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Output Regulation of the Chen Attractor

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Abstract

In this paper, we solve the problem of regulating the output of the Chen attractor, which is one of the paradigms of the chaotic attractors studied by G. Chen and T. Ueta (1999). Explicitly, we construct state feedback control laws to regulate the output of the Chen attractor so as to track constant reference signals. The control laws are derived using the regulator equations of Byrnes and Isidori (1990), who have solved the output regulation of nonlinear systems involving neutrally stable exosystem dynamics. The output regulation of the Chen attractor has important applications in several branches of Science and Engineering. We also discuss the simulation results in detail.

Keywords*:* Nonlinear control systems, feedback stabilization, output regulation and Chen attractor.

Introduction

Output regulation of nonlinear control systems is one of the very important problems in nonlinear control theory. The output regulation problem is the problem of controlling a fixed linear or nonlinear plant in order to have its output tracking reference signals produced by some external generator (*exosystem*). For linear control systems, the output regulation problem has been solved by Francis and Wonham [1]. For nonlinear control systems, the output regulation problem has been solved by Byrnes and Isidori [2] generalizing the internal model principle obtained by Francis and Wonham [1]. Byrnes and Isidori [2] have made an important assumption in their work which demands that the exosystem dynamics generating the reference and/or disturbance signals is a neutrally stable system (Lyapunov stable in both forward and backward time). This class of exosystem signals includes the important particular cases of constant reference signals as well as sinusoidal reference signals. Using Centre Manifold Theory [3], Byrnes and Isidori have derived regulator equations, which completely characterize the solution of the output regulation problem of nonlinear control systems.

The Chen attractor is one of the paradigms of the chaotic dynamical systems studied by the mathematicians G. Chen and T. Ueta (1999, [4]). It is important to note that Chen attractor is not topologically equivalent to the famous Lorenz attractor [5].

In this paper, we solve the output regulation problem for the Chen attractor using the regulator equations [2] to derive the state feedback control laws for regulating the output of the Chen attractor for the case of constant reference signals (set-point signals).

This paper is organized as follows. In Section 2, we present a review of the solution of the output regulation for nonlinear control systems and the Byrnes-Isidori regulator equations [2]. In Section 3, we detail our solution of the output regulation problem for the Chen chaotic attractor. In Section 4, we discuss the simulation results. In Section 5, we present the conclusions of this paper.

Review of the Output Regulation for Nonlinear Control Systems

In this section, we consider a multivariable nonlinear control system modelled by equations of the form

$$\dot{x} = f(x) + g(x)u + p(x)\omega \tag{1a}$$

$$\dot{\omega} = s(\omega) \tag{1b}$$

$$e = h(x) - q(\omega) \tag{2}$$

Here, the differential equation (1a) describes the *plant dynamics* with state x defined in a neighbourhood X of the origin of $\boldsymbol{R}^n$ and the input u takes values in $\boldsymbol{R}^m$ subject to the effect of a disturbance represented by the vector field $p(x)\omega$. The differential equation (1b) describes an autonomous system, known as the *exosystem*, defined in a neighbourhood W of the origin of $\boldsymbol{R}^k$, which models the class of disturbance and reference signals taken into consideration. The equation (2) describes the error between the actual plant output $h(x)$ and the reference signal $q(\omega)$, which models the class of disturbance and reference signals taken into consideration.

We also assume that all the constituent mappings of the system (1) and the error equation (2) namely, f, g, p, s, h and q are C^1 mappings vanishing at the origin, *i.e.*

$$f(0) = 0,\; g(0) = 0,\; p(0) = 0,\; h(0) = 0 \quad \text{and} \quad q(0) = 0.$$

Thus, for $u = 0$, the composite system (1) has an equilibrium state $(x, \omega) = (0, 0)$ with zero error (2).

A state feedback controller for the composite system (1) has the form

$$u = \alpha(x, \omega) \tag{3}$$

where α is a C^1 mapping defined on $X \times W$ such that $\alpha(0, 0) = 0$. Upon substitution of the feedback law (3) in the composite system (1), we get the closed-loop system given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)\ \alpha(x,\omega) + p(x)\ \omega \\ \dot{\omega} &= s(\omega) \end{aligned} \tag{4}$$

The purpose of designing the state feedback controller (3) is to achieve both *internal stability* and *output regulation*. Internal stability means that when the input is disconnected from (4) (*i.e.* when $\omega = 0$), the closed-loop system (4) has an exponentially stable equilibrium at $x = 0$. Output regulation means that for the closed-loop system (4), for the initial states $(x(0), \omega(0))$ sufficiently close to the origin, $e(t) \to 0$ asymptotically as $t \to \infty$. Explicitly, we can summarize the requirements as follows.

State Feedback Regulator Problem [2]:
Find, if possible, a state feedback control law $u = \alpha(x,\omega)$ such that
(OR1) [*Internal Stability*] The equilibrium $x = 0$ of the dynamics

$$\dot{x} = f(x) + g(x)\ \alpha(x,0)$$

is locally asymptotically stable.

(OR2) [*Output Regulation*] There exists a neighbourhood $U \subset X \times W$ of $(x,\omega) = (0,0)$
such that for each initial condition $(x(0),\omega(0)) \in U$, the solution $(x(t),\omega(t))$ of the closed-loop system (4) satisfies

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} [h(x(t)) - q(\omega(t))] = 0. \qquad \blacksquare$$

Byrnes and Isidori [2] have solved this problem under the following assumptions:
The exosystem dynamics $\dot{\omega} = s(\omega)$ is neutrally stable at $\omega = 0$, i.e. if the system is Lyapunov stable in both forward and backward time at $\omega = 0$.

The pair $(f(x), g(x))$ has a stabilizable linear approximation at $x = 0$, *i.e.* if

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} \quad \text{and} \quad B = \left[\frac{\partial g}{\partial x}\right]_{x=0},$$

then (A, B) is stabilizable, which means that we can find a gain matrix K such that $A + BK$ is Hurwitz. $\blacksquare$

Next, we recall the solution of the output regulation problem derived by Byrnes and Isidori [2].

Theorem 1. [2] Under the hypotheses (H1) and (H2), the state feedback regulator problem is solvable if, and only if, there exist C^1 mappings $x = \pi(\omega)$ with $\pi(0) = 0$ and $u = \varphi(\omega)$ with $\varphi(0) = 0$, both defined in a neighbourhood of $W^0 \subset W$ of $\omega = 0$ such that the following equations (called the *Byrnes-Isidori regulator equations*) are satisfied:

$$\frac{\partial \pi}{\partial \omega}\ s(\omega) = f(\pi(\omega)) + g(\pi(\omega))\ \varphi(\omega) + p(\pi(\omega))\ \omega.$$

$$h(\pi(\omega)) - q(\omega) = 0.$$

When the Byrnes-Isidori regulator equations (1) and (2) are solved, a control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)], \tag{5}$$

where K is any gain matrix such that $A + BK$ is Hurwitz. ■

Output Regulation of the Chen Attractor

Chen attractor is one of the paradigms of the chaotic systems discovered by Chen and Ueta ([4], 1993) and described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c - a)x_1 - x_1x_3 + cx_2 + u \\ \dot{x}_3 &= x_1x_2 - bx_3 \end{aligned} \tag{6}$$

where $a > 0,$ $b > 0$ and $c > 0$ are parameters and u is the control.

Chen and Ueta studied the chaotic attractor (6) with $a = 35,$ $b = 3,$ $c = 28$ and $u = 0.$ The chaotic portrait of the unforced Chen attractor is illustrated in Figure 1.

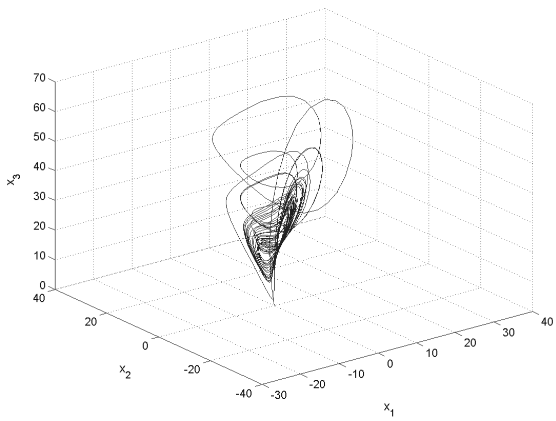

Figure 1: Chaotic Portrait of the Unforced Chen Attractor

In this paper, we solve the output regulation problem for the Chen attractor (6) for the tracking of constant reference signals (set-point signals).

The constant or set-point reference signals are generated by the exosystem dynamics

$$\dot{\omega} = 0 \tag{7}$$

It is important to observe that the exosystem given by (7) is neutrally stable. This follows simply because the differential equation (7) admits only constant solutions, *i.e.*

$$\omega(t) \equiv \omega(0) = \omega_0 \quad \text{for all} \quad t \in R.$$

Thus, the assumption (H1) of Theorem 1 holds trivially.

Linearizing the dynamics of the Chen attractor (6), we get the system matrices

$$A = \begin{bmatrix} -a & a & 0 \\ c-a & c & 0 \\ 0 & 0 & -b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using Kalman's rank test for controllability ([6], p.378), it can be easily seen that the pair (A, B) is not controllable. However, we can easily show that the pair (A, B) is stabilizable. If $K = [k_1,\ k_2,\ k_3]$ is any 1×3 gain matrix, then the closed-loop system matrix $A + BK$ has the characteristic equation given by

$$(\lambda + b)\left[\lambda^2 + \lambda(a - c - k_2) - a(2c - a + k_1 + k_2)\right] = 0 \tag{8}$$

Since $b > 0,$ it is immediate that $\lambda = -b$ is always an eigenvalue of $A + BK$ and that the matrix $A + BK$ is Hurwitz provided that k_1 and k_2 satisfy the inequalities

$$a - c - k_2 > 0 \quad \text{and} \quad a(2c - a + k_1 + k_2) < 0 \tag{9}$$

Since $a > 0,$ the requirement (9) can be simplified as

$$a - c - k_2 > 0 \quad \text{and} \quad k_1 + k_2 < a - 2c \tag{10}$$

Since k_3 does not play any role in the above calculations, we can always take $k_3 = 0.$

Thus, we shall assume that $K = [k_1,\ k_2,\ 0]$, where k_1 and k_2 are chosen so that the inequalities (10) are satisfied, *i.e.* such that $A + BK$ is Hurwitz. This shows that (A, B) is *stabilizable*.

Case (A): The error equation is $e = x_1 - \omega$

Solving the Byrnes-Isidori regulator equations (Theorem 1), we get

$$\pi_1(\omega) = \omega,\ \pi_2(\omega) = \omega,\ \pi_3(\omega) = \frac{\omega^2}{b} \text{ and } \varphi(\omega) = \omega\left(a - 2c + \frac{\omega^2}{b}\right)$$

By Theorem 1, a control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)] = \omega\left(a - 2c + \frac{\omega^2}{b}\right) + k_1(x_1 - \omega) + k_2(x_2 - \omega),$$

where k_1 and k_2 satisfy the inequalities given in (10).

Case (B): The error equation is $e = x_2 - \omega$

Solving the Byrnes-Isidori regulator equations (Theorem 1), we get

$$\pi_1(\omega) = \omega,\ \pi_2(\omega) = \omega,\ \pi_3(\omega) = \frac{\omega^2}{b} \text{ and } \varphi(\omega) = \omega\left(a - 2c + \frac{\omega^2}{b}\right)$$

By Theorem 1, a control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)] = \omega\left(a - 2c + \frac{\omega^2}{b}\right) + k_1(x_1 - \omega) + k_2(x_2 - \omega),$$

where k_1 and k_2 satisfy the inequalities given in (10).

Case (C): The error equation is $e = x_3 - \omega$

Solving the Byrnes-Isidori regulator equations (Theorem 1), we get

$$\pi_1(\omega) = \sqrt{b\omega},\ \pi_2(\omega) = \sqrt{b\omega},\ \pi_3(\omega) = \omega \text{ and } \varphi(\omega) = \sqrt{b\omega}\ (a - 2c + \omega)$$

By Theorem 1, a control law solving the state feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)] = \sqrt{b\omega}\ (a - 2c + \omega) + k_1(x_1 - \sqrt{b\omega}) + k_2(x_2 - \sqrt{b\omega}),$$

where k_1 and k_2 satisfy the inequalities given in (10)

Simulation Results

For simulation, we consider the classical chaotic case considered by Chen and Ueta (1999), namely $a = 35,\ b = 3$ and $c = 28.$ We also consider the set-point control as $\omega_0 = 2.$

By Eq. (9), it follows that $\lambda = -b$ is always an eigenvalue of the closed-loop matrix $A + BK$ and the other two eigenvalues of $A + BK$ are given by the characteristic equation

$$\lambda^2 + \lambda(a - c - k_2) - a(2c - a + k_1 + k_2) = 0.$$

Thus, $\lambda_1 = -b = -3$ is always an eigenvalue of $A + BK$ and we choose k_1 and k_2 so that $A + BK$ has two other eigenvalues at $-4, -4.$ A simple calculation gives

$$k_2 = a - c - 8 = -1 \quad \text{and} \quad k_1 = -\frac{16}{a} + a - 2c - k_2 = -20.4571.$$

Hence, we take the gain matrix as $K = [k_1,\ k_2,\ 0] = [-20.4571,\ -1,\ 0].$

Case (A): The error equation is $e = x_1 - \omega$

Suppose that we take $x(0) = (x_1(0), x_2(0), x_3(0)) = (5, 6, 2)$ and $\omega_0 = 2.$

The simulation graph is depicted in Figure 2 from which it is clear that the state $x_1(t)$ tracks the constant reference signal $\omega = 2$ in about 3 seconds.

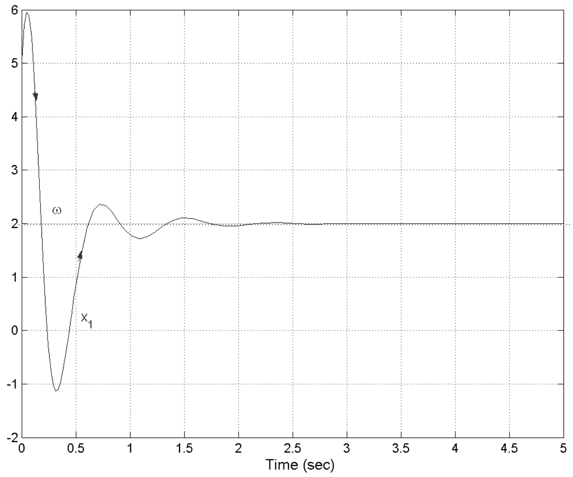

Figure 2: x_1 tracks the set-point signal $\omega = 2$

Case (B): The error equation is $e = x_2 - \omega$

Suppose that we take $x(0) = (x_1(0), x_2(0), x_3(0)) = (4, 8, 3)$ and $\omega_0 = 2.$

The simulation graph is depicted in Figure 3 from which it is clear that the state $x_2(t)$ tracks the constant reference signal $\omega = 2$ in about 3 seconds.

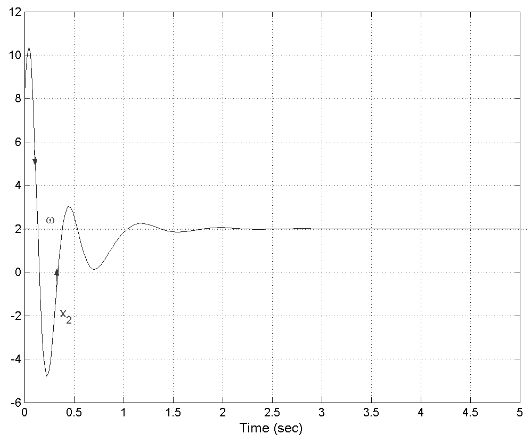

Figure 3: x_2 tracks the set-point signal $\omega = 2$

Case (C): The error equation is $e = x_3 - \omega$

Suppose that we take $x(0) = (x_1(0), x_2(0), x_3(0)) = (4, 5, 8)$ and $\omega_0 = 2.$

The simulation graph is depicted in Figure 4 from which it is clear that the state $x_3(t)$ tracks the constant reference signal $\omega = 2$ in about 3 seconds.

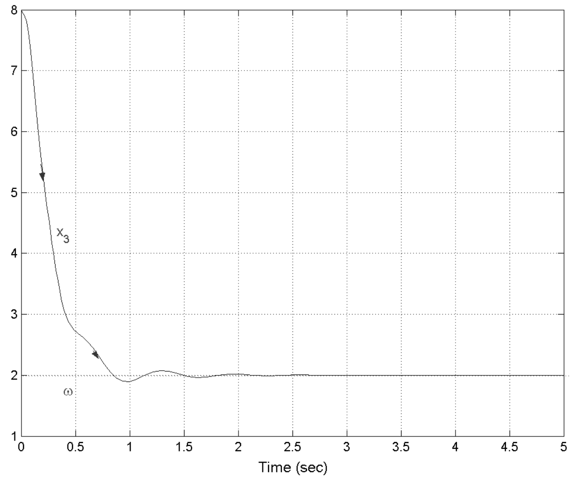

Figure 4: x_3 tracks the set-point signal $\omega = 2$

Conclusions

In this paper, we have studied in detail the output regulation of the Chen attractor (1999) and we have also obtained a complete solution of the output regulation problem for the Chen attractor. Explicitly, using the Byrnes-Isidori regulator equations (1990), we have presented new feedback control laws for regulating the output of the Chen attractor. As reference signals to be tracked, we have considered constant reference signals (set-point signals) and we have derived feedback control laws regulating the output of the Chen attractor. We have also given the simulation results for the various cases of the output regulation problem of the classical chaotic case of the Chen attractor studied by G. Chen and T. Ueta (1999).

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