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Uniidmensional Flow through Unsaturated Porous Media: A Problem of Groundwater Recharge with Perturbation Technique

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Introduction

In This paper an appropriate solution of unidimensional and unsteady fluid flow through unsaturated porous media with small diffusivity coefficient has been obtained by using a singular perturbation technique, viz. method of composite expansions. The hydrological situation of one dimensional vertical groundwater recharge by spreading [1] confirms to this model. Such flows have a great practical importance in water resource science, spreading of contaminated water, chemical engineering and agricultural purposes. Engineers in several fields have of late, endeavored to learn the mechanics of drainage and to apply to problems of water supply, land reclamation and stabilization of foundations and sub grade, and also to the field of petroleum production and agriculture. Provision for adequate drainage is an essential part of planning, construction and operation of an irrigation project. For agriculture purpose, the continued presence of water in excess of that needed for vegetation is harmful. Prolonged saturation of soil excludes air essentially for healthy plant growth and soil becomes cold, and unproductive. Consequently unsaturated or irrigated soil is a necessary evil, so far as irrigation is concerned. The problem discussed here refers to this type of drainage where original saturation conditions are existing up to the top [5], [6], [7], [8].

Statement of the Problem

For definiteness of the physical problem, we consider, here, that the recharge takes place over a large basin of such geological configuration that the sides are limited by rigid boundaries while the bottom is confined by a thick layer of water table. In this circumstances water will flow vertically downwards, through unsaturated porous medium. It is assumed that the diffusivity coefficient is equivalent to the average value of over the whole range of moisture content is small enough to be regarded as a perturbation parameter.

Mathematical Formulation of the Problem

For the motion of the water in porous medium, Darcy’s law is given by

$$\vec{v} = K\nabla\phi \tag{1}$$

Where $\nabla\phi$ represents the gradient of the whole moisture potential, $\vec{v}$ the volume flux of moisture and *K* the coefficient of aqueous conductivity. The equation of continuity for unsaturated porous media is given by

$$\frac{\partial}{\partial t}(\rho_s\theta) = -\nabla . M \tag{2}$$

Where ρ_s is the bulk density of the medium , θ is moisture content on a dry weight basis, and M is the mass flux of moisture.

Using g $= \rho \vec{v}$, ρ is the flux density and considering that the flow takes only in vertical direction the combination equations (1) and (2) yields

$$\rho_s\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left\{\rho K\frac{\partial\psi}{\partial t}\right\} - \frac{\partial}{\partial z(\rho K g)} \tag{3}$$

Where ψ is the pressure potential related with ϕ by $\phi = \psi - gz$, g is the gravitational constant and z is the direction (vertical) of the flow. The positive direction of z-axis is the same as that of gravity.

Considering θ and ψ to be connected by a single valued function, we may write equation (3) as,

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left\{D\frac{\partial\theta}{\partial z}\right\} + \frac{\rho}{\rho_s}g\frac{\partial K}{\partial z} \tag{4}$$

Where $D = \frac{\rho}{\rho_s}K$ and is called the diffusivity coefficient. For definiteness, we consider an average value D_a of the diffusivity coefficient over the whole range of moisture content, which is small enough to consider as a perturbation parameter.

Assuming $K = \frac{k_o\theta}{\sqrt{t}}$, where k_o is the slope of the permeability versus weighted moisture content plot [2]. Equation (4), now, reduces to

$$\frac{\partial\theta}{\partial t} = D_a \frac{\partial^2\theta}{\partial z^2} + \frac{\rho}{\rho_s} g \frac{k_0}{\sqrt{t}} \frac{\partial\theta}{\partial z} \tag{5}$$

The appropriate boundary conditions are given by

$$\theta(0,t) = \theta_0 \quad and \quad \theta(l,t) = 1 \tag{6}$$

Solution of the Problem

We choose a new variable $Z = \frac{z}{2\sqrt{t}}$ [3] and assuming that the moisture content is expressible in the variable separable form $\theta(z,t) = H(t)F(Z)$, equation (5) reduce to

$$\frac{H(t)}{4t}\left[D_a F''(Z) + 2F'(Z)(M_1 + Z)\right] = H'(t)F(Z) \tag{7}$$

Where $M_1 = \frac{g k_0}{\rho_s}$. For physical consistent, we consider that $H(t) = c_2 t^n$, where n is any integer and c_2 is a constant. Then equation (7) becomes

$$D_a F''(Z) + 2F'(Z)(M_1 + Z) - 4nF(Z) = 0 \tag{8}$$

Again the related boundary conditions becomes

$$F(0) = \theta_0 \,, \qquad F(L) = 1 \quad where\ L = \frac{l}{2\sqrt{t}}, \qquad t > 0 \tag{9}$$

For definiteness we consider, here, the small parameter D_a as a perturbation parameter ϵ. We solve equation (8) together with the boundary conditions (9) by using the method of composite expansions[4].

Since the coefficient of *F'* is positive in *(0,L)*, the nonuniformity occurs near *Z=0*. To describe *F* in the region of nonuniformity, we need a stretching transformation $\zeta = \frac{Z}{\epsilon}$, and the inner expansion is described in terms of the special function $\exp(-\zeta)$. we assume that

$$F(Z,\epsilon) = H(Z,\epsilon) + G(\zeta,\epsilon)$$

$$= H_0(Z) + G_0(\zeta) + \epsilon[H_1(Z) + G_1(\zeta)] + \cdots \tag{10}$$

Where $G(\zeta,\epsilon)$ is negligible outside the inner region [4] , so that

$$F^o(Z,\epsilon) = H_0(Z) + \epsilon H_1(Z) + \cdots \tag{11}$$

Since $Z = \epsilon\zeta$, we have

$$F^i(Z,\epsilon) = H_0(Z) + G_0(\zeta) + \epsilon[H_1'(0)\zeta + H_1(0) + G_1(\zeta)] + \dots \tag{12}$$

The boundary conditions corresponding to (9) will now give the following set of conditions

$$H_0(L) = 1\,, \quad H_n(L) = 0 \qquad for\, n \geq 1 \tag{13}$$

$$H_0(0) + G_0(0) = \theta_0\,, \qquad H_n(0) + G_n(0) = \theta_0 \tag{14}$$

To determine the equations governing H_n, we substitute (11) into (8), and equating the coefficients of like powers of ϵ, we obtain

$$(M_1 g + Z)H_0'' - 2nH_0 = 0 \tag{15}$$

$$2(M_1 g + Z)H_1'' - 4nH_1 = -H_0'' \tag{16}$$

To determine the equations governing G_n, we first express (8) in terms of the inner variable ζ as follows

$$\frac{d^2F}{d\xi^2} + 2(M_1 g + \epsilon\zeta)\frac{dF}{d\zeta} - 4n\epsilon F = 0 \tag{17}$$

Substituting (12) in (17) and equating coefficients of like powers of ϵ, we obtain

$$G_0'' + 2M_1 g G_0' = 0 \tag{18}$$

$$G_1'' + 2M_1 g G_1' = -2\zeta G_0' + 4nG_0 + 4nH_0(0) - 2M_1 g H_1'(0) \tag{19}$$

The solution of (15) subject to (13) is given by

$$H_0(Z) = \left[\frac{M_1 g + Z}{M_1 g + L}\right]^{2n} \tag{20}$$

Hence from (14),

$$G_0(0) = \theta_0 - \left[\frac{M_1 g}{M_1 g + L}\right]^{2n} \tag{21}$$

And thus the solution of (18) that tends to zero as $\zeta \to \infty$ is given by

$$G_0(\zeta) = \left[\theta_0 - \left\{\frac{M_1 g}{M_1 g + L}\right\}^{2n}\right]\exp(-2M_1 g\zeta) \tag{21}$$

The solution of (16) subject to (13) is

$$H_1(Z) = \frac{n(2n-1)}{2}\left[\frac{M_1 g + Z}{M_1 g + L}\right]^{2n}(2M_1 g + L + Z)(L - Z) \tag{22}$$

Which when combined with (14) gives

$$G_1(0) = \frac{-n(2n-1)}{2}\left[\frac{M_1 g}{M_1 g + L}\right]^{2n}(2M_1 g + L)L \tag{23}$$

Substituting for H_0 and G_0 into (16) the solution of the resulting equation that tends to zero as $\xi \to \infty$, corresponding to above condition is given by

$$G_1(\xi) = A(\xi)\exp(-2M_1 g\xi) \tag{24}$$

Where

$$A(\xi) = \left[\theta_0 - \left\{\frac{M_1 g}{M_1 g + L}\right\}^{2n}\right]\left(2n + 1 + M_1 g\xi(1 + 2n + M_1 g\xi)\right) - \frac{n(2n-1)}{2}\left[\frac{M_1 g}{M_1 g + L}\right]^{2n}(2n + L)L \tag{25}$$

Thus an approximate solution of the (8) is the higher ordered solution containing two terms and is given by

$$F(Z,\varepsilon) = \left[\frac{M_1 g + Z}{M_1 g + L}\right]^{2n} + \left[\theta_0 - \left\{\frac{M_1 g}{M_1 g + L}\right\}^{2n}\right]\exp(-2M_1 g\xi) + \varepsilon\left[\frac{n(2n-1)}{2}\left[\frac{M_1 g + Z}{M_1 g + L}\right]^{2n}(2M_1 g + L + Z)(L - Z) + A(\xi)\exp(-2M_1 g\xi)\right] + \cdots$$

Where $A(\xi)$ is given by (25) and $\xi = \frac{Z}{\varepsilon}$

In general

$$\theta(Z,t,\epsilon) = c_2 t^n F(Z,\epsilon) = c_2 t^n F\left(\frac{Z}{2\sqrt{t}}\right) \tag{26}$$

Concluding Remarks

Equation (26) gives an analytical expression for the distribution of the moisture content in the investigated problem of unidimensional flow through unsaturated porous media. It is an expansion in terms of perturbation parameter containing exponential function, shows that moisture distributes exponentially .For the sake of definiteness in the analysis; we have assumed certain specific relationship which are consistent with physical problems. For different values of time '*t*' at various depths we can obtain the corresponding values of moisture content using (26).

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