Generalized k-normal Matrices

S. Krishnamoorthy and R. Subash

1Department of Mathematics, Government Arts College (Autonomous), Kumbakonam, Tamilnadu, India 612 001.
2Department of Mathematics, A.V.C. College of Engineering, Mannampandal, Mayiladuthurai, Tamilnadu, India 609305.
E-mail: subash_ru@rediffmail.com

Abstract

A matrix $A \in \mathbb{C}_{n \times n}$ is called generalized k-normal provided that there is a positive definite k-hermite matrix $H$ such that $HAH$ is k-normal. In this paper, these matrices are investigated and their canonical form, invariants and relative properties in the sense of congruence are obtained.

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Introduction

Let $\mathbb{C}_{n \times n}$ be the space of n xn complex matrices. Let ‘k’ be a fixed product of disjoint transposition in $S_n = \{1,2,\ldots,n\}$ (hence, involutary) and let $K$ be the associated permutation matrix of $k$. Two matrices $A, B \in \mathbb{C}_{n \times n}$ are called k-unitarily similar provided that there exists a k-unitarily matrix $U$ such that $B = KU^*AU$, where $U^*$ is the conjugate transpose matrix of $U$. For a matrix $A \in \mathbb{C}_{n \times n}$, there exists a k-unitary matrix $U$ such that $KU^*AU$ is an upper triangular matrix and $A$ is k-unitarily similar to a k-diagonal matrix if and only if $A$ is k-normal. Two matrices $A, B \in \mathbb{C}_{n \times n}$ are congruent if there exists a nonsingular matrix $P \in \mathbb{C}_{n \times n}$ such that $B = KP^*AP$. It is easy to see that, for a given matrix $A \in \mathbb{C}_{n \times n}$, there exists a nonsingular matrix
$P \in \mathbb{C}_{n \times n}$ such that $KP^*AP$ is an upper triangle matrix since k-unitary similar matrices $A, B$ are congruent.

In this paper, we obtain the canonical form and invariants for generalized k-normal matrices in the sense of congruence. We give some determinantal inequalities for generalized k-normal matrices and their k-hermite part and skew k-hermite part.

**Basic Definitions**

**Definition 2.1:** A matrix $A \in \mathbb{C}_{n \times n}$ is said to be k-hermitian, if $A = K A^* K$.

That is, $a_{ij} = \overline{a_{k(j)k(i)}}, i, j = 1, 2, \ldots, n$.

**Example 2.2:** $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & i \\ -i & 0 & 0 \end{bmatrix}$ is k-hermitian.

**Definition 2.3:** A matrix $A \in \mathbb{C}_{n \times n}$ is said to be skew k-hermitian, if $A = -K A^* K$.

That is, $a_{ij} = -\overline{a_{k(j)k(i)}}, i, j = 1, 2, \ldots, n$.

**Example 2.4:** $A = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & -i \end{bmatrix}$ is skew k-hermitian.

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**Definition 3.1[4]:** A matrix $A \in \mathbb{C}_{n \times n}$ is said to be k-normal, if $A A^* K = K A^* A$.

That is, $a_{ij} \overline{\alpha_{n-k(j)+k(i)}} = \overline{\alpha_{k(j) n-k(i)+1}} a_{ij}; i, j = 1, 2 \ldots n$.

**Example 3.2:** $A = \begin{bmatrix} 0 & -i & -3i \\ -i & -3i & 0 \\ -3i & 0 & -i \end{bmatrix}$ is k-normal.

**Definition 3.3[4]:** A matrix $A \in \mathbb{C}_{n \times n}$ is said to be k-unitary, if $A A^* K = K A^* A = K$.

**Example 3.4:** $A = \begin{bmatrix} i & 1 \\ \sqrt{2} & \sqrt{2} \\ 1 & i \end{bmatrix}$ is k-unitary.
\textbf{Definition 3.5:} A matrix $A \in \mathbb{C}_{n \times n}$ is called generalized k-normal if there is a positive definite k-hermite matrix $H$ such that $HAH$ is k-normal.

\textbf{Example 3.6:} $H = \begin{bmatrix} 1-i & 1 \\ 1 & 1+i \end{bmatrix}$ is a positive definite k-hermite matrix.

Then $A = \begin{bmatrix} -i & 1 \\ 0 & i \end{bmatrix}$ is generalized k-normal.

\textbf{Theorem 3.7:} $A \in \mathbb{C}_{n \times n}$ is generalized k-normal if and only if there exists a positive definite k-hermite matrix $X$ such that $KAXA^*A = AXA^*K$.

\textbf{Proof:} Assume that $A$ is generalized k-normal matrix. Then by definition (3.5), there exists a positive definite k-hermite matrix $H$ such that $HAH$ is k-normal.

Therefore,

$$K(HAH)^*H = (HAH)(HAH)^*K$$

$$\Rightarrow K H^* A^* H A H = H A H H^* A^* H^* K$$

$$\Rightarrow K(KHK)A^*(KHK)H = HAH(KHK)A^*(KHK)K$$

$$\Rightarrow H(KA(KH)^2A) = H(A(HK)^2A^*KH)$$

$$\Rightarrow K(A(KH)^2A = A(HK)^2A^*K) \tag{1}$$

Since $H$ is a positive definite, $H$ is non-singular. Then $H^2$ is positive definite. Let $X = H^2 = H^2K^2 = (HK)^2 = (HK)^2$. Then $X$ is a positive definite k-hermite matrix.

Therefore (1) $\Rightarrow K A^* A = A X A^* K$.

Conversely, Assume that $K A^* A = A X A^* K$ and $X$ is a positive definite k-hermite matrix. Then there exists a positive definite k-hermite matrix $H$ such that $X = H^2 = H^2K^2 = (HK)^2 = (HK)^2$.

Thus

$$K A^*(HK)^2A = A(HK)^2A^*K$$

$$\Rightarrow HKA^*KH = HAHKHK A^*KH$$

$$\Rightarrow (KH^2K)A^*KH = HAHKH^* A^*K$$

$$\Rightarrow K(HA^*KH)A^*KH = HAHKH^* A^*KH$$

$$\Rightarrow K(HAH)^*(HK) = (HAH)(HK)^2K$$

Hence $HAH$ is k-normal and then $A$ is generalized k-normal.

\textbf{Theorem 3.8:} $A \in \mathbb{C}_{n \times n}$ is generalized k-normal if and only if $A$ is congruent to a k-diagonal matrix.
Proof: Assume that A is a generalized k-normal matrix. Then by definition (3.5), there exists a positive definite k-hermite matrix H such that HAH is k-normal. By Schur’s theorem [2], there exists a k-unitary matrix U such that \( KU^*(HAH)U \) is k-diagonal matrix. Let \( P = HU \Rightarrow P' = U^*HK \Rightarrow P' = U^*KH \Rightarrow P' = U^*(KH^2) \Rightarrow P' = U^*H^2 \Rightarrow P' = U^*H \). Then P is nonsingular and \( KP'AP \) is k-diagonal matrix. That is, A is congruent to a k-diagonal matrix.

Conversely, Assume that A is congruent to a k-diagonal matrix. Then there exists a nonsingular matrix P such that \( KU^*(HAH)U = D \) is a k-diagonal matrix. By the polar factorization theorem of matrices, there exists a positive definite k-hermite matrix H and a k-unitary matrix U such that \( P = HU \Rightarrow P' = U^*HKH \Rightarrow P' = U^*(KH^2) \Rightarrow P' = U^*H^2 \Rightarrow P' = U^*H \). Thus \( KU^*(HAH)U = D \) is a k-diagonal matrix. By Schur’s theorem [2] and definition (3.5), A is generalized k-normal.

Theorem 3.9: Let \( A \in \mathbb{C}^{n \times n} \) be a generalized k-normal matrix. Then there exists a nonsingular matrix P such that \( A = KP' \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P \), where \( D = diag(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}) \), \( r = \text{rank}(A) \), \( -\pi \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_r < \pi \) and \( e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r} \) are all nonzero k-eigen values of \( P^{-1}(K-P)^{-1}A \).

Proof: Since A is a generalized k-normal matrix. By Theorem (3.8) there exists a nonsingular matrix Q such that \( A = KQ' diag(\lambda_1, \lambda_2, \ldots, \lambda_r, 0, 0, \ldots, 0)Q \), where \( r = \text{rank}(A) \) is the rank of A, \( \lambda_1, \lambda_2, \ldots, \lambda_r \) are nonzero complex numbers.

Let \( \lambda_j = \rho_j e^{i\theta_j} \), \( \rho_j > 0 \), \( j = 1, 2, \ldots, r \) and \( -\pi \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_r < \pi \).

Denote \( R = diag(\rho_1^{1/2}, \rho_2^{1/2}, \ldots, \rho_r^{1/2}, 1, \ldots, 1) \) and \( P = RQ \Rightarrow P' = Q'R^* \).

Then P is nonsingular and \( A = KP' \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P \).

Since \( P^{-1}K \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KP)^{-1}A = P^{-1}K \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KP)^{-1}(KP') \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P \)

\( = P^{-1} \begin{pmatrix} D^2 & 0 \\ 0 & 0 \end{pmatrix}P, e^{2i\theta_1}, e^{2i\theta_2}, \ldots, e^{2i\theta_r} \) are all nonzero k-eigen values of \( P^{-1} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}(KP)^{-1}A \).
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Without loss of generality, \(-\pi \leq 2\theta_{j_1} \leq 2\theta_{j_2} \leq \ldots \leq 2\theta_{j_r} < \pi \pmod{2\pi}\).

Then \(2\theta_r - 2\theta_i \leq 2\theta_r - 2\theta_{j_1} < 2\pi \pmod{2\pi}\). Thus \(\theta_r - \theta_i < \pi\).

Since \(A = KP^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P\),
\[
P^{-1}(KP^*)^{-1}A = P^{-1}(KP^*)^{-1}(KP^*) \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = P^{-1} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P.
\]

Hence \(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}\) are all nonzero k-eigen values of \(P^{-1}(KP^*)^{-1}A\).

**Theorem 3.10:** Let \(A \in \mathbb{C}_{n \times n}\) be a generalized k-normal matrix. If there exist a nonsingular matrix \(P\) and \(Q\) such that \(A = KP^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = KQ^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q\), where
\[
D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}), \quad -\pi \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_r < \pi, \quad \theta_r - \theta_i < \pi, \quad r = \text{rank}(A),
\]
\[
\tilde{D} = \text{diag}(e^{iw_1}, e^{iw_2}, \ldots, e^{iw_r}), \quad -\pi \leq w_1 \leq w_2 \leq \ldots \leq w_r < \pi \quad \text{and} \quad w_r - w_i < \pi.
\]

Then \(\theta_j = w_j\) for \(1 \leq j \leq r\).

**Proof:** Given \(A = KP^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = KQ^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q\)
\[
\Rightarrow KP^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = KQ^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q.
\]

Pre multiply by \(K\) on both sides.

Therefore, \(P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P = Q^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q\)
\[
\Rightarrow \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = (P^*)^{-1} Q^* \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} Q P^{-1} \tag{2}
\]

Denote \(QP^{-1} = R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}\), where \(R_{11}\) is an \(r \times r\) complex matrix.
\[
\Rightarrow (QP^{-1})^* = (P^*)^{-1} Q^* = R^* = \begin{pmatrix} R^*_{11} & R^*_{12} \\ R^*_{21} & R^*_{22} \end{pmatrix} \Rightarrow
\]
\[
(2) \Rightarrow \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R^*_{11} & R^*_{12} \\ R^*_{21} & R^*_{22} \end{pmatrix} \begin{pmatrix} \tilde{D} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} R^*_{11} \tilde{D} & 0 \\ R^*_{21} \tilde{D} & 0 \end{pmatrix} \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = \begin{pmatrix} R^*_{11} \tilde{D} R_{11} & R^*_{11} \tilde{D} R_{12} \\ R^*_{21} \tilde{D} R_{11} & R^*_{21} \tilde{D} R_{12} \end{pmatrix}.
\]
Thus \( D = R_i^* \tilde{D} R_i \). Since \( D \) is nonsingular, \( R_i \) and \( R_i^* \) are also nonsingular.

Hence
\[
D^2 = D(D^*)^{-1} = (R_i^* \tilde{D} R_i)(R_i^* \tilde{D}^* (R_i^*)^{-1})^* = R_i^* \tilde{D} (\tilde{D}^*)^{-1} (R_i^*)^{-1} = R_i^* \tilde{D}^2 (R_i^*)^{-1}.
\]
That is, \( D^2 \) and \( \tilde{D}^2 \) are similar. Thus \( D^2 \) and \( \tilde{D}^2 \) have the same k-eigen values.

That is, \( e^{2i\theta_j} = e^{2i\omega_j} \) for \( 1 \leq j \leq r \).

Therefore, \( 2\theta_j = 2\omega_j \) (mod \( 2\pi \)) for \( 1 \leq j \leq r \) and hence \( \theta_j = \omega_j \) (mod \( \pi \)) for \( 1 \leq j \leq r \). By the conditions of the theorem, \( \theta_j = \omega_j \) for \( 1 \leq j \leq r \).

**Remark 3.11:** Theorem (3.10) indicates that when a generalized k-normal matrix \( A \) satisfies \( A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P \), then \( D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}) \) is independent of the choice of the nonsingular matrix \( P \), that is, the k-diagonal matrix \( D \) is only determined by \( A \). So we have the following definitions.

**Definition 3.12:** Let \( A \) be a generalized k-normal matrix satisfying \( A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P \). Denote \( D(KA) = D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}) \), where \( r = \text{rank}(KA) \), \(-\pi \leq \theta_1 \leq \theta_2 \leq \ldots \leq \theta_r < \pi\) and \( \theta_r - \theta_1 < \pi \). Then \( \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} \) is called the canonical form of the generalized k-normal matrix \( A \) in the sense of congruence and \( \sigma_k(A) = (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}) \) is called the generalized k-spectrum of \( A \).

**Theorem 3.13:** Let \( A, B \in \mathbb{C}_{n \times n} \) be generalized k-normal matrices. Then \( A \) and \( B \) are congruent if and only if \( \sigma_k(A) = \sigma_k(B) \), that is \( A \) and \( B \) have the same generalized k-spectrum.

**Proof:** Assume that \( A \) and \( B \) are congruent. Then there exists a nonsingular matrix \( Q \) such that \( B = KQ^* A Q \).

By Theorem (3.9), there exists a nonsingular matrix \( P \) such that
\[
A = K P^* \begin{pmatrix} D(A) & 0 \\ 0 & 0 \end{pmatrix} P.
\]

Thus \( B = KQ^* KP^* \begin{pmatrix} D(A) & 0 \\ 0 & 0 \end{pmatrix} PQ \). (3)

Denote \( R = PQ \), then \( R \) is nonsingular. \( \Rightarrow R = PKQK \Rightarrow R^* = KQ^* KP^* \).
Therefore (3) \( \Rightarrow \) \( B = R^* \begin{pmatrix} D(A) & 0 \\ 0 & 0 \end{pmatrix} R \Rightarrow KB = R^* \begin{pmatrix} D(KA) & 0 \\ 0 & 0 \end{pmatrix} R \).

By a Theorem (3.10), \( D(KB) = D(KA). \) Hence \( \sigma(KA) = \sigma(KB) \) \( \Rightarrow \) \( \sigma_k(A) = \sigma_k(B). \) Conversely, let \( \sigma_k(A) = \sigma_k(B) \) \( \Rightarrow \) \( \sigma(KA) = \sigma(KB) \) \( \Rightarrow \) \( D(KA) = D(KB) = D \) and there exists a nonsingular matrices \( P, Q \) such that

\[ A = K P^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} P \tag{4} \]

and \( B = K Q^* \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} Q \tag{5} \)

Pre multiply by \( (KP^*)^{-1} \) and post multiply by \( P^{-1} \) in (4). Therefore (4) \( \Rightarrow \) \( (KP^*)^{-1} AP^{-1} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}. \) Hence \( B = (K Q^*)(KP^*)^{-1} AP^{-1} Q. \) Therefore \( A \) and \( B \) are congruent.

**Determinantal properties**

Let \( A \in \mathbb{C}_{n \times n}. \) Alternatively, let \( H(A) = \frac{1}{2}(A + KA^*K), \) the k-hermite part and

\[ S(A) = \frac{1}{2}(A - KA^*K), \] the skew k-hermite part. It is known that \( A = H(A) + S(A) \) and \( KA^*K = H(A) - S(A). \) \( A \) is called positive definite provided that \( H(A) \) is positive definite k-hermite matrix. It is easy to know that \( A \) is positive definite if and only if \( \text{Re}(Kx^*, Ax) > 0 \) for each nonzero complex column vector \( x = (x_1, x_2, ..., x_n)^T, \) where \( \text{Re}(a) \) denotes the real part of a complex number ‘ \( a \) ’.

**Theorem 4.1:** Let \( A \in \mathbb{C}_{n \times n} \) be a positive definite matrix, then \( A \) is generalized k-normal.

**Proof:** since \( A \) is positive definite, \( H(A) \) is nonsingular.

Then

\[ KA^*K[H(A)]^{-1}A = [H(A) - S(A)][H(A)]^{-1}[H(A) + S(A)] \]

\[ = \{H(A)[H(A)]^{-1} - S(A)[H(A)]^{-1}\}[H(A) + S(A)] \]

\[ = \{I - S(A)[H(A)]^{-1}\}[H(A) + S(A)] \]

\[ = H(A) + S(A) - S(A)[H(A)]^{-1}[H(A) + S(A)] \]

\[ = H(A) + S(A) - S(A)[H(A)]^{-1}H(A) - S(A)[H(A)]^{-1}S(A) \]
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\[ H(A) + S(A) = H(A) - S(A) \]

\[ A(H(A))^{-1} K = [H(A) + S(A)][H(A)]^{-1}(H(A) - S(A)] \]

\[ = (I + S(A)[H(A)]^{-1})[H(A) - S(A)] \]

\[ H(A) - S(A) + S(A)[H(A)]^{-1}[H(A) - S(A)] \]

\[ = H(A) - S(A) + S(A)[H(A)]^{-1}S(A) \]

\[ = H(A) - S(A)[H(A)]^{-1}S(A) \]  \hspace{1cm} (5)

From (5) and (6), we get

\[ KA'K[H(A)]^{-1}A = A[H(A)]^{-1}KA'K \]  \hspace{1cm} (6)

Since H is a positive definite k-hermite matrix. Hence \([H(A)]^{-1}\) is a positive definite k-hermite matrix. By a Theorem (3.7), \(X = K[H(A)]^{-1} = [H(A)]^{-1}K\) is a positive definite k-hermite matrix. Therefore (7) \(\Rightarrow KA'XA = AXA'K\). Hence A is generalized k-normal matrix.

**Theorem 4.2:** Let ‘n’ be an integer with \(n \geq 3\) and \(A \in \mathbb{C}_{n \times n}\) a generalized k-normal matrix. Then \(\det(A) \geq \det(H(A)) + \det(S(A))\). Further, the equality holds if and only if \(r = \text{rank}(A) < n\) or A is a k-hermite matrix or A is a skew k-hermite matrix.

**Proof:** There exists a nonsingular matrix \(P\) such that \(A = KP'\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P\). It is easy to see that \(H(A) = KP'H(D)P\) and \(S(A) = KP'S(D)P\). Thus \(\text{rank}[H(A)]=\text{rank}[H(D)]\) and \(\text{rank}[S(A)]=\text{rank}[S(D)]\). We distinguish the following cases.

**Case (1):** If \(\text{rank}(A) < n\). Then \(\text{rank}[H(A)]=\text{rank}[H(D)] \leq \text{rank}(A) < n\) and \(\text{rank}[S(A)]=\text{rank}[S(D)] \leq \text{rank}(A) < n\). Thus \(\det(A) = \det(H(A)) = \det(S(A)) = 0\) and so \(\det(A) = \det(H(A)) + \det(S(A))\).

**Case (2):** If \(\text{rank}(A) = n\). Now, \(D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_n})\),

\[ \Rightarrow D = \text{diag}(\cos \theta_1 + i \sin \theta_1, \cos \theta_2 + i \sin \theta_2, ..., \cos \theta_n + i \sin \theta_n) \].

\[ \Rightarrow H(D) = \text{diag}(\cos \theta_1, \cos \theta_2, ..., \cos \theta_n), S(D) = i \text{diag}(\sin \theta_1, \sin \theta_2, ..., \sin \theta_n) \] and \(A = KP'DP\). Then \(\det(A) = (\det(KP)) e^{i\sum_{j=1}^{n} \theta_j}\). \(\Rightarrow \det(H(A)) = (\det(KP)) - \sum_{j=1}^{n} \cos \theta_j\) and \(\det(S(A)) = i^n (\det(KP))^2 \sum_{j=1}^{n} \sin \theta_j\).
**Sub case (2.1):** If \( \cos \theta_j = 0 \) for some \( j \).

Then \( |\det H(A)| + |\det S(A)| = |\det KP| \prod_{j=1}^{n} |\sin \theta_j| \leq |\det KP| \leq |\det (A)| \). Further, the above equality holds if and only if \( |\sin \theta_j| = 1 \) for all \( j \), if and only if \( \cos \theta_j = 0 \) for all \( j \), if and only if \( \text{H}(A)=0 \) if and only if \( A \) is skew \( k \)-hermitian.

**Sub case (2.2):** If \( \sin \theta_j = 0 \) for some \( j \).

Then \( |\det H(A)| + |\det S(A)| = |\det KP| \prod_{j=1}^{n} |\cos \theta_j| \leq |\det KP| \leq |\det (A)| \). Further, the above equality holds if and only if \( |\cos \theta_j| = 1 \) for all \( j \), if and only if \( \sin \theta_j = 0 \) for all \( j \), if and only if \( \text{S}(A)=0 \) if and only if \( A \) is \( k \)-hermitian.

**Sub case (2.3):** If \( \cos \theta_j \neq 0 \) and \( \sin \theta_j \neq 0 \) for all \( j \).

Now \( |\det H(A)| + |\det S(A)| = |\det KP| \left( \prod_{j=1}^{n} |\cos \theta_j| + \prod_{j=1}^{n} |\sin \theta_j| \right) \)

\( \Rightarrow |\det H(A)| + |\det S(A)| = |\det KP| \left( |\cos \theta_1| |\cos \theta_2| + |\sin \theta_1| |\sin \theta_2| \right) \leq |\det KP| = |\det (A)| \)

By summing up the above cases the proof of this theorem is completed.

**Remark 4.3:** Let \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \) be non negative real numbers. Then \( \left( x_1, x_2, \ldots, x_n \right)^\frac{1}{n} + \left( y_1, y_2, \ldots, y_n \right)^\frac{1}{n} \leq \left( x_1 + y_1 \right)^\frac{1}{n} \left( x_2 + y_2 \right)^\frac{1}{n} \cdots \left( x_n + y_n \right)^\frac{1}{n} \) and the equality holds if and only if there exists and integer \( j \) such that \( x_j = y_j = 0 \) or \( (x_1, x_2, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \) are linearly dependent.

**Theorem 4.4:** Let \( 'n' \) be an integer with \( n \geq 3 \) and \( A \in \mathbb{C}_{n \times n} \) a generalized \( k \)-normal matrix with generalized \( k \)-spectrum \( \sigma_k(A) = (e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_r}) \), where \( r = \text{rank}(A) \).

Then \( |\det (A)|^\frac{2}{n} \geq |\det \text{H}(A)|^\frac{2}{n} + |\det \text{S}(A)|^\frac{2}{n} \). Further, the equality holds if and only if \( r<n \) or \( (\cos^2 \theta_1, \cos^2 \theta_2, \ldots, \cos^2 \theta_r) \) and \( (\sin^2 \theta_1, \sin^2 \theta_2, \ldots, \sin^2 \theta_r) \) are linearly dependent.

**Proof:** There exists a nonsingular matrix \( P \) such that \( A = KP\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}P \). It is easy to see that \( H(A) = KP^2H(D)P \) and \( S(A) = KP^2S(D)P \). Thus \( \text{rank} \ [\text{H}(A)] = \text{rank} \ [\text{H}(D)] \) and \( \text{rank} \ [\text{S}(A)] = \text{rank} \ [\text{S}(D)] \). We distinguish the following cases.

**Case (1):** If \( \text{rank} \ (A)<n \).

Then \( \text{rank} \ [\text{H}(A)] = \text{rank} \ [\text{H}(D)] \leq \text{rank} \ (A)<n \) and \( \text{rank} \ [\text{S}(A)] = \text{rank} \ [\text{S}(D)] \leq \text{rank} \ (A)<n \).
Thus \( \det(A) = \det(H(A)) = \det(S(A)) = 0 \).

Therefore \( \left| \det(A) \right|^2 \geq \left| \det(H(A)) \right|^2 + \left| \det(S(A)) \right|^2 \).

**Case (2):** If \( \text{rank}(A) = n \). Now, \( D = \text{diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \),
\[ H(D) = \text{diag}(\cos(\theta_1), \cos(\theta_2), \ldots, \cos(\theta_n)), S(D) = i \text{diag}(\sin(\theta_1), \sin(\theta_2), \ldots, \sin(\theta_n)) \] and \( A = KP^*DP \). Then \( \det(A) = (\det(KP))^2 \prod_{j=1}^{n} e^{i\theta_j} \). \( \Rightarrow \) \( \det(H(A)) = (\det(KP))^2 \prod_{j=1}^{n} \cos(\theta_j) \) and 
\[ \det(S(A)) = i^n (\det(KP))^2 \prod_{j=1}^{n} \sin(\theta_j). \]

Thus \( \left| \det(H(A)) \right|^2 + \left| \det(S(A)) \right|^2 = \left| \det(KP) \right|^4 \left( \prod_{j=1}^{n} \cos(\theta_j) \right)^2 + \left( \prod_{j=1}^{n} \sin(\theta_j) \right)^2 \).

By Remark (4.3), \( \left| \det(H(A)) \right|^2 + \left| \det(S(A)) \right|^2 \leq \left| \det(A) \right|^2 \) and the equality holds if and only if there exists an integer \( j \) such that \( \cos^2(\theta_j) = \sin^2(\theta_j) = 0 \) or \( \cos^2(\theta_1), \cos^2(\theta_2), \ldots, \cos^2(\theta_n) \) and \( \sin^2(\theta_1), \sin^2(\theta_2), \ldots, \sin^2(\theta_n) \) are linearly dependent. Since \( \cos^2(\theta_j) + \sin^2(\theta_j) = 1 \), the equality holds if and only if \( \cos^2(\theta_1), \cos^2(\theta_2), \ldots, \cos^2(\theta_n) \) and \( \sin^2(\theta_1), \sin^2(\theta_2), \ldots, \sin^2(\theta_n) \) are linearly dependent. Summing up the above cases, we finish the proof of this theorem.

**References**