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On Mixed Quadrature Rules for Numerical Integration of Real Definite Integrals

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Abstract

The concept of mixed quadrature rules has been used for construction of such types of rules of precision 5, 7, and 9. The error associated with these rules has been analyzed and some definite real integrals have been approximately evaluated by the rules and found to yield good approximation to the exact values of the integrals otherwise obtained.

Keywords: Mixed quadrature rule, degree of precision, error bound, Newton-cotes rule, Gauss quadrature rule.

Introduction
Basically, there are two types of quadrature rules, which are:
1. Newton-Cotes type of rules and
2. Gauss- type rules, for evaluating definite integrals:

$$I(f) = \int_{-1}^{1} f(x)dx \qquad (1.1)$$

numerically.

It is a well-known fact that the Gauss-type rules integrate more accurately than the Newton-Cotes type of rule even though both have same number of nodes and moreover the degree of precision of the Gauss- type rules is greater than the Newton-Cotes type of rule having equal number of nodes.

However, if Gauss- type rule with certain precision (say) d is suitably coupled with Newton-cotes types of rules having same precision, an integration rule of higher precision is produced. The precision of such rule is at least two more than each of the quadrature rules used for the construction of the new rules. Such rules have been defined as mixed quadrature rules by Das and Pradhan [1]. For example we state here the mixed quadrature rule:

$$R_{1,2}(f)=\frac{1}{5}\left[2R_1(f)+3R_2(f)\right] \tag{1.2}$$

formulated by Das and Pradhan [1]. This rule is the weighted mean of the well-known Simpsons $\frac{1}{3}rd$ rule:

$$R_1(f)=\frac{1}{3}[f(-1)+4f(0)+f(1)] \tag{1.2a}$$

of Newton-cotes type and Gauss-Legendre two point rule:

$$R_2(f)=[f(-\frac{1}{\sqrt{3}})+f(\frac{1}{\sqrt{3}})] \tag{1.2b}$$

And it is to be noted that the precision of these rules given in equations (1.2a) and (1.2b) is three; where as the precision of the mixed quadrature rule $R_{1,2}(f)$ is five; two more than that of Simpsons $\frac{1}{3}rd$ rule $R_1(f)$ and Gauss-Legendre two point rule: $R_2(f)$.

It may be noted here that the coefficients of $R_1(f)$ and $R_2(f)$ in the mixed quadrature rule given in equation (1.2) are simple fractions $\frac{2}{5}=0.4\, and\, \frac{3}{5}=0.6\, respectively.$ So there is no addition of errors like truncation error, round off error or machine error due to finite precision of computing machine, if the integral given in equation (1.1) is numerically integrated by this rule or by any other rules of this class of rules.

Further, it may be mentioned here that the degree of accuracy to a desired decimal place of the approximate value of the integral by a single quadrature rule can not be ascertained; but the same may be assured of to some extent from the approximate values obtained by numerically integrating the integral by the two rules and the mixed quadrature rule constructed out of these rules.

Here it is noteworthy that, no additional evaluation of function is required while numerically integrating the integral by a mixed quadrature rule. The formulation of mixed quadrature rule from the existing rules of numerical integration is quite simple but yields result of greater accuracy.

Thus, in this paper we intend to construct a quadrature rule of mixed types of precision 5, other than one given in equation (1.2),and two rules of precision 7 in the same vain, as it is done by Das and Pradhan [1] in succeeding articles.

In addition to the formation of mixed quadrature rules a complete program in C^{++} is appended at the end of this paper to facilitate the evaluation of definite integrals by mixed quadrature rules and to study the differences in accuracies by such rules and the rules which have been used in construction of such mixed quadrature rules.

A set of five definite integrals: I_1 to I_6 whose exact values are otherwise known have been numerically integrated by the mixed quadrature rules constructed in this paper and results have been depicted in table's no. 3.1 – 3.6.

Again all the numerical calculations have done by machine (PIV) in double precision using the program appended in this paper.

Formulation of mixed quadrature rule of precision 5:

For the construction of 2nd mixed quadrature rule of precision 5, we choose the following rule of Newton-cotes type:

$$R_3(f) = \frac{1}{4}\left[f(-1) + 3\left[f\left(-\tfrac{1}{3}\right) + f\left(\tfrac{1}{3}\right)\right] + f(1)\right];$$

which is well known Simpson's (3/8) th rule of precision three, besides the Gauss-Legendre two point rule: $R_2(f)$ given in equation (1.2b).

Let $E_2(f)$ and $E_3(f)$ respectively denote the error in approximating the real definite integral given in (1.1) by the qudrature rules $R_2(f)$ and $R_3(f)$. Thus,

$$I(f) = R_2(f) + E_2(f) \tag{2.1}$$

and

$$I(f) = R_3(f) + E_3(f) \tag{2.2}$$

Let us assume here that, the function $f(x)$ is sufficiently differentiable in the range of integration $[-1, 1]$. Then expanding $f(x)$ about x = 0, in Taylor's series we have:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ... \tag{2.3}$$

where

$$c_n = \frac{f^{(n)}(0)}{(n)!} \quad ; \quad n = 1, 2, 3.....$$

As the series given in (2.3) is uniformly convergent in [-1, 1], we obtain by integrating both sides of the series (2.3) term by term

$$I(f) = 2c_0 + \frac{2}{3}c_2 + \frac{2}{5}c_4 + ... \tag{2.4}$$

where

$$c_{2n} = \frac{f^{(2n)}(0)}{(2n)!}; \quad n = 0, 1, 2, 3.....$$

Further substituting $x = {}^{-1}\!\!/_{\sqrt{3}}$ and $x = {}^{1}\!\!/_{\sqrt{3}}$ successively in the Taylor's expansion of $f(x)$ given in equation (2.3) we get

$$\left.\begin{aligned} f\left(-1/\sqrt{3}\right) &= c_0 - \frac{c_1}{\sqrt{3}} + \frac{c_2}{3} - \frac{c_3}{3\sqrt{3}} + \ldots. \\ f\left(1/\sqrt{3}\right) &= c_0 + \frac{c_1}{\sqrt{3}} + \frac{c_2}{3} + \frac{c_3}{3\sqrt{3}} + \ldots. \end{aligned}\right\} \tag{2.5}$$

Further, putting the Taylor's expansions given in equation (2.5) in $R_2(f)$ we have:

$$R_2(f) = 2c_0 + \frac{2}{3}c_2 + \frac{2}{9}c_4 + \ldots. \tag{2.6}$$

Proceeding in the same way we obtained a result similar to equation (2.6) for the rule $R_3(f)$ as:

$$R_3(f) = 2c_0 + \frac{2}{3}c_2 + \frac{14}{27}c_4 + \ldots. \tag{2.7}$$

and the error terms corresponding to the rules $R_2(f)$ and $R_3(f)$ as

$$E_2(f) = \frac{8}{45}c_4 + \frac{40}{189}c_6 + \ldots\ldots..$$
$$E_3(f) = \frac{-16}{135}c_4 - \frac{368}{1701}c_6 - \ldots.$$

respectively.

Therefore,

$$I(f) = R_2(f) + \frac{8}{45}c_4 + \frac{40}{189}c_6 + \ldots. \tag{2.8}$$

$$I(f) = R_3(f) - \frac{16}{135}c_4 - \frac{368}{1701}c_6 - \ldots. \tag{2.9}$$

Here it is pertinent to note that the coefficients of c_4 in $E_2(f)$ and $E_3(f)$ are of opposite sign which is a very basic requirement for improvement in accuracy of the approximation by the mixed quadrature rule to be formulated as the linear combination of rules $R_2(f)$ and $R_3(f)$. In such a situation, the sign of the coefficients of rules $R_2(f)$ and $R_3(f)$ do not change in sign in mixed quadrature rule. As a result, there is no chance of lose of significant digits when the definite integral given in equation (1.1) is numerically integrated by the mixed quadrature rule claimed to be of higher precision 5.

Now multiplying the equation (2.8) by 2 and equation (2.9) by 3; then adding the resulting series, we obtained after simplification:

$$I(f)=\frac{1}{5}\left[2R_2(f)+3R_3(f)\right]-\frac{1}{5}\left[\frac{128}{567}c_6+\frac{320}{729}c_8+\ldots\right] \tag{2.10}$$

and we claim that:

$$\frac{1}{5}\left[2R_2(f)+3R_3(f)\right]$$

is the new quadrature rule of precision 5. If this rule is denoted by $R_{2,3}(f)$, then

$$R_{2,3}(f)=\frac{1}{5}\left[2R_2(f)+3R_3(f)\right] \tag{2 .11}$$

Thus,

$$I(f)=R_{2,3}(f)+E_{2,3}(f)$$

where $E_{2,3}(f)$ is the analytical error associated with the rule $R_{2,3}(f)$ and it is given as:

$$E_{2,3}(f)=\frac{1}{5}\left[2E_2(f)+3E_3(f)\right] \tag{2.12}$$

Further from equations (2.10) – (2.12) it follows that

$$E_{2,3}(f)=-\frac{1}{5}\left[\frac{128}{567}c_6+\frac{320}{729}c_8+\ldots\right] \tag{2.13}$$

Degree of precision of $R_{2,3}(f)$:

For $f(x)=x^i;\ i=0,1,2,3\ \ we\,have:$

$$E_{2,3}(x^i)=\frac{1}{5}\left[2E_2(x^i)+3E_3(x^i)\right]$$
$$=0,$$

since

$$E_2(x^i)=0,\quad E_3(x^i)=0$$

as each of these rules: $R_2(f)$ and $R_3(f)$ is of degree of precision three.

Further

$$E_2(x^4)=\int_{-1}^{1}x^4dx-R_2(x^4)=\frac{8}{45}$$

and

$$E_3(x^4)=\int_{-1}^{1}x^4dx-R_3(x^4)=\frac{-16}{135}$$

$$\Rightarrow$$

$$E_{2,3}\left(x^4\right) = \frac{1}{5}\left[\frac{16}{45} - \frac{16}{45}\right] \qquad = 0$$

Further the mixed quadrature rule $R_{2,3}(f)$ integrates all monomials of odd degree since it is a symmetric quadrature rule.

Also

$$E_{2,3}\left(x^6\right) = -\frac{377}{8505} \neq 0,$$

suggests that the degree of precision of the rule $R_{2,3}(f)$ is 5.

Error bound of $R_{2,3}(f)$

Let

$$\text{M} = \underset{-1 \le x \le 1}{Max}\left|f^{(iv)}(x)\right|$$

Now

$$\left|E_{2,3}(f)\right| = \frac{1}{5}\left|\left[2E_2(f) + 3E_3(f)\right]\right|$$

$$\le \frac{1}{5}\left[2\left|E_2(f)\right| + 3\left|E_3(f)\right|\right] \tag{2.14}$$

But

$$E_2(f) = \frac{1}{135} f^{(4)}(\eta_1); \qquad -1 < \eta_1 < 1$$

and

$$E_3(f) = -\frac{2}{405} f^{(4)}(\eta_2); \qquad -1 < \eta_2 < 1$$

(Ref. [5]; page 266,276) along with (2.14), leads to the following theorem on the error bound for the rule $R_{2,3}(f)$.

Theorem: 2.1 If $f(x)$ is assumed to be sufficiently differentiable in the interval [-1, 1] and its fourth derivative is bounded by M in this interval [-1, 1], then

$$\left|E_{2,3}(f)\right| \le \frac{4}{675} M \tag{2.15}$$

In the next sub article of this article we construct such type of rules of precision seven.

Formulation of mixed quadrature rules of precision 7

For the construction of quadrature rules of precision 7, we take Gauss-Legendre three point rule denoted as $R_4(f)$ and given by:

$$R_4(f) = \frac{1}{9}\left[5f\left(-\sqrt{3/5}\right) + 8f(0) + 5f\left(\sqrt{3/5}\right)\right]$$

which is of precision 5.

We combine this rule and the rule $R_{1,2}(f)$ given in equation (1.2) to produce one mixed quadrature rule of precision 7 and the rule $R_4(f)$ is again combined with the rule $R_{2,3}(f)$ given in equation (2.11) to produce 2nd mixed quadrature rule of precision 7. These rules are respectively denoted as $R_{1,2;4}(f)$ and $R_{2,3;4}(f)$.

Now without repeating the technique that we adopted in the formulation of mixed quadrature rule of precision 5, we simply state the rules below.

$$R_{1,2;4}(f) = \frac{1}{14}\left[9R_{1,2}(f)+5R_4(f)\right] \tag{2.16}$$

and

$$R_{2,3;4}(f) = \frac{1}{161}\left[81R_{2,3}(f)+80R_4(f)\right] \tag{2.17}$$

Denoting the corresponding error terms by $E_{1,2;4}(f)$ and $E_{2,3;4}(f)$ we have:

$$E_{1,2;4}(f) = \frac{1}{14}\left[9E_{1,2}(f)+5E_4(f)\right] \tag{2.18}$$

and

$$E_{2,3;4}(f) = \frac{1}{161}\left[81E_{2,3}(f)+80E_4(f)\right] \tag{2.19}$$

Degree of precision $R_{1,2;4}(f)$ and $R_{2,3;4}(f)$

Now

$$E_{1,2;4}\left(x^i\right) = \frac{1}{14}\left[9E_{1,2}\left(x^i\right)+5E_4\left(x^i\right)\right] \quad = 0\text{ ; for} \quad i = 0(1)5$$

since both

$$E_{1,2}\left(x^i\right) = 0 = E_4\left(x^i\right);\ i = 0(1)5$$

as each rule $R_{1,2}(f)$ and $R_4(f)$ is of precision 5.

Further

$$E_{1,2;4}\left(x^6\right) = 0$$

since

$$E_{1,2}\left(x^6\right) = \frac{-8}{315} \text{ and } E_4\left(x^6\right) = \frac{8}{175}$$

when substituted in $E_{1,2;4}\left(x^6\right)$ makes it zero.

Also, since the mixed quadrature rule $R_{1,2;4}(f)$ integrates all monomials of odd degree as it is a symmetric quadrature rule

and

$$E_{1,2;4}\left(x^8\right) = \frac{-16}{1575} \neq 0\,;$$

it follows that the degree of precision of the rule $R_{1,2;4}(f)$ is 7.

It is not difficult to show that the degree of precision of the rule $R_{2,3;4}(f)$ given in equation (2.17) is seven.

Error bound of $\boldsymbol{R_{1,2;4}(f)}$ and $\boldsymbol{R_{2,3;4}(f)}$

The error bounds of these rules are given in the following theorems.

Theorem: 2.2 Suppose the derivatives of all order of $f(x)$ exist in the interval [-1 , 1] and $M = \max\limits_{-1\le x\le 1}\left[\left|f^{(iv)}(x)\right|,\left|f^{(vi)}(x)\right|\right]$.Then

$$\left|E_{1,2;4}(f)\right| \le \frac{253}{3150}M \tag{2.20}$$

and

$$\left|E_{2,3;4}(f)\right| \le \frac{52078}{206325}M \tag{2.21}$$

Proof:

Since

$$\left|E_{1,2}(f)\right| \le \frac{2}{225}M \tag{2.22}$$

and

$$\left|E_{4}(f)\right| \le \frac{1}{15750}M \tag{2.23}$$

From the inequalities (2.22) and (2.23) and from equation (2.18) it follows that:

$$\left|E_{1,2;4}(f)\right| \le \frac{253}{3150}M$$

and it can also be shown from $E_{2,3}(f)$ given in equation (2.15) and $E_4(f)$ given in (2.23)

$$\left|E_{2,3;4}(f)\right| \le \frac{52078}{206325}M\,.$$

Numerical verifications

To test the accuracy of the formula, we have taken the following integrals:

$$I_1 = \int_{-1}^{1} e^{x}dx \qquad\qquad I_2 = \int_{0}^{1} e^{-x^2}dx$$

$$I_3 = \int_0^1 e^{x^2} dx \qquad I_4 = \int_1^3 \frac{\sin^2 x}{x} dx$$

$$I_5 = \int_0^1 \frac{dx}{1+e^x} \qquad I_6 = \int_{-1}^1 \frac{dx}{1+x^2}$$

Table-3.1

Rules	Approximate value of $\boldsymbol{I}_1$
$\boldsymbol{R}_1(\boldsymbol{f})$	2.3620538
$\boldsymbol{R}_2(\boldsymbol{f})$	2.3426961
$\boldsymbol{R}_3(\boldsymbol{f})$	2.3556481
$\boldsymbol{R}_4(\boldsymbol{f})$	2.3503369
$\boldsymbol{R}_{1,2}(\boldsymbol{f})$	2.3504392
$\boldsymbol{R}_{2,3}(\boldsymbol{f})$	2.3504673
$\boldsymbol{R}_{1,2;4}(\boldsymbol{f})$	2.3504027
$\boldsymbol{R}_{2,3;4}(\boldsymbol{f})$	2.3504025
Exact Value	2.3504024

Table-3.2

Rules	Approximate value of $\boldsymbol{I}_2$
$\boldsymbol{R}_1(\boldsymbol{f})$	0.747180
$\boldsymbol{R}_2(\boldsymbol{f})$	0.746595
$\boldsymbol{R}_3(\boldsymbol{f})$	0.746992
$\boldsymbol{R}_4(\boldsymbol{f})$	0.746815
$\boldsymbol{R}_{1,2}(\boldsymbol{f})$	0.746829
$\boldsymbol{R}_{2,3}(\boldsymbol{f})$	0.746833
$\boldsymbol{R}_{1,2;4}(\boldsymbol{f})$	0.746824
$\boldsymbol{R}_{2,3;4}(\boldsymbol{f})$	0.746824
Exact Value	0.746824

Table-3.3

Rules	Approximate value of I_3
$R_1(f)$	1.4757
$R_2(f)$	1.4542
$R_3(f)$	1.4687
$R_4(f)$	1.4624
$R_{1,2}(f)$	1.4628
$R_{2,3}(f)$	1.4629
$R_{1,2;4}(f)$	1.4627
$R_{2,3;4}(f)$	1.4627
Exact Value	1.4627

Table-3.4

Rules	Approximate value of I_4
$R_1(f)$	0.7894517
$R_2(f)$	0.7985600
$R_3(f)$	0.7926145
$R_4(f)$	0.7946527
$R_{1,2}(f)$	0.7949167
$R_{2,3}(f)$	0.7949927
$R_{1,2;4}(f)$	0.7948224
$R_{2,3;4}(f)$	0.7948238
Exact Value	0.7948251

Table-3.5

Rules	Approximate value of I_4
$R_1(f)$	0.3798507
$R_2(f)$	0.3799089
$R_3(f)$	0.3798702
$R_4(f)$	0.3798853
$R_{1,2}(f)$	0.3798856

$\boldsymbol{R}_{2,3}(\boldsymbol{f})$	0.3798857
$\boldsymbol{R}_{1,2;4}(\boldsymbol{f})$	0.3798855
$\boldsymbol{R}_{2,3;4}(\boldsymbol{f})$	0.3798855
Exact Value	0.3798855

The authors are investigating possibility of some other mixed quadrature rules of higher precision for numerical integration of real and complex integrals of analytic functions on the line segment in the complex plane $\mathbb{C}$.

Further, we have taken the definite integral:

$$I_6(f) = \int_{-1}^{1} \frac{dx}{1+x^2};$$

for its peculiar behavior in response to evaluation of this integral by numerical quadrature.

This has attracted the attention of researchers since long since the approximation of this integral by standard quadrature rules does not yield good approximation to its exact value $\left(22/14\right)$. Although, the integrand:

$$f(x) = \frac{1}{1+x^2};$$

does not have any singularity on the path of the integration. Researchers are of opinion that the integrand behaves in a manner not conducive to the rules of numerical integration. Because of its complex singularities i.e. simple poles at the points $z = \pm i$ near to the path of the integration.

Thus, it is motivated us to test the response of this particular integral to numerical integration by the mixed quadrature rules constructed in this paper.

For the approximate values given in Table- 3.6, we observe that the adverse effect on numerical integration gradually diminishes as it is numerically integrated by rules of increasing precision and it is to be noted that the approximate values so obtained is satisfactory, at least two or three decimal places.

Thus, it appears that mixed quadrature rules of increasing precision nullifies to some extent to the adverse effect of near by singular points for numerical integration.

Programming

This article is devoted to the computer program in C^{++} in order to facilitate the researchers for evaluation of definite integrals by the mixed quadrature rules that we have developed in this paper. It is hoped that the program will immensely help students and researchers to appreciate the new rules in the field of numerical quadrature of definite integrals.

```
//	A complete program in C++ for Approximation of integrals by
```

```
//                  Mixed_Quadrature rules.
//      *********************************************************
//   Change the limits of integration to [-1,1] if it is other than this.
#include<iostream.h>
#include<conio.h>
#include<math.h>;
float fun(float);
void main()
{
clrscr();
float r1,r2,r3,r4,r12,r23,r124,r234;

     r1=(1.0/3.0)*(fun(-1)+ 4*fun(0)+ fun(1));
     r2=fun(-1.0/sqrt(3.0)) +  fun(1.0/sqrt(3.0));
     r3=(fun(-1)+3*(fun(-1.0/3.0)+fun(1.0/3.0))+fun(1))/4.0;
     r4=(8*fun(0)+5*(fun(-sqrt(0.6))+fun(sqrt(0.6))))/9.0;
    r12=(2*r1+3*r2)/5.0;
    r23=(2*r2+3*r3)/5.0;
   r124=(9*r12+5*r4)/14.0;
   r234=(81*r23+80*r4)/161.0;
cout<<"            "<<  "YOUR OUTPUT"   <<endl;
cout<<"            "<<  "***********"   <<endl;
cout<<"     approximate values are :"   <<endl;
cout<<"----------------------------------------"<<endl<<endl;
cout<<"     r1="<<r1<<"     :     "<<"r2="<<     r2     <<"     :     "<<"r3="<<     r3<<"     :
"<<"r4="<<r4<<endl<<endl;
cout<<"     "  <<"   r12="<<r12<<":" <<" r23="<<r23<<endl<<endl;
cout<<"     "  <<"  r124="<<r124<<":"<<"r234="<<r234<<endl<<endl;
cout<<"-----------HAVE A NICE DAY--------------";
// Rules of precision 3:  r1,r2,r3.
// Rules of precision 5:  r12,r23.
// Rules of precision 7:  r124,r234.
getch();
}
float fun(float x)
{
  float t, y;
  t=0.25*(pow(1+x, 2));//Here is your integrand:exp(-x*x);limits:0,1
  y=0.5*exp(-t);//for limits other than [-1,1], suitably change the integrand.
   return(y);
}
```

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