamma * \delta \geq \alpha * \beta \, \forall \, \alpha, \beta, \gamma, \delta \in [0,1] \, with \, \gamma \geq \alpha \, and \, \delta \geq \beta$,
4. $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma) \, \forall \, \alpha, \beta, \gamma \in [0,1]$.

A triangular norm is also denoted by t-norm. For any $a, b \in [0,1]$, if we define $a * b = \min\{a, b\}$, then $*$ is a t-norm and is denoted by 'min'.

Definition1.3: [4] Let X be a non-empty set and let $F: X \times X \to \mathfrak{D}$ (The set of distribution functions). For $p, q \in X$, we denote the image of the pair (p, q) by $F_{p,q}$ which is a distribution function so that $F_{p,q}(x) \in [0,1]$, for every real x. Suppose F satisfies:

1. $F_{p,q}(x) = 1$ for all $x > 0$ if and only if $p = q$,
2. $F_{p,q}(0) = 0$,
3. $F_{p,q}(x) = F_{q,p}(x)$,
4. If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$ where $p, q, r \in X$.

Then (X, F) is called a probabilistic metric space.

Definition1.4: [4] Let X be a non empty set, $*$ a t-norm and $F : X \times X \to \mathfrak{D}$ satisfies:

1. $F_{p,q}(0) = 0 \, \forall \, p, q \in X$,
2. $F_{p,q}(x) = 1 \, \forall \, x > 0$ if and only if $p = q$,
3. $F_{p,q}(x) = F_{q,p}(x) \, \forall \, p, q \in X$,
4. $F_{p,r}(x + y) \geq *(F_{p,q}(x), F_{q,r}(y)) \, \forall \, x, y \geq 0 \, and \, p, q, r \in X$.

Then the triplet $(X, F, *)$ is called a Menger space.

Definition 1.5: [5]

1. Let $(X, F, *)$ be a Menger space and $p \in X$. For $\varepsilon > 0, 0 < \lambda < 1$, the (ε, λ)-neighbourhood of p is defined as $U_p(\varepsilon, \lambda) = \{q \in X: F_{p,q}(\varepsilon) > 1 - \lambda\}$. It may be observed that, if $*$ is continuous then the topology induced by the family $\{U_p(\varepsilon, \lambda): p \in X, \varepsilon > 0, 0 < \lambda < 1\}$ is a Hausdorff topology on X and is known as the (ε, λ) - topology.
2. A sequence $\{x_n\}$ in X is said to converge to $p \in X$ in the (ε, λ) -topology, if for any $\varepsilon > 0 \, and \, 0 < \lambda < 1$ there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,p}(\varepsilon) > 1 - \lambda$ where $n > N$.
3. A sequence $\{x_n\}$ in X is said to be a Cauchy sequence in the (ε, λ)-topology, if for $\varepsilon > 0$ and $0 < \lambda < 1$ there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_m,x_n}(\varepsilon) > 1 - \lambda$ for all $m, n > N$.
4. A Menger space $(X, F, *)$, where $*$ is continuous, is said to be complete if

every Cauchy sequence in X is convergent in the (ε, λ) -topology.

Definition 1.6: [1] Let $*$ be a t-norm. For any $a \in [0,1]$, write $*_0(a) = 1$ and $*_1(a) = *(*_0(a), a) = *(1, a) = a$

In general define $*_{n+1}(a) = *(*_n(a), a)$ $for\ n = 0, 1, 2 \ldots\ldots$

If $\{*_n\}$ is equicontinuous at 1, that is given $\varepsilon > 0$ $there\ exists\ \delta > 0\ such\ that\ x > 1 - \delta\ implies\ *_n(x) > 1 - \varepsilon\ \forall\ n \in N.$ then we say that $*$ is a Hadzic type t-norm.

We observe that 'min' t-norm is of Hadzic type.

Definitions 1.7: [6] Two self mappings A and B of a Menger space $(X, F, *)$ are said to be

1. compatible of type (P) if $F_{ABx_n, BBx_n}(t) \to 1$ and $F_{BAx_n, AAx_n}(t) \to 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.
2. compatible of type (P_1) if $F_{ABx_n, BBx_n}(t) \to 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.
3. compatible of type (P_2) if $F_{BAx_n, AAx_n}(t) \to 1$ for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Bx_n \to z$ for some z in X as $n \to \infty$.

We need the following lemma.

Lemma 1.8: [2] Let $(X, F, *)$ be a Menger space with Hadzic-type t-norm $*$ and $0 < a < 1$. Suppose $\{x_n\}$ is a sequence in X such that for any $s > 0$, $F_{x_n, x_{n+1}}(s) \geq F_{x_0, x_1}(\frac{s}{a^n})$.

Then $\{x_n\}$ is a Cauchy sequence.

Definition 1.9: [3] Let $(X, F, *)$ be a Menger space such that $F_{x,y}(t)$ is strictly increasing in t whenever $x \neq y$. Then $(X, F, *)$ is called a strict Menger space.

Example 1.10: [3] Let (X, d) be a metric space. Define $F_{x,y}(t) = \frac{t}{t + d(x,y)}\ \forall\ t > 0$ and $x, y \in X$. If t-norm $*$ is $a * b = \min\{a, b\}\ \forall\ a, b \in [0,1]$, then $(X, F, *)$ is a strict Menger space.

Main results

The following theorem is given in Sastry. et. al [3].

Theorem 2.1: [3] Let P, Q, R and C be self maps of a complete strict Menger space $(X, F, *)$ with min t-norm $*$ satisfying:

1. $P(X) \subseteq R(X), Q(X) \subseteq C(X)$,
2. there exists a constant $k \in (0,1)$ such that $F_{Px,Qy}(kt) \geq F_{Cx,Ry}(t) * F_{Px,Cx}(t) * F_{Qy,Ry}(t) * F_{Px,Ry}(2t) * F_{Qy,Cx}(2t)$ for all $x, y \in X, t > 0$,

3. either P or C is continuous,
4. the pairs (P, C) and (Q, R) are both compatible of type (P_1) or type (P_2).

Then P, Q, R and C have a unique common fixed point.
The following open problem is posed in [3].

Open Problem 2.2: [3] Is Theorem 2.1 valid if $2t$ in condition (b) is replaced by αt where $\alpha \in (1,2)$?

Now we prove a fixed point theorem for four self maps on a complete strict Menger space which gives a partial solution to the open problem 2.2.

Theorem 2.3: Let P, Q, R and C be self maps of a complete strict Menger space $(X, F, *)$ with min t-norm $*$ satisfying:

1. $P(X) \subseteq R(X), Q(X) \subseteq C(X)$, there exists a constant $k \in (0,1)$ and $\alpha \in (2k, 2)$ such that $F_{Px,Qy}(kt) \geq F_{Cx,Ry}(t) * F_{Px,Cx}(t) * F_{Qy,Ry}(t) * F_{Px,Ry}(\alpha t) * F_{Qy,Cx}(\alpha t)$ for all $x, y \in X, t > 0$,
2. either P or C is continuous,
3. the pairs (P, C) and (Q, R) are both compatible of type (P_1) or type (P_2).

Then P, Q, R and C have a unique common fixed point.

Proof: Let $x_0 \in X$. By (a), there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$Px_{2n} = Rx_{2n+1} = y_{2n}$ and $Qx_{2n+1} = Cx_{2n+2} = y_{2n+1}$ for $n = 0, 1, 2$ ...
Step 1: By taking $x = x_{2n}, y = x_{2n+1}$ for all $t > 0$ in (b), we get

$$F_{Px_{2n},Qx_{2n+1}}(kt) \geq F_{Cx_{2n},Rx_{2n+1}}(t) * F_{Px_{2n},Cx_{2n}}(t) * F_{Qx_{2n+1},Rx_{2n+1}}(t) * F_{Px_{2n},Rx_{2n+1}}(\alpha t) * F_{Qx_{2n+1},Cx_{2n}}(\alpha t)$$

$$\Rightarrow F_{y_{2n},y_{2n+1}}(kt) \geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n-1}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n},y_{2n}}(\alpha t) * F_{y_{2n+1},y_{2n-1}}(\alpha t)$$

$$\geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+1},y_{2n-1}}(\alpha t)$$

$$\geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}(t) * F_{y_{2n+1},y_{2n}}\left(\frac{\alpha t}{2}\right) * F_{y_{2n},y_{2n-1}}\left(\frac{\alpha t}{2}\right)$$

$$\geq F_{y_{2n-1},y_{2n}}\left(\frac{\alpha t}{2}\right) \text{ } (\because \text{ X is a strict Menger space})$$

Similarly, we can prove that $F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}\left(\frac{\alpha t}{2}\right)$

Hence $F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}\left(\frac{\alpha t}{2}\right)$ for all $n \in N$

Therefore $F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}\left(\frac{\alpha t}{2k}\right) \ \forall\ t > 0, n \in N \text{ and } \frac{\alpha}{2k} > 1$

By Lemma 1.8, $\{y_n\}$ is a Cauchy sequence.

Since $(X, F, *)$ is complete, it converges to a point z in X. Also its sub sequences $\{Px_{2n}\} \to z$, $\{Cx_{2n}\} \to z$, $\{Qx_{2n+1}\} \to z$ and $\{Rx_{2n+1}\} \to z$.

Case (i): C is continuous, (P, C) and (Q, R) are compatible of type (P_2)
$CCx_{2n} \to Cz$, $CPx_{2n} \to Cz$ ($\because$ C is continuous)

and $PPx_{2n} \to Cz$ ($\because$ (P, C) is compatible of type (P_2))
By taking $x = Px_{2n}, y = x_{2n+1}$in (b), we get
$$F_{PPx_{2n},Qx_{2n+1}}(kt)$$
$$\geq F_{CPx_{2n},Rx_{2n+1}}(t) * F_{PPx_{2n},CPx_{2n}}(t) * F_{Qx_{2n+1},Rx_{2n+1}}(t) * F_{PPx_{2n},Rx_{2n+1}}(\alpha t) *$$
$$F_{Qx_{2n+1},CPx_{2n}}(\alpha t)$$
On letting $n \to \infty$, we get
$$F_{Cz,z}(kt) \geq F_{Cz,z}(t) * F_{Cz,Cz}(t) * F_{z,z}(t) * F_{Cz,z}(\alpha t) * F_{z,Cz}(\alpha t)$$
$$\geq F_{Cz,z}(t) * F_{Cz,z}(\alpha t)$$
If $Cz \neq z$, $F_{Cz,z}(kt) < F_{Cz,z}(t)$ ($\because$ $0 < k < 1$)
and $F_{Cz,z}(kt) < F_{Cz,z}(\alpha t)$ ($\because k < 2k < \alpha$)
$\therefore F_{Cz,z}(kt) < F_{Cz,z}(t) * F_{Cz,z}(\alpha t) \leq F_{Cz,z}(kt)$, a contradiction.
Hence $Cz = z$.

Step 3: By taking $x = z, y = x_{2n+1}$ in (b), we get
$$F_{Pz,Qx_{2n+1}}(kt) \geq F_{Cz,Rx_{2n+1}}(t) * F_{Pz,Cz}(t) * F_{Qx_{2n+1},Rx_{2n+1}}(t) * F_{Pz,Rx_{2n+1}}(\alpha t) * F_{Qx_{2n+1},Cz}(\alpha t)$$
On letting $n \to \infty$, we get
$$F_{Pz,z}(kt) \geq F_{z,z}(t) * F_{Pz,z}(t) * F_{z,z}(t) * F_{Pz,z}(\alpha t) * F_{z,z}(\alpha t)$$
$$\geq F_{Pz,z}(t) * F_{Pz,z}(\alpha t)$$
Thus $Pz = z$.

Step 4: Since $P(X) \subseteq R(X)$, there exists $w \in X$ such that $z = Pz = Rw$
By taking $x = x_{2n}, y = w$ in (b), we get
$$F_{Px_{2n},Qw}(kt) \geq F_{Cx_{2n},Rw}(t) * F_{Px_{2n},Cx_{2n}}(t) * F_{Qw,Rw}(t) * F_{Px_{2n},Rw}(\alpha t) * F_{Qw,Cx_{2n}}(\alpha t)$$
On letting $n \to \infty$, we get
$$F_{z,Qw}(kt) \geq F_{z,z}(t) * F_{z,z}(t) * F_{Qw,z}(t) * F_{z,z}(\alpha t) * F_{Qw,z}(\alpha t)$$
$$\geq F_{Qw,z}(t) * F_{Qw,z}(\alpha t)$$
Thus $Qw = z$
$\therefore Rw = Qw = z$
Since (Q, R) is compatible of type (P_2), we have $RQw = QQw$.

Therefore $Rz = Qz$.

Step 5: By taking $x = x_{2n}, y = z$ in (b), we get
$$F_{Px_{2n},Qz}(kt) \geq F_{Cx_{2n},Rz}(t) * F_{Px_{2n},Cx_{2n}}(t) * F_{Qz,Rz}(t) * F_{Px_{2n},Rz}(\alpha t) * F_{Qz,Cx_{2n}}(\alpha t)$$
On letting $n \to \infty$, ewe get
$$F_{z,Qz}(kt) \geq F_{z,Qz}(t) * F_{z,z}(t) * F_{Qz,Qz}(t) * F_{z,Qz}(\alpha t) * F_{Qz,z}(\alpha t)$$
$$\geq F_{Qz,z}(t) * F_{Qz,z}(\alpha t)$$
Thus $Qz = z$.
$\therefore Pz = Qz = Cz = Rz = z$.

i.e. z is a common fixed point for P, Q, R and C.

Case (ii): P is continuous and (P, C), (Q, R) are both compatible of type (P_2)
$PPx_{2n} \to Pz$, $PCx_{2n} \to Pz$ ($\because P$ is continuous)
$CPx_{2n} \to Pz$ ($\because$ (P, C) is compatible of type (P_2))

Step 6: By taking $x = Px_{2n}, y = y_{2n+1}$in (b), we get

$$F_{PPx_{2n},Qx_{2n+1}}(kt) \geq F_{CPx_{2n},Rx_{2n+1}}(t) * F_{PPx_{2n},CPx_{2n}}(t) * F_{Qx_{2n+1},Rx_{2n+1}}(t) * F_{PPx_{2n},Rx_{2n+1}}(\alpha t) * F_{Qx_{2n+1},CPx_{2n}}(\alpha t)$$

On letting $n \to \infty$, we get

$$F_{Pz,z}(kt) \geq F_{Pz,z}(t) * F_{Pz,Pz}(t) * F_{z,z}(t) * F_{Pz,z}(\alpha t) * F_{z,Pz}(\alpha t)$$
$$\geq F_{Pz,z}(t) * F_{Pz,z}(\alpha t)$$

Thus $Pz = z$.
Using step 4 and step 5, we get $z = Qz = Rz$.

Step 7: Since $Q(X) \subseteq C(X)$, there exists $w \in X$ such that $z = Qz = Cw$.
By taking $x = w, y = x_{2n+1}$ in (b), we get

$$F_{Pw,Qx_{2n+1}}(kt) \geq F_{Cw,Rx_{2n+1}}(t) * F_{Pw,Cw}(t) * F_{Qx_{2n+1},Rx_{2n+1}}(t) * F_{Pw,Rx_{2n+1}}(\alpha t) * F_{Qx_{2n+1},Cw}(\alpha t)$$

On letting $n \to \infty$, we get

$$F_{Pw,z}(kt) \geq F_{z,z}(t) * F_{Pw,z}(t) * F_{z,z}(t) * F_{Pw,z}(\alpha t) * F_{z,z}(\alpha t)$$
$$\geq F_{Pw,z}(t) * F_{Pw,z}(\alpha t)$$

Thus $z = Pw$, since $z = Qz = Cw$, hence $Pw = Cw$.
(P, C) is compatible of type (P_2), we have $CPw = PPw$ i.e. $Cz = Pz$.
$\therefore z = Pz = Cz = Qz = Rz$
i.e. z is a common fixed point for P, Q, R and C.
$\therefore$ z is a common fixed point for P, Q, R and C when C is continuous(or P is continuous) and (P, C), (Q, R) are compatible of type P_2 (or P_1)

Step 8: For uniqueness v be common fixed point for P, Q, R and C.
Take $x = z, y = v$ in the condition (b), we get

$$F_{Pz,Qv}(kt) \geq F_{Cz,Rv}(t) * F_{Pz,Cz}(t) * F_{Qv,Rv}(t) * F_{Pz,Rv}(\alpha t) * F_{Qv,Cz}(\alpha t)$$
$$\Rightarrow F_{z,v}(kt) \geq F_{z,v}(t) * F_{z,z}(t) * F_{v,v}(t) * F_{z,v}(\alpha t) * F_{v,z}(\alpha t)$$
$$\geq F_{z,v}(t) * F_{z,v}(\alpha t)$$

Thus v = z.
We conclude our paper with an open problem.
Open Problem 2.4: If $1 > k > \frac{1}{2}$ and $\alpha \in (2k, 2)$, is the result valid ?

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