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Application of Sumudu Transform in Fractional Differential Equation Associated with RLC Electrical Circuit

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Abstract

In this paper, we obtain the solution of a fractional differential equation associated with a RLC electrical circuit. The solution is derived by the application of the Sumudu transform. The results are obtained in compact and elegant forms in terms of the generalized Mittag-Leffler function and H-function, which are suitable for numerical computation.

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Introduction

Fractional differential equations have attracted in the recent years a considerable interest due to their frequent appearance in various field and their more accurate models of systems under consideration provided by fractional derivatives. For example, fractional derivatives have been used successfully to model frequency dependent damping behavior of many viscoelastic materials. They are also used in modeling of many chemical processes, mathematical biology and many other problems in Physics and Engineering. In this connection, one can refer to the monographs by Hilfer [11], Kilbas et al. [12], Kiryakova [13], Podlubny [20] and the various works cited therein. Debnath [7-9] considered solutions of fractional order homogeneous and non-homogeneous differential equations and integral equations in fluid mechanics. Magin and Ovadia [15] proposes modeling the cardiac tissue electrode interface using fractional calculus by means of a convenient three element electrical circuit. Camargo et al. [5] discuss the so-called telegraph equation in a fractional version whose solution is given in terms of a three-parameter Mittag-Leffler function and present also two new theorems involving the two and three parameter Mittag-Leffler functions. In a recent paper Soubhia et al. [24] studied a theorem involving series in the three-parameter Mittag-Leffler function and obtained the solution of a fractional differential equation associated with a RLC electrical circuit by the application of Laplace transform. Watugala [25] introduced a new integral transform, called the Sumudu transform defined for functions of exponential order. Over the set of functions,

$$A=\{f(t)\,|\,\exists M,\tau_1,\tau_2>0\,|f(t)|<M\,e^{|t|/\tau_j},\text{if } t\in(-1)^j[0,\infty)\}, \tag{1}$$

the Sumudu transform is defined by

$$G(u)=S[f(t)]=\int_0^\infty f(ut)\,e^{-t}\,dt,\ u\in(-\tau_1,\tau_2). \tag{2}$$

The Riemann-Liouville fractional integral of order ν is defined by (Miller and Ross [18], p.45; Kilbas et al. [12])

$$_0D_t^{-\nu}\ f(t)=\frac{1}{\Gamma(\nu)}\int_0^t (t-u)^{\nu-1}f(u)\,du, \tag{3}$$

where Re(ν)> 0.

The following fractional derivative of order $\alpha > 0$ is introduced by Caputo [6]; see also Kilbas et al. [12] in the form

$$_0D_t^{\alpha} f(t)=\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)\,d\tau}{(t-\tau)^{\alpha+1-m}},\ m-1<\alpha\le m,\mathrm{Re}(\alpha)>0,\ m\in N,$$

$$=\frac{d^m f(t)}{dt^m},\ \text{if } \alpha=m \tag{4}$$

where $\frac{d^m f(t)}{dt^m}$ is the m-th order derivative of f(t) with respect to t.

From Caputo [6] and Belgacem et al. [3] it follows that the Sumudu transform of the Caputo derivative is given by

$$S\{_0D_t^{\alpha}\ f(t);u\}=u^{-\alpha}\ F(u)-\sum_{k=0}^{m-1}\frac{f^k(0)}{u^{\alpha-k}},\ (m-1<\alpha\le m). \tag{5}$$

In view of the results Kilbas et al. [12] and Belgacem et al. [3], we can easily find that

$$S^{-1}[u^{\gamma-1}(1-\omega u^{\beta})^{-\delta}]=t^{\gamma-1}E_{\beta,\gamma}^{\delta}(\omega t^{\beta}). \tag{6}$$

Now, we will establish the following result which is directly applicable in the solution of fractional differential equation

$$S^{-1}\left\{\frac{u^{-\rho}}{u^{-\alpha}+au^{-\beta}+b}\right\}=\sum_{r=0}^{\infty}(-a)^r\, t^{(\alpha-\beta)r+\alpha-\rho}\, E^{r+1}_{\alpha,\alpha+(\alpha-\beta)r-\rho+1}(-bt^{\alpha}). \tag{7}$$

To prove this, we have

$$\frac{u^{-\rho}}{u^{-\alpha}+au^{-\beta}+b}=\frac{u^{-\rho+\beta}}{(u^{-\alpha+\beta}+b\,u^{\beta})\left[1+\dfrac{a}{u^{-\alpha+\beta}+b\,u^{\beta}}\right]}$$

$$=\sum_{r=0}^{\infty}(-a)^r\, u^{(\alpha-\beta)r+\alpha-\rho}\,(1+bu^{\alpha})^{-r-1}. \tag{8}$$

Finally, using the result (6), it gives the required result.

To prove our main result, we also need the following:

Theorem 1.1 [24]. For the three-parameter Mittag-Leffler function with Re(α) > 0 and Re (β) > 0 and x, y $\in$ C, x $\neq$ y, we have the following explicit representation of the series

$$\sum_{k=0}^{\infty}(-xy)^k\, E^{k+1}_{\alpha,2\alpha k+\beta}(x+y)=\frac{x\,E_{\alpha,\beta}(x)-y\,E_{\alpha,\beta}(y)}{x-y} \tag{9}$$

in terms of the two-parameter Mittag-Leffler function $E_{\mu,\nu}(.)$.

RLC Electrical Circuit

In this section, we present a RLC electrical with a capacitor and an inductor are connected in parallel and this set is connected in series with a resistor and a voltage. A similar circuit was recently studied by Soubhia [24]. The capacitance, C, the inductance, L and the resistance, R, are considered positive constant and $\theta(t)$ is the Heaviside function.

The constitute equations associated with the three-elements of the RLC electrical circuit are: the voltage drop $U_C(t)=\frac{1}{C}\int^{t} I(\xi)\,d\xi,$ across a capacitor; the voltage drop $U_L(t)=L\frac{d}{dt}I(t),$ across an inductor; the voltage drop $U_R(t)$ = R I(t), across a resistor, and where I(t) is the current.

Now using the Kirchhoff's voltage law and the constitutive equations associated with the three elements, we can write the non-homogeneous second order differential equation

$$RC\frac{d^2}{dt^2}U_C(t)+\frac{d}{dt}U_C(t)+\frac{R}{C}U_C(t)=\frac{d}{dt}\theta(t), \tag{10}$$

where $U_C(t)$ is the voltage on the capacitor which is the same on the inductor, as we can see in figure 1, because they are connected in parallel.

On the other hand, we obtain another non-homogeneous second order differential equation associated with the current on the inductor

$$RLC\frac{d^2}{dt^2}i_L(t)+L\frac{d}{dt}i_L(t)+R\,i_L(t)=\theta(t). \tag{11}$$

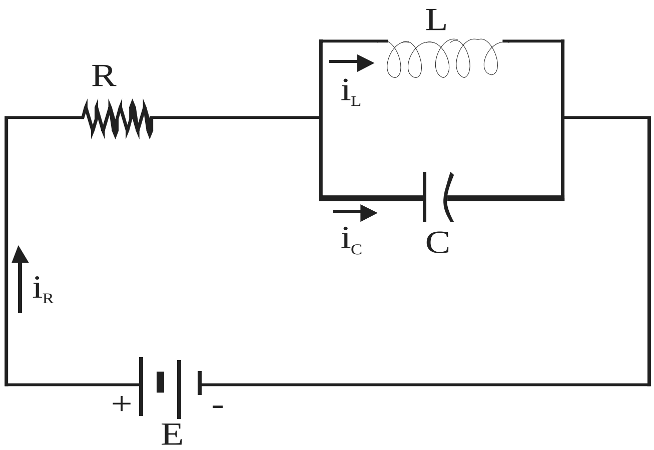

Figure 1: Three-Element Electrical Circuit.

Again, using the constitutive equation for the inductor, these two non-homogeneous second order differential equations can be led to the correspondent integro-differential equations,

$$R\frac{d}{dt}i_C(t)+\frac{1}{C}i_C(t)+\frac{R}{LC}\int_0^t i_C(\xi)\,d\xi=\frac{d}{dt}\theta(t) \tag{12}$$

and

$$RC\frac{d}{dt}U_L(t)+U_L(t)+\frac{R}{L}\int_0^t U_L(\xi)\,d\xi=\theta(t), \tag{13}$$

respectively. We observe that, both integro-differential equations (12) and (13) have the same form.

Fractional integro-differential equation

In this section, we investigate the solution of the fractional generalization of equation (12). The result is given in the form of the following theorem

Theorem 3.1. Consider the following fractional integro-differential equation associated with the current on the capacitor,

$$R\,\frac{d^{\alpha}}{dt^{\alpha}} i_C(t) + \frac{1}{C} i_C(t) + \frac{R/LC}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} i_C(\xi)d\xi = \frac{d}{dt}\theta(t),\ 0<\alpha \le 1, \quad (14)$$

with the initial condition

$$i_C(0) = 0, \quad (15)$$

where $\theta(t)$ is the Heaviside function. Then for the solution of (14), subject to the initial condition (15), there holds the formula

$$i_C(t) = \frac{t^{\alpha-1}}{R} \frac{\mu E_{\alpha,\alpha}(\mu t^{\alpha}) - \nu\ E_{\alpha,\alpha}(\nu t^{\alpha})}{\mu - \nu} \theta(t)$$

$$= \frac{t^{\alpha-1}}{R(\mu-\nu)} \left\{ \mu H^{1,1}_{1,2}\left[-\mu t^{\alpha} \middle| \begin{matrix} (0,1) \\ (0,1),(1-\alpha,\alpha) \end{matrix} \right] - \nu H^{1,1}_{1,2}\left[-\nu t^{\alpha} \middle| \begin{matrix} (0,1) \\ (0,1),(1-\alpha,\alpha) \end{matrix} \right] \right\} \theta(t), \quad (16)$$

where $H^{1,1}_{1,2}$ is the H-function for a detailed comprehensive account of the H-function, see [10], and μ and ν are the roots of the algebraic system $\mu + \nu = -\frac{1}{RC}$ and $\mu\nu = \frac{1}{LC}$.

Proof. Applying the Sumudu transform with respect to the variable t, we get

$$R\,u^{-\alpha}F(u) + \frac{F(u)}{C} + \frac{R/LC}{u^{-\alpha}}F(u) = \frac{1}{u}. \quad (17)$$

Solving for F(u), it gives

$$F(u) = \frac{1}{R}\frac{u^{-\alpha-1}}{u^{-2\alpha} + a\,u^{-\alpha} + b}, \quad (18)$$

where we have introduced the positive parameters $a \equiv 1/RC$ and $b \equiv 1/LC$.

On taking the inverse Sumudu transform of (18) and applying the formula (7), it is seen that

$$i_C(t) = \frac{t^{\alpha-1}}{R} \sum_{r=0}^{\infty} (-a)^r\ t^{\alpha r}\ E^{r+1}_{2\alpha,\alpha+\alpha r}(-b\,t^{2\alpha})\theta(t), \quad (19)$$

where $E^{\rho}_{\mu,\nu}(.)$ is the three-parameter Mittag-Leffler function [21] and $\theta(t)$ is the

Heaviside function.

Now, using the Theorem 1.1, in (19), we get

$$i_C(t) = \frac{t^{\alpha-1}}{R} \frac{\mu E_{\alpha,\alpha}(\mu t^{\alpha}) - \nu\ E_{\alpha,\alpha}(\nu t^{\alpha})}{\mu - \nu} \theta(t), \tag{20}$$

where μ and ν are the roots of the algebraic system $\mu + \nu = -\frac{1}{RC}$ and $\mu\nu = \frac{1}{LC}$. Finally, applying the identity [23, p.291]

$$E_{\alpha,\beta}(z) = H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\beta,\alpha) \end{matrix} \right], \tag{21}$$

the above expression (20) becomes,

$$i_C(t) = \frac{t^{\alpha-1}}{R(\mu-\nu)} \left\{ \mu H^{1,1}_{1,2}\left[-\mu t^{\alpha} \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\alpha,\alpha) \end{matrix} \right] - \nu H^{1,1}_{1,2}\left[-\nu t^{\alpha} \,\middle|\, \begin{matrix} (0,1) \\ (0,1),(1-\alpha,\alpha) \end{matrix} \right] \right\} \theta(t). \tag{22}$$

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