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Domination in Fuzzy Graph: A New Approach

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Abstract

In the existing literature, as for as authors knowledge, *u* dominates *v* always implies that *v* dominates *u* in fuzzy graph [6]. In this paper, the concept of domination in fuzzy graph is introduced in a different approach. That is *u* dominates *v* need not imply *v* dominates *u* and vice versa. Using this new concept on domination in fuzzy graph some important theorems have been proved in this paper. The independent domination is also defined with this concept. The domination number (γ), the independent domination number (*i*) and the bounds on these parameters are discussed.

Keywords: Fuzzy graph, Effective edge, Domination, Independent Domination.

Introduction

The concept of domination and determines the domination number for several fuzzy graphs are discussed in [6]. In this paper, we introduced the concept of domination in fuzzy graph which is different from the existing literature. Also we discussed about the domination number (γ), independence set, independent domination number *(i)* and bounds on the domination number. For terminologies and definitions one can refer [1-7]. The fuzzy graph $G(\sigma, \mu)$ used here is a simple connected fuzzy graph.

Preliminaries

Let V be a finite non empty set and E be the collection of all two element subsets of V. A *fuzzy graph* $G(\sigma, \mu)$ is a set with a pair of relations $\sigma: V \to [0, 1]$ and $\mu : VXV \to [0, 1]$ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all u, v in V. The scalar cardinality of a fuzzy set V is the sum of the membership grades of the elements of the fuzzy set and is denoted by $|V|$. That is $|V| = \sum_{u\in V}\sigma(v)$ [3]. The *order* and *size* of a fuzzy graph is defined by the scalar cardinality of *V* and the scalar cardinality of *VXV* and are denoted by *'p'* and *'q'* respectively. That is $\sum_{v\in V}\sigma(v) = p$ and $\sum_{u,v\in V}\mu(u,v) = q$. Let *G* (σ, μ) be a fuzzy graph on *V* and $V_1 \subseteq V$. Define σ_1 on V_1 by $\sigma_1(u) = \sigma(u)$ for all $u \in V_1$ and define μ_1 on V_1XV_1 by $\mu_1(u,v) = \mu(u,v)$ for all $u, v \in V_1$. Then $G_1(\sigma_1, \mu_1)$ is called a *fuzzy sub graph* of *G* induced by V_1 and is denoted by $<V_1>$. $\bar{G}(\sigma, \bar{\mu})$ is called the *complement* of a fuzzy graph $G(\sigma, \mu)$, where $\bar{\mu}(u,v) = \sigma(u)\wedge\sigma(v) - \mu(u,v)$. An edge $(u, v) = e$ is called an *effective edge* if $\mu(u, v) = \sigma(u)\wedge\sigma(v)$. If (u, v) is an effective edge, then *u* and *v* are *adjacent effective edges*.

A fuzzy graph is said to be *strong fuzzy graph* if $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all *u, v* in *V*. A vertex *u* is said to be *isolated vertex* $\mu(u, v) < \sigma(u)\wedge\sigma(v)$ for all $v \in V-\{u\}$. A fuzzy graph $G(\sigma, \mu)$ is said to be *complete fuzzy graph,* if all the edges are effective between every pair of vertices and is denoted by K_σ. $N(u)=\{v\in V/ \mu(u, v) = \sigma(u)\wedge\sigma(v)\}$ is called the *open neighborhood of u* and $N[u] = N(u) \cup\{u\}$ is the *closed neighborhood* of *u*. If an edge (u, v) is an effective edge, then its incident is said to be *effective incident* with both the vertices. The *effective incident degree of a vertex u* of a fuzzy graph is defined to be the sum of the scalar cardinality of the effective edges incident at *u* and is denoted by $d_E(u)$. The minimum effective incident degree is $\delta_E(G) = min\{d_E(u) / u\in V\}$ and the maximum effective incident degree is $\Delta_E(G) = max\{d_E(u) / u\in V\}$. A fuzzy graph *G* is said to be *bipartite* if the vertex set *V* can be partitioned into two sets V_1 and V_2 such that $\mu(v_1,v_2)=0$ if $(v_1,v_2) \in V_1 X V_1$ or $(v_1,v_2) \in V_2 X V_2$ Further if $\mu(u, v) = \sigma(u)\wedge\sigma(v)$ for all $u\in V_1$ and $v \in V_2$, then $G(\sigma, \mu)$ is called *complete bipartite fuzzy graph* and is denoted by K_{σ_1,σ_2}. Hereafter we write *G* for $G(\sigma, \mu)$.

Domination in Fuzzy Graph

Definition 3.1: Let $G(\sigma,\mu)$ be a fuzzy graph on *V* and let $u, v \in V$. If $\mu(u, v)=\sigma(u)\wedge\sigma(v)= \sigma(v)$ then *u dominates v* (or *v is dominated by u*) in *G*. A subset *D* of *V* is called a *dominating set* in *G* if for every $v \in V-D$ then there exist $u \in D$ such that *u* dominates *v*. A dominating set *S* of a fuzzy graph *G* is said to be a *minimal dominating set (MDS)* if there is no dominating set *S'* of *G* such that $S' \subset S$. A dominating set *D* of a fuzzy graph *G* is said to be a *minimum dominating set,* if there is no dominating set *D'* of G such that $|D'| < |D|$. The minimum fuzzy cardinality of a minimum dominating set of *G* is called the *domination number* of *G* and is denoted by $\gamma(G)$ or γ.

Example 3.1: Consider the fuzzy graph $G(\sigma, \mu)$, where $\sigma = \{v_1/0.1, v_2/0.5, v_3/0.8\}$, and $\mu = \{(v_1, v_2)/0.4, (v_2, v_3)/0.5\}$. $\{v_1, v_3\}$ is a dominating set but $\{v_2\}$ is not a dominating set, because v_2 dominates v_1 but not dominates v_3.

Example 3.2: Since $\{u\}$ is a dominating set of K_σ for all $u \in V$ we have $\gamma(K_\sigma) = \max_{u \in V} \sigma(u)$. $\gamma(G) = p$ if and only if $\mu(u, v) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$. In particular $\gamma(\overline{K_\sigma}) = p$. $\gamma(K_{\sigma_1, \sigma_2}) = \max_{u \in V_1} \sigma_1(u) + \max_{v \in V_2} \sigma_2(v)$.

Remark 3.1: For any $u, v \in V$, if u dominates v need not imply v dominate u.

For any $u, v \in V$, if $\mu(u, v) = \sigma(u) \wedge \sigma(v) = \sigma(u) = \sigma(v)$ then u dominates v and also v dominates u.

For any $u \in V$, $N(u)$ is precisely the set of all $v \in V$ which is adjacent to u.

If $\mu(u, v) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$ then clearly the only dominating set in G is V.

Theorem 3.1: *For any fuzzy graph G.* $\gamma + \overline{\gamma} = 2p$ *if and only if* $0 < \mu(u, v) < \sigma(u) \wedge \sigma(v)$ *for all* $u, v \in V$ *where* $\overline{\gamma}$ *is the domination number of* $\overline{G}$.

Proof: From the definition of isolated vertex, $\gamma = p \Leftrightarrow \mu(u, v) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$ and $\overline{\gamma} = p \Leftrightarrow \mu(u, v) - \sigma(u) \wedge \sigma(v) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

Hence $\gamma + \overline{\gamma} = 2p$ if and only if $0 < \mu(u, v) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$.

Remark 3.2: For any fuzzy graph, $\gamma + \overline{\gamma} \leq 2p$.

Theorem 3.2: *A dominating set D is a minimal dominating set (MDS) if and only if for each* $d \in D$ *one of the following two conditions holds d is not adjacent to any vertex in D*

(i) For some $u \in D$*: (i)* $\sigma(d) < \sigma(u)$ *(or)*

(ii) there exists a vertex $c \in V - D$ *such that* $N(c) \cap D = \{u\}$.

Proof: Assume that D is an MDS of G. Then for every vertex $d \in D$, $D - \{d\}$ is not a dominating set. This means that some vertex c in $(V-D) \cup \{d\}$ is not dominated by any vertex in $D-\{d\}$. Then there are two cases.

Case (i): If $c = d$ in which case d is adjacent to any vertex u in D such that $\sigma(d) < \sigma(u)$.

Case (ii): If $c \in V-D$ then c is not dominated by any vertex in $D-\{d\}$, but it is dominated by some vertex u in D then u is adjacent to only vertex c in $V-D$. Thus $N(c) \cap D = \{u\}$.

Conversely, suppose that *D* is a dominating set and for each vertex $d \in D$ one of the two conditions holds. We now show that *D* is an MDS. Suppose that *D* is not an MDS, and then there exists a vertex $d \in D$ such that *D-{d}* is a dominating set. Hence d is adjacent to at least one vertex in *D-{d}*, and then condition (a) does not hold. Also if *D-{d}* is a dominating set then every vertex of *V-D* is adjacent to at least one vertex in *D-{d}* that is conditions (i) or (ii) does not hold. This is a contradiction to our assumption that at least one of the condition holds.

Remark 3.3: Let *G* be a fuzzy graph without isolated vertices and let *D* be a dominating set of *G*.

Then *V-D* need not be a dominating set of *G*.

If all the vertices having the same membership grade then *V-D* is also a dominating set.

Examples 3.3: Consider the fuzzy graph $G(\sigma, \mu)$, where $\sigma = \{v_1/0.8, v_2/0.2\}$, and $\mu = \{(v_1, v_2)/0.2\}$. $D = \{v_1\}$ is a dominating set of G but $V\text{-}D = \{v_2\}$ is not a dominating set of *G*.

Consider the fuzzy graph $G(\sigma, \mu)$, Where $\sigma = \{v_1/0.8, v_2/0.8\}$, and $\mu = \{(v_1, v_2)/0.8\}$. Then $D=\{v_1\}$ and $V\text{-}D=\{v_2\}$ are the dominating sets of G.

Theorem 3.3: *For any fuzzy graph,* $p-q \leq \gamma \leq p - \delta_E$, *Where p, q and* δ_E *are the order, size and minimum effective incident degree of G respectively.*

Proof: Let *D* be a dominating set and γ be the minimum domination number in G. Then the scalar cardinality of *V-D* is less than or equal to the scalar cardinality of $V \times V$. Hence $p-q \leq \gamma$.

Now, let *u* be the vertex with minimum effective incident degree δ_E. Clearly *V-{u}* is a dominating set and hence $\gamma \leq p - \delta_E$. Hence $p-q \leq \gamma \leq p - \delta_E$ is true for any fuzzy graph.

Remark 3.4: If all the vertices having the same membership grade, then $p-q \leq \gamma \leq p - \Delta_E$.

Example 3.4: Consider the fuzzy graph $G(\sigma, \mu)$, where $\sigma = \{v_1/0.5, v_2/0.8\ v_3/0.3, v_4/0.1\}$ and $\mu = \{(v_1, v_2)/0.5, (v_2, v_3)/0.3, (v_3, v_4)/0.1, (v_4, v_1)/0.1, (v_1, v_3)/0.3\}$. $D_1 = \{v_2, v_4\}$, $D_2 = \{v_1, v_2\}$, are all dominating set but D_1 is a minimum dominating set. Here $p = 1.7$, $q = 1.3$, $\gamma = 0.9$ and $\delta_E = 0.2$.

$$\therefore\ p - q \leq \gamma \leq p - \delta_E$$
$$1.7 - 1.3 \leq 0.9 \leq 1.7 - 0.2$$
$$0.4 \leq 0.9 \leq 1.5.$$

Definition 3.2: The domination number $\gamma(G)$ of a fuzzy graph equals the minimum scalar cardinality of a set in MDS (G). The upper domination number Γ *(G)* of a fuzzy garaph G equals the maximum scalar cardinality of MDS (G).

Example 3.5: From the example 3.4, $\gamma(G)$ = 0.9 and Γ *(G)* = 1.3.

Independent Domination in Fuzzy Graph

Definition 4.1: A set $S \subseteq V$ in a fuzzy graph G is said to be *independent* if $\mu(u,v) < \sigma(u) \wedge \sigma(v)$ for all $u, v \in S$.

Definition 4.2: A Dominating set is called an *independent dominating set* if *D* is independent. An independent dominating set *S* of a fuzzy graph *G* is said to be a *maximal independent dominating set (MIDS)* if there is no independent dominating set *S'* of *G* such that $S' \subset S$. An independent dominating set *S* of a fuzzy graph *G* is said to be a *maximum independent dominating set* if there is no independent dominating set *S'* of *G* such that $|S'| > |S|$. The minimum scalar cardinality of an maximum independent dominating set *G* is called the *independent domination number* of *G* and is denoted by i *(G)*.

Example 4.1: Consider the fuzzy graph $G(\sigma, \mu)$, where $\sigma = \{v_1/0.2, v_2/0.3\ v_3/0.6, v_4/0.8\ v_5/0.4, v_6/0.1\}$ *and* $\mu = \{(v_1, v_3)/ 0.2, (v_2, v_3)/ 0.3, (v_3, v_4)/ 0.6, (v_4, v_5)/ 0.4, (v_4, v_6)/ 0.1\}$. $S1=\{v_3,v_5,v_6\}$, $S2 = \{v_1,v_2,v_4\}$ and $S3=\{v_1,v_2,v_5,v_6\}$ are the independent dominating sets. The independent domination number $i(G) = 1.1$

Remark 4.1: For any fuzzy graph *G*, $\gamma(G) \leq i(G)$.

Definition 4.3: The *independent domination number* i *(G)* of a fuzzy graph *G* equals the minimum scalar cardinality of the set in MIDS. The *upper domination number* *I(G)* of a fuzzy graph *G* equals the maximum scalar cardinality of a set in MIDS.

Example 4.2: From the fuzzy graph given in the example 4.1, $i(G)$=1.1 and $I(G)$=1.3.

Theorem 4.1: An independent set is a maximal independent set in a fuzzy graph and the membership grades of all vertices are equal if and only if it is an independent dominating set.

Proof: Every maximal independent set is an independent dominating set because all the vertices having the membership grades are equal. Conversely, Let *S* be an independent dominating set. Now we have to show that it is maximal independent set. Suppose *S* is not a maximal independent set then there exists a vertex $u \in V-S$ such that $S \cup \{u\}$ is an independent set. But if $S \cup \{u\}$ is an independent set, then no vertex is adjacent to *u*. Thus *S* is not a dominating set. This is a contradiction. Hence *S* is a maximal independent set.

Theorem 4.2: Every maximal independent set in a fuzzy graph G need not be a dominating set.

Proof: Let S be a maximum independent set. We show that S need not be a dominating set. Suppose the membership grades of the vertices are different, then some of the vertex $u \in V\text{-}S$ are adjacent to some of the vertices $v \in S$ and $\sigma(u) > \sigma(v)$.

Thus, S is not a dominating set. Example 4.3: Consider the fuzzy graph $G(\sigma, \mu)$, where $\sigma = \{v_1/0.4, v_2/0.8, v_3/0.5\}$, and $\mu = \{(v_1, v_2)/0.4, (v_2, v_3)/0.5\}$. $I_1=\{v_2\}$ and $I_2=\{v_1,v_3\}$ are independent sets. I_1 is a dominating set but I_2 is not a dominating set. Here I_2 is a maximum independent set but not a dominating set.

Remark 4.2: Every maximum independent set in a fuzzy graph *G* is a dominating set, if all the vertices having the same membership values.

Theorem 4.3: *For any fuzzy graph G without isolated vertex then* $\frac{p-\Delta_E}{2} \leq \gamma \leq i$.

Proof: Let *D* be a dominating set of r elements and $D = \{v_1, v_2, v_3, \ldots v_r\}$. Since every element in *V-D* is adjacent to some vertex v_i in *D* such that $\sigma(c) < \sigma(v_i)$, $c \in V\text{-}D$ we have

$$|V-D| \leq \sum_{i=1}^{r} d_E(v_i) \leq \gamma + \Delta_E$$

Thus $p - \gamma \leq \gamma + \Delta_E$

$$\Delta_E \leq 2\gamma$$

$$\Rightarrow \frac{p-\Delta_E}{2} \leq \gamma .$$

From the remark 4.4, $\gamma \leq i$. Thus, $\frac{p-\Delta_E}{2} \leq \gamma \leq i.$

Example 4.4: From the figure given in example 3.4. $p = 1.7$, $\Delta_E = 0.9$, $\gamma = 0.9$ and $i = 0.9$

Therefore, $\frac{p-\Delta_E}{2} \leq \gamma \leq i. \Rightarrow \frac{1.7-0.9}{2} \leq 0.9 \leq 0.9 \Rightarrow 0.4 \leq 0.9 \leq 0.9.$

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