International Journal of Computational Science and Mathematics.
ISSN 0974-3189 Volume 4, Number 1 (2012), pp. 49-56

http://www.irphouse.com

# Anti Fuzzy Implicative Filters in Lattice W-Algebras

**M. Basheer Ahamed[1] and A. Ibrahim[2]**

[1]*Department of Mathematics, Faculty of Science,*
*University of Tabuk, Tabuk, Kingdom of Saudi Arabia.*
*E-mail: bahamed@ut.edu.sa*
[2]*Department of Mathematics, Texcity Arts & Science College,*
*Coimbatore-641 105, Tamilnadu, India.*
*E-mail: ibrahimaadhil@yahoo.com*

**Abstract**

In this paper, we introduce the concept of an anti fuzzy implicative filters of lattice Wajsberg algebras. We find characterizations of fuzzy implicative filters and anti fuzzy implicative filters. We discuss a relation between fuzzy filters and anti fuzzy implicative filters in lattice Wajsberg algebra. Finally, we obtain an extension property of anti fuzzy implicative filters.

**Keywords:** Wajsberg algebras, Implicative filters, Fuzzy implicative filters, Anti fuzzy implicative filters.

**Mathematical subject Classification 2010:** 03E72, 03G25, 06F35.

## Introduction

In the fileld of logical system whose propositional value is given in a lattice, Modchaj Wajsberg proposed the concept of lattice Wajsberg algebras (W- algebras), in 1935. A. Rose et al. published the proof of W- algebras. Josep. M. Font et al. proposed the notion of implicative filters in lattice W- algebras and discussed some of their properties in [4]. The aim of this paper is to introduce the notion of anti fuzzy implicative filters and investigate some useful properties. We obtain characterizations of fuzzy filters and anti fuzzy implicative filters. We discuss a relation between fuzzy filters and anti fuzzy implicative filters in lattice W- algebras. Also we give an extension of anti fuzzy implicative filters.

## Preliminaries

The following some useful results and properties which are help for our main results.

**Definition 2.1[4]:** Let (A, $\rightarrow$, $*$ , 1) be an algebra with complement "$*$" and a binary operation "$\rightarrow$" is called Wajsberg algebra (W-algebra) if and only if it satisfies the following axioms, $\forall x, y, z \in$ A.

(i) $1 \rightarrow x = x$

(ii) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$

(iii) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$

(iv) $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1$

**Proposition 2.2[4].** The W-algebra (A, $\rightarrow$, $*$, 1) satisfies the following equations and implications, $\forall x, y, z \in$ A.

(i) $x \rightarrow x = 1$

(ii) If $x \rightarrow y = y \rightarrow x = 1$ then $x = y$

(iii) $x \rightarrow 1 = 1$

(iv) $x \rightarrow (y \rightarrow x) = 1$

(v) If $x \rightarrow y = y \rightarrow z = 1$ then $x \rightarrow z = 1$

(vi) $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

(vii) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$

(viii) $x \rightarrow 0 = x \rightarrow 1^* = x^*$

(ix) $(x^*)^* = x$

(x) $x^* \rightarrow y^* = y \rightarrow x$

**Proposition 2.3[4].** The W-algebra (A, $\rightarrow$, $*$ , 1) satisfies the following equations and implications, $\forall x, y, z \in$ A.

(i) If $x \leq y$ then $x \rightarrow z \geq y \rightarrow z$

(ii) If $x \leq y$ then $z \rightarrow x \leq z \rightarrow y$

(iii) $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$

(iv) $(x \vee y)^* = (x^* \wedge y^*)$

(v) $(x \wedge y)^* = (x^* \vee y^*)$

(vi) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$

(vii) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$

(viii) $(x \rightarrow y) \vee (y \rightarrow x) = 1$

(ix) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$

(x) $(x \wedge y) \rightarrow z = (x \rightarrow y) \vee (x \rightarrow z)$

(xi) $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$

(xii) $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$

**Definition 2.4[4].** The W-algebra A is called lattice W-algebra if it satisfies the following conditions, $\forall\ x,\ y \in \mathrm{A}$.

(i) A partial ordering $\leq$ on a lattice W – algebra A, such that $x \leq y$ if and only if $x \rightarrow y = 1$

(ii) $(x \vee y) = (x \rightarrow y) \rightarrow y$

(iii) $(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$. Thus, we have (A, $\vee$, $\wedge$, 0, $*$, 1) is a lattice

W-algebra with lower bound 0.

**Definition 2.5[4].** Let A be W- algebra. A subset F of A is called an implicative filter of A if it satisfies the following axioms for all $x, y \in \mathrm{A}$.

(i) $1 \in \mathrm{F}$

(ii) $x \in \mathrm{F}$ and $x \rightarrow y \in \mathrm{F}$ imply $y \in \mathrm{F}$

**Definition 2.6[10].** Let X be a set. A function $\mu : \mathrm{X} \rightarrow [0, 1]$ is called a fuzzy subset on X, for each $x \in$ X, the value of $\mu(x)$ describes a degree of membership of $x$ in $\mu$.

**Definition 2.7[10].** Let $\mu$ be a fuzzy set in a set X. For $t \in [0, 1]$, the set $\mu_t = \{x \in \mathrm{X}$ / $\mu(x) \geq t\}$ is called a level subset of $\mu$.

**Definition 2.8[10].** Let $\mu$ be fuzzy subset of a W- algebra A. Then for $t \in [0, 1]$, the set $\mu^t = \{\ x \in \mathrm{A}\ /\ \mu(x) \leq t\ \}$ is called the lower $t$- level cut of $\mu$.

**Definition 2.9[1]** A fuzzy subset $\mu$ of lattice W- algebra A is called a fuzzy implicative filter of A if it satisfies the following axioms.

(i) $\mu(1) \geq \mu(x)$, for all $x \in \mathrm{A}$

(ii) $\mu(y) \geq \min\{\mu(x \rightarrow y),\ \mu(x)\}$, for all $x,\ y \in \mathrm{A}$.

**Proposition 2.10[11].** Let $\mathrm{I}_1$ and $\mathrm{I}_2$ be subsets of lattice W- algebra A with $\mathrm{I}_1 \subseteq \mathrm{I}_2$, if $\mathrm{I}_1$ is implicative filter then so is $\mathrm{I}_2$.

## Main results

In this section, we introduce anti fuzzy implicative filter of lattice W-algebra and investigate some properties.

**Definition 3.1.** A fuzzy subset $\mu$ of lattice W- algebra A is said to be an anti fuzzy implicative filter of A if it satisfies the following axioms.

(i) $\mu(1) \leq \mu(x)$

(ii) $\mu(y) \leq \max\{\ \mu(x \rightarrow y),\ \ \mu(x)\ \}$ for all $x,\ y \in \mathrm{A}$

**Example 3.2.** Let A = {0, $a$, $b$, $c$, 1} be a set with Figure (1) as partial ordering. Define unary operation "$*$" and a binary operation "$\rightarrow$" on A as in the Table (1) and Table (2)

| $x$ | $x^*$ |
|---|---|
| 0 | 1 |
| $a$ | $b$ |
| $b$ | $a$ |
| $c$ | $c$ |
| 1 | 0 |

Table (1)

| $\rightarrow$ | 0 | $a$ | $b$ | $c$ | 1 |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | $c$ | 1 | 1 |
| $b$ | $a$ | $c$ | 1 | 1 | 1 |
| $c$ | $c$ | $c$ | $c$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | 1 |

Table (2)

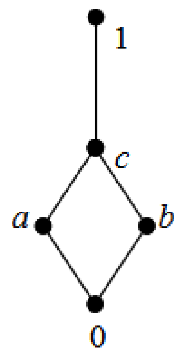

Figure (1)

Define $\vee$ and $\wedge$ operations on A as follow:

$(x \vee y) = (x \rightarrow y) \rightarrow y$,

$(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$, for all $x, y \in$ A. Then, we have A is lattice W-algebra.

Consider the fuzzy subset $\mu$ on A is defined by,

$$\mu(x) = \begin{cases} 0.2 \text{ } if \text{ } x = 1 \\ 0.7 \text{ } otherwise \end{cases} \quad \forall x \in \text{A}$$

Then, $\mu$ is anti fuzzy implicative filter of A.

**Proposition 3.3.** Let $\mu$ be an anti fuzzy implicative filters of lattice W- algebra A then for all $x, y \in$ A, we have, $x^* \geq y^* \Rightarrow \mu(x^*) \geq \mu(y^*)$.

**Proof:** For any $x, y \in$ A, we have $x^* \geq y^*$ if and only if $y^* \rightarrow x^* = 1$

Now $\mu(y^*) = \mu(1 \rightarrow y^*)$

$= \mu((y^* \rightarrow x^*) \rightarrow y^*)$

$\mu(x^*) = \mu(1 \rightarrow x^*)$

$= \mu((y^* \rightarrow x^*) \rightarrow x^*)$

$\geq \mu(y^*)$

Thus, we get $x^* \geq y^* \Rightarrow \mu(x^*) \geq \mu(y^*)$.

**Proposition 3.4.** A fuzzy subset $\mu$ of A is an anti fuzzy implicative filter if and only if for any $x, y \in$ A, $x \leq y \to z$ implies $\mu(z) \leq \max\{\mu(x), \mu(y)\}$.

**Proof:** If $\mu$ is fuzzy subset of A and satisfying the condition $x \leq y \to z$ implies $\mu(z) \leq \max\{\mu(x), \mu(y)\}$, $\forall\, x, y, z \in$ A.

Since $x \leq x \to 1$ implies $\mu(1) \leq \max\{\mu(x), \mu(x)\} = \mu(x)$

Thus, we get $\mu(1) \leq \mu(x)$

Since $x \leq y \to z$ it follows that replacing $z$ by $x$ we have the following condition $\mu(y) \leq \max\{\mu(y \to x), \mu(x)\}$. Thus, $\mu$ is anti fuzzy implicative filter of A By the definition 3.1 of anti fuzzy implicative filter of A, we get the converse part immediately.

**Proposition 3.5.** Any anti fuzzy implicative filter of lattice W- algebra A is implicative filter of A.

**Proof:** We have, $\mu(1) \leq \mu(x)$ and $\mu(y) \leq \max\{\mu(x \to y), \mu(x)\}, \forall\, x, y \in$ A. Clearly, we get any anti fuzzy implicative filter of A is a implicative filter of A.

**Note:** The converse of the proposition 3.5 is not true.

**Example 3.6.** Let A = {0, $a$, $b$, $c$, $d$, 1} be a set with Figure (2) as partial ordering. Define unary operation "$*$" and a binary operation "$\to$" on A as in the tables (3) and (4).

| $x$ | $x^*$ |
|---|---|
| 0 | 1 |
| $a$ | $c$ |
| $b$ | $d$ |
| $c$ | $a$ |
| $d$ | $b$ |
| 1 | 0 |

Table (3)

| $\to$ | 0 | $a$ | $b$ | $c$ | $d$ | 1 |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $c$ | 1 | $b$ | $c$ | $b$ | 1 |
| $b$ | $d$ | $a$ | 1 | $b$ | $a$ | 1 |
| $c$ | $a$ | $a$ | 1 | 1 | $a$ | 1 |
| $d$ | $b$ | 1 | 1 | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | 1 |

Table (4)

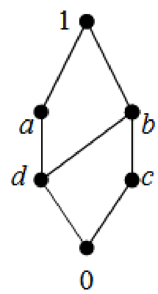


Figure (2)

Define $\vee$ and $\wedge$ operations on A as follow:

$(x \vee y) = (x \to y) \to y$,

$(x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in$ A. Then we have A is lattice W-algebra.

A fuzzy subset $\mu$ on A defined by, $\mu(x) = \begin{cases} 0.8 & if\ \ x=1 \\ 0.3 & otherwise \end{cases}$ $\forall x \in$ A

Then, we have $\mu$ is fuzzy implicative filter of lattice W-algebra A, but $\mu$ is not a anti fuzzy implicative filter of A since $\mu(1) \not\leq \mu(x)$.

**Proposition 3.7.** Any fuzzy subset $\mu$ of lattice W- algebra A is anti fuzzy implicative filter of A if and only if for each $t \in [0, 1]$, $\mu^t$ is either empty of an implicative filter of A.

**Proof:** If $\mu$ is anti fuzzy implicative filter of A and $\mu^t \neq \phi,\ \forall t \in [0, 1]$

Obviously, we get, $1 \in \mu^t$, since $\mu(1) \leq t$

Let $x, y, z \in$ A be such that $x \in \mu^t$ and $x \to y \in \mu^t$. Then, we have $\mu(x) \leq t$ and $\mu(x \to y) \leq t$. Thus, by the definition 3.1 we get $\mu(y) \leq \max\{ \mu(x), \ \mu(x \to y) \} \leq t$

Therefore, by the proposition 3.5, we have, any anti fuzzy implicative filter of lattice W-algebra A is a implicative filter A. Hence, we get $\mu^t$ is an implicative filter of A.

Conversely, if for each $t \in [0, 1]$, $\mu^t$ is either empty or an implicative filter of A

**Part (1):** For any $x \in$ A, let $\mu(x) = t$ then, we get $x \in \mu^t$

Since $\mu^t \neq \phi$ is an implicative filter of A, $1 \in \mu^t$

Hence, we have $\mu(1) \leq t = \mu(x)$, $\mu(1) \leq \mu(x)$.

**Part (2):** We show that $\mu(y) \leq \max\{ \mu(x \to y), \ \mu(x) \}$

If not, $\exists\ x^*, y^*, z^* \in$ A such that $\mu(y^*) > \max\{ \mu(x^* \to y^*), \ \mu(x^*) \}$

Let $t_1 = \frac{1}{2}[\mu(y^*) + \max\{ \mu(x^* \to y^*), \ \mu(x^*)\}]$

Then, we get $\mu(x^*) > t_1 > \max\{ \mu(x^* \to y^*), \ \mu(x^*) \}$

Hence, we have $x^* \notin \mu_{t_1}$ and $x^* \to y^* \in \mu_{t_1}$ is not an implicative filter of A

Which is a contradiction to implicative filter

Therefore, we have $\mu$ is anti fuzzy implicative filter of A.

**Proposition 3.8.** Let $\mu_1$ and $\mu_2$ be any fuzzy subsets of lattice W- algebra of A with $\mu_1 \leq \mu_2$ and $\mu_1(1) = \mu_2(1)$. If $\mu_1$ is anti fuzzy implicative filter of A then so $\mu_2$.

**Proof:** Let us show that $\mu_2$ is an anti fuzzy implicative filter of A. It is enough to show for any $t \in [0, 1]$, ${\mu_2}^t$ is either empty or an implicative filter of A.

If ${\mu_2}^t \neq \phi$ and ${\mu_1}^t \subseteq {\mu_2}^t$, Let $x \in {\mu_1}^t$ then $t \geq \mu_1(x)$, $\forall x \in$ A and so $t \geq \mu_2(x)$, $\forall x \in$ A That is, $x \in {\mu_2}^t$. Since $\mu_1$ is anti fuzzy implicative filter of A, from the proposition 3.7, we have ${\mu_1}^t$ is either empty or an implicative filter of A. Since ${\mu_1}^t \neq \phi$, ${\mu_1}^t$ is an implicative filter of A. Thus by the proposition 2.10, we get ${\mu_2}^t$ is also an implicative filter of A. Hence $\mu_2$ is anti fuzzy implicative filter of A.

**Proposition 3.9.** If *I* is an implicative filter of lattice W- algebra A, then there is a anti fuzzy implicative filter $\mu$ of A such that $\mu_t =$ F for some $t \in$ (0, 1).

**Proof:** Let $\mu$ be a fuzzy subset of A defined by $\mu(x) = \begin{cases} t & if \quad x \in I \\ 0 & if \quad x \notin I \end{cases} \quad \forall x \in \mathrm{A}$

Where *t* is a fixed number (0 < *t* < 1). We verify that $\mu$ is an anti fuzzy implicative filter of A. For any $x$, $y \in$ A such that $x \in$ F and $(x \to y) \in$ F then by the definition 2.5 of implicative filter, we have, $y \in$ F. Then, we get $\mu(x \to y) = \mu(x) = \mu(y) = t$ and so $\mu(y) = \max\{\, \mu(x \to y),\ \mu(x)\, \}$. If at least one of $\mu(x \to y)$ or $\mu(x)$ is 0

Hence, we have $\mu(y) \leq \max\{\, \mu(x \to y),\ \mu(x)\, \}$ for all $x$, $y \in$ A

Since $1 \in$ A, $\mu(1) = t \leq \mu(y)$, for all $y \in$ A

Therefore, $\mu$ is a fuzzy implicative filter of A. Obviously, $\mu_t =$ F.

## References


[1] M. Basheer Ahamed and A. Ibrahim, Fuzzy implicative filters of lattice Wajsberg Algebras, Advances in Fuzzy Mathematics, volume 6, Number 2 (2011), pp. 235-243.

[2] C. C. Chang, Algebraic analysis of many-valued logics, Transactions of the American Math. Soc, 88 (1958) 467-490.

[3] C. C. Chang, A new proof of the completeness of the Lukaiewicz axioms, Transactions of the American. Math. Soc. 93 (1959) 74-80.

[4] Josep. M. Font et al. Wajsberg Algebras, STOCHASTICA Vol VIII, No 1, (1984), 5-31.

[5] S. M. Hong and Y. B. Jun, Anti fuzzy ideals in *BCK*- algebras, Kyungpook Math. J. 38 (1998), 145-150.

[6] Y. B. Jun, Implicative filters of lattice implication algebras, Bull. Korean Math. Soc. 4 (1997), no. 2, 193-198.

[7] Y.B. Jun, E. H. Roh, J. Meng and X. L. Xin, Fuzzy prime and fuzzy irreducible ideals in *BCK*-algebras, SOOCHOW JOURNAL OF MATHEMATICS, Volume 21, No. 1, (1995), pp.49-56.

[8] J. Liu and Y. Xu, Filters and structure of lattice implication algebra, Chinese Science Bulletin 42 (1997), no. 18, 1517-1520.

[9] Y. Komori, super-Lukasiewicz propositional logics, Nagoya Mth. J, 84 (1981), 119-133.

[10] Won Kyun Jeong, On anti fuzzy prime ideals in *BCK* – algebras, JOURNAL OF THE CHUNGCHEONG MATHEMATICAL SOCIETY, Volume 12, (1999), 15-21.

[11] Yiquan Zhu, Qun Zhang and Eun Hwan Roh, On fuzzy implication filters of lattice implication algebras, Commun. Korean Math. Soc. 18 (2003), no. 4, 621-628.

[12] L. A. Zadeh, Fuzzy sets Inform and Control 8 (1965), 338-353.