

Unsteady Couette Flow through a Porous Medium in a Rotating System

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Abstract

Investigation on an unsteady Couette flow through a porous medium of a viscous incompressible fluid between parallel plates, rotating with a uniform angular velocity about an axis normal to the plate is presented. The fluid saturated porous media and plates rotate in unison with the same angular velocity. Laplace transform technique has been applied to obtain an exact solution of the governing equations for small as well as large times and presented graphically. Shear stress for primary as well as secondary velocities is computed analytically.

Keywords: Couette flow. Porous medium. Ekman number. Shear stress. Rotating system.

Introduction

Rotation has an immense importance in various phenomena such as in cosmical fluid dynamics, meteorology, geophysical fluid dynamics, gaseous and nuclear reactors and many engineering applications, that is why, the study of Couette flow through porous media in a rotating system enhances an interest to the researchers due to its applications in the aforesaid area. Such a study has a greater importance in the design of turbines and turbo mechanics, in estimating the flight path of rotating wheels and spin-stabilized missiles. Furthermore, such flows are useful for petroleum engineers, hydrologists and aero-dynamists to serve their purposes. In this regard Greenspan [1] has presented world a pioneer work on the theory of rotating fluids and theoretical

presentation of Stokes and Rayleigh layers in rotating systems has been given by Thornely [2]. Berker [3] has investigated the flow between two disks, rotating about a common axis with the same angular velocity. Vidyanidhi and Nigam [4] studied the Couette flow between rotating parallel plates under constant pressure gradient. Later, Erdogan [5,6] established the initial conditions which make the flow two-dimensional. He [6] has obtained an exact solution for the flow due to parallel disks rotating about non-coincident axes when one of the disks is executing non-torsional oscillations. He [7] has also considered the flow due to non-coaxial rotations of a disk oscillating in its own plane and a fluid rotating at infinity. Erdogan [7] studied the unsteady hydrodynamic viscous flow between eccentric rotating disks. An exact solution for the unsteady flow in which the eccentric disks execute non-torsional oscillations is shown by Rao and Kasiviswanathan [8] and an extension to this flow to the unsteady heat transfer is presented by Kasiviswanathan and Rao [9]. They [10] also presented an exact solution of the unsteady Navier-Stokes equations for the flow due to an eccentrically rotating porous disk oscillating in its own plane and the fluid at infinity.

A rich variety of important analytical, numerical, and experimental results have been published on this topic, for example, Batchelor [11], Greenspan [1], Pop and Soundalgekar [12], Puri [13], and Vidyanidhi and Nigam [2], have done excellent work analytically. Jana and Dutta [14] studied the steady Couette flow of a viscous incompressible fluid between two infinite parallel plates, one stationary and the other moving with uniform velocity, in a rotating frame of reference and discussed it for small as well as large times.

A large number of investigations made on rotating fluid and they are important to better understand the flow through a porous medium in a rotating system. In general, most of solutions for unsteady flows of viscous fluids are in a series form. These series may be rapidly convergent for large values of the time but slowly convergent for small values of the time or vice versa. Sometimes, it can be difficult to obtain the solution for small values of the time but it can be easy to obtain it for large values of the time and the opposite can also be true. Abbott and Walters [15] presented an exact solution of the hydrodynamic flow between two disks, rotating with the same angular velocity about non-coincident axes. Mohanty [16] discussed the magnetohydrodynamics flow between two disks, rotating with the same angular velocity about two different axes. Again Rao and Kasiviswanathan [17] extended it to the problem for micropolar fluid. Other extensions were made by Ersoy [18], Rajagopal [19]. Rao and Kasiviswanathan [20] considered the flow of an incompressible viscous fluid between two eccentric rotating disks for unsteady cases.

In this paper, we consider the unsteady Couette flow through porous media in a rotating frame of reference and obtain an exact solution of velocity field and shear stress for large values of the time as well as for small values of the time, Laplace transform technique has been used to calculate these results for small and large times. The flow through porous media between two parallel plates distant ' d ' apart where the upper plate is suddenly set into motion with uniform velocity U . Velocity field and shear stresses due to primary and secondary flows for small as well as large time τ are obtained analytically and depicted graphically.

Mathematical Analysis

A Newtonian fluid saturated porous media is assumed to flow between two infinitely long parallel plates distant 'd' apart rotating with uniform angular velocity Ω . The upper plate moves with a uniform velocity U in x -direction while the lower plate is kept stationary. In the coordinate system the stationary plate is taken along x -axis and z -axis in the direction normal to it.

The flow of the rotating fluid is governed by the Navier-Stokes equation whose two components are

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} + 2\Omega v - \frac{\nu}{k} u, \quad (2.1)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} - 2\Omega u - \frac{\nu}{k} v, \quad (2.2)$$

where $\nu (= \frac{\mu}{\rho})$ is known as kinematic co-efficient of viscosity and the velocity components u and v are taken along x - and y - directions respectively.

Here the initial and boundary conditions are

$$u = v = 0 \text{ for } t \leq 0, 0 \leq z \leq d, \quad (2.3)$$

and

$$\begin{aligned} u = v = 0 \text{ at } z = 0, \text{ for } t > 0, \\ u = U, v = 0 \text{ at } z = d, \text{ for } t > 0. \end{aligned} \quad (2.4)$$

Introducing non-dimensional variables

$$\eta = \frac{z}{d}, \tau = \frac{\nu t}{d^2}, u_1 = \frac{u}{U}, v_1 = \frac{v}{U} \text{ and } K = \frac{k}{d^2}. \quad (2.5)$$

Eqs. (2.1) and (2.2) reduce to

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial^2 u_1}{\partial \eta^2} + 2Pv_1 - \frac{u_1}{K}, \quad (2.6)$$

$$\frac{\partial v_1}{\partial \tau} = \frac{\partial^2 v_1}{\partial \eta^2} + 2Pu_1 - \frac{v_1}{K}, \quad (2.7)$$

where $P = E^{-1} = \frac{\Omega d^2}{\nu}$; E is the Ekman number.

These two equations can be expressed as

$$\frac{\partial q}{\partial \tau} = \frac{\partial^2 q}{\partial \eta^2} - 2iPq - \frac{q}{K}, \quad (2.8)$$

where $q = u_1 + iv_1$ and i , the imaginary number (2.9)

Eqs. (2.3) and (2.4) become

$$q = 0 \text{ for } \tau \leq 0, 0 \leq \eta \leq 1, \quad (2.10)$$

and

$$\begin{aligned} q &= 0 \text{ at } \eta = 0, \tau > 0, \\ q &= 1 \text{ at } \eta = 1, \tau > 0. \end{aligned} \quad (2.11)$$

Consider

$$q(\eta, \tau) = F(\eta, \tau)e^{-2iP\tau}. \quad (2.12)$$

Eqs. (2.8) with initial and boundary conditions (2.10) and (2.11) be transformed into the following equations respectively

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial \eta^2} - \frac{F}{K}, \quad (2.13)$$

$$F(0, \tau) = 0, \text{ and } F(1, \tau) = e^{2iP\tau}. \quad (2.14)$$

Applying Laplace transform technique to Eq. (2.13), we get

$$s\bar{F} = \frac{d^2 \bar{F}}{d\eta^2} - \frac{\bar{F}}{K}, \quad (2.15)$$

where

$$\bar{F} = \int_0^\infty F(\eta, \tau)e^{-s\tau} d\tau.$$

Eq. (2.14) becomes

$$\bar{F}(0) = 0 \text{ and } \bar{F}(1) = \frac{1}{s-2iP}, \quad (2.16)$$

Using the boundary conditions (2.16), we get the solution of Eq. (2.15) in the form

$$\bar{F}(\eta, s) = \frac{\sinh \sqrt{s + \frac{1}{K}} \eta}{(s-2iP) \sinh \sqrt{s + \frac{1}{K}}}. \quad (2.17)$$

Analytical Solution

Solutions for small time τ

Eq. (2.17) can be expressed as

$$\bar{F}(\eta, s) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2iP)^n}{s^{n+1}} \left[e^{-(1+2m-\eta)\sqrt{s+\frac{1}{K}}} - e^{-(1+2m+\eta)\sqrt{s+\frac{1}{K}}} \right]. \quad (3.1)$$

The inverse Laplace transform of (3.1) is

$$F(\eta, \tau) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(2iP)^n}{2\sqrt{\pi}n!} \times \left[(1+2m-\eta) \int_0^\tau \frac{(\tau-\xi)^n}{\xi^{3/2}} e^{-\left(\frac{\xi}{K} + \frac{(1+2m-\eta)^2}{4\xi}\right)} d\xi - \right.$$

$$(1 + 2m + \eta) \int_0^\tau \frac{(\tau - \xi)^n}{\xi^{3/2}} e^{-\left(\frac{\xi}{k} + \frac{(1+2m+\eta)^2}{4\xi}\right)} d\xi \tag{3.2}$$

When $k \rightarrow \infty$, Eq (3.2) becomes

$$\begin{aligned} F(\eta, \tau) &= \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{(2iP)^n}{2\sqrt{\pi n!}} \left[(1 + 2m - \eta) \int_0^\tau \frac{(\tau - \xi)^n}{\xi^{3/2}} e^{-\left(\frac{(1+2m-\eta)^2}{4\xi}\right)} d\xi - \right. \\ &(1 + 2m + \eta) \int_0^\tau \frac{(\tau - \xi)^n}{\xi^{3/2}} e^{-\left(\frac{(1+2m+\eta)^2}{4\xi}\right)} d\xi \left. \right] \\ &= \sum_{m=0}^\infty \sum_{n=0}^\infty (2iP)^n (4\tau)^n \times \\ &\left[i^{2n} \operatorname{erfc}\left(\frac{1+2m-\eta}{2\sqrt{\tau}}\right) - i^{2n} \operatorname{erfc}\left(\frac{1+2m+\eta}{2\sqrt{\tau}}\right) \right]. \end{aligned} \tag{3.3}$$

Which agrees with those of Guria[39].

Alternately we can find the Laplace inversion ([20], page 297, [22]) of Eq. (2.8) subject to the boundary conditions (2.9) and (2.10) as

$$F(\eta, \tau) = \sum_{m=0}^\infty \sum_{n=0}^\infty \left(2iP + \frac{1}{k}\right)^n (4\tau)^n \left[i^{2n} \operatorname{erfc}\left(\frac{1+2m-\eta}{2\sqrt{\tau}}\right) - i^{2n} \operatorname{erfc}\left(\frac{1+2m+\eta}{2\sqrt{\tau}}\right) \right], \tag{3.4}$$

When $k \rightarrow \infty$, Eq (3.4) reduces to

$$\begin{aligned} F(\eta, \tau) &= \sum_{m=0}^\infty \sum_{n=0}^\infty (2iP)^n (4\tau)^n \times \\ &\left[i^{2n} \operatorname{erfc}\left(\frac{1+2m-\eta}{2\sqrt{\tau}}\right) - i^{2n} \operatorname{erfc}\left(\frac{1+2m+\eta}{2\sqrt{\tau}}\right) \right], \end{aligned} \tag{3.5}$$

This is same as the above Eq. (3.3).

Where

$$\begin{aligned} i^n \operatorname{erfc}(x) &= \int_x^\infty i^{n-1} \operatorname{erfc}(\xi) d\xi, \\ i \operatorname{erfc}(x) &= \int_x^\infty \operatorname{erfc}(\xi) d\xi, \\ i^0 \operatorname{erfc}(x) &= \operatorname{erfc}(x) \end{aligned}$$

Solutions for large time τ

For the large time we apply the method used by Batchelor[12] and the solution of Eq.(2.8) can be written as

$$q(\eta, \tau) = \frac{\sinh \sqrt{2iP + \frac{1}{K}} \eta}{\sinh \sqrt{2iP + \frac{1}{K}}} + F_1(\eta, \tau), \tag{3.6}$$

where $F_1(\eta, \tau)$ is the solution of the differential equation

$$\frac{\partial F_1}{\partial \tau} = \frac{\partial^2 F_1}{\partial \eta^2} - 2iPF_1 - \frac{F_1}{K}, \tag{3.7}$$

subject to the condition

$$F_1(0, \tau) = 0, F_1(1, \tau) = 0,$$

and

$$F_1(\eta, 0) = -\frac{\sinh \sqrt{2iP + \frac{1}{K}} \eta}{\sinh \sqrt{2iP + \frac{1}{K}}}. \quad (3.8)$$

Solving Eq. (3.7), we have,

$$F_1(\eta, \tau) = A_n e^{-\lambda_n^2 \tau} \sin n\pi\eta, \quad (3.9)$$

where A_n can be obtained from

$$\sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 \tau} \sin n\pi\eta = -\frac{\sinh \sqrt{2iP + \frac{1}{K}} \eta}{\sinh \sqrt{2iP + \frac{1}{K}}}, \quad (3.10)$$

And

$$\lambda_n^2 = (n\pi)^2 + 2iP + \frac{1}{K}. \quad (3.11)$$

Consequently the fluid velocity

$$q(\eta, \tau) = \frac{\sinh \sqrt{2iP + \frac{1}{K}} \eta}{\sinh \sqrt{2iP + \frac{1}{K}}} + 2 \sum_{n=1}^{\infty} \frac{n\pi(-1)^n e^{-\lambda_n^2 \tau}}{(n\pi)^2 + 2iP + \frac{1}{K}} \sin n\pi\eta. \quad (3.12)$$

Its real and imaginary parts are given by

$$u_1 = \frac{S(\theta\eta)S(\theta) + C(\theta\eta)C(\theta)}{(S(\theta))^2 + (C(\theta))^2} + 2 \sum_{n=0}^{\infty} \frac{n\pi(-1)^n \sin n\pi\eta}{((n\pi)^2 + \frac{1}{K})^2 + (2P)^2} \times \left\{ \left((n\pi)^2 + \frac{1}{K} \right) \cos 2P\tau - 2P \sin 2P\tau \right\} e^{-((n\pi)^2 + \frac{1}{K})\tau}, \quad (3.13)$$

$$v_1 = \frac{C(\theta\eta)S(\theta) - S(\theta\eta)C(\theta)}{(S(\theta))^2 + (C(\theta))^2} - 2 \sum_{n=0}^{\infty} \frac{n\pi(-1)^n \sin n\pi\eta}{((n\pi)^2 + \frac{1}{K})^2 + (2P)^2} \times \left\{ 2P \cos 2P\tau - \left((n\pi)^2 + \frac{1}{K} \right) \sin 2P\tau \right\} e^{-((n\pi)^2 + \frac{1}{K})\tau}. \quad (3.14)$$

Where

$$S(\theta\eta) = \sinh \alpha \eta \cdot \cos \beta \eta, C(\theta\eta) = \cosh \alpha \eta \cdot \sin \beta \eta,$$

$$S(\theta) = \sinh \alpha \cdot \cos \beta, C(\theta) = \cosh \beta \cdot \sin \alpha,$$

$$\alpha = \sqrt{R} \cos \theta / 2, \beta = \sqrt{R} \sin \theta / 2, \theta = \tan^{-1} 2PK,$$

$$R = \sqrt{(2P)^2 + \frac{1}{K^2}} \quad (3.15)$$

Shear Stress

$$\left(\frac{\partial q}{\partial \eta}\right)_{\eta=0} = \frac{\sqrt{2iP+\frac{1}{K}}}{\sinh\sqrt{2iP+\frac{1}{K}}} + 2 \sum_{n=1}^{\infty} \frac{(n\pi)^2(-1)^n e^{-((n\pi)^2+2iP+\frac{1}{K})\tau}}{(n\pi)^2+2iP+\frac{1}{K}}. \tag{4.1}$$

On separating into real and imaginary parts we get the shear components as

$$\begin{aligned} \tau_{x0} &= \frac{\alpha S(\theta)+\beta C(\theta)}{(S(\theta))^2+(C(\theta))^2} + 2 \sum_{n=1}^{\infty} \frac{(n\pi)^2(-1)^n e^{-((n\pi)^2+\frac{1}{K})\tau}}{((n\pi)^2+\frac{1}{K})^2+(2P)^2} \\ &\times \left\{ \left((n\pi)^2 + \frac{1}{K} \right) \cos 2P\tau - 2P \sin 2P\tau \right\}, \end{aligned} \tag{4.2}$$

$$\begin{aligned} \tau_{y0} &= \frac{-\alpha C(\theta)+\beta S(\theta)}{(S(\theta))^2+(C(\theta))^2} - 2 \sum_{n=1}^{\infty} \frac{(n\pi)^2(-1)^n e^{-((n\pi)^2+\frac{1}{K})\tau}}{((n\pi)^2+\frac{1}{K})^2+(2P)^2} \\ &\times \left\{ 2P \cos 2P\tau - \left((n\pi)^2 + \frac{1}{K} \right) \sin 2P\tau \right\}. \end{aligned} \tag{4.3}$$

Conclusive Discussions

The study of unsteady Couette flow through porous media in a rotating system is addressed in this chapter. Solution of the equation of complex velocity (3.12) is obtained by the use of Laplace transform technique. The primary and the secondary flows of the velocity profile are depicted against η for different values of the porosity parameter K and the inverse Ekman number ($E^{-1} = P$) for large time. Figures 5.1 & 5.2 are represented for different values of P and fixed $\tau = 0.5$ & $K = 0.1$. The primary velocity profile displays that an increasing inverse Ekman number P decreases the velocity profile for $P = 3, 4$ and have opposite behavior for smaller $P (= 1, 2)$, whereas the secondary velocity profile overlaps one another and reaches its peak between the middle of the stationary and the moving plates (at about $\eta = 0.53$). For higher rotation parameter ($E^{-1} = 4$), the secondary velocity profile (fig. 5.2) behaves in an oscillatory manner near the stationary plate and leads to a back flow in the region $0 \leq \eta \leq 0.7$. The next figures have been erected for varied porosity parameter K . It is stated from fig. 5.3 that the primary velocity u_1 for small time $\tau = 0.5$ increases with increase in the porosity parameter K and becomes stagnant in form of slope for higher K . Consequently this profile approaches its peak near the moving plate. In the next figure velocity profile overlaps each other. It is noticed that a back flow influence occurs in the region $0 \leq \eta \leq 0.5$. The secondary velocity profile v_1 (fig.5.4) are overlapping in nature and reaches its peak at $\eta = 0.5$ for small porosity parameter $K = 0.1$. Figures 5.5 & 5.6 are erected for large time $\tau = 10$ and increasing porosity parameter K . They show the same behavior as that of previous pairs of velocity profiles. The primary profile (fig. 5.5) shows their nature increasingly with increase in porosity parameter K and having fixed slope against increasing η while a back flow exists in the region $0 \leq \eta \leq 0.7$. The secondary velocity profile v_1 (fig. 5.6) increase in magnitude with increase in η and get its maximum peak at $\eta = 0.5$ for $K = 0.1$.

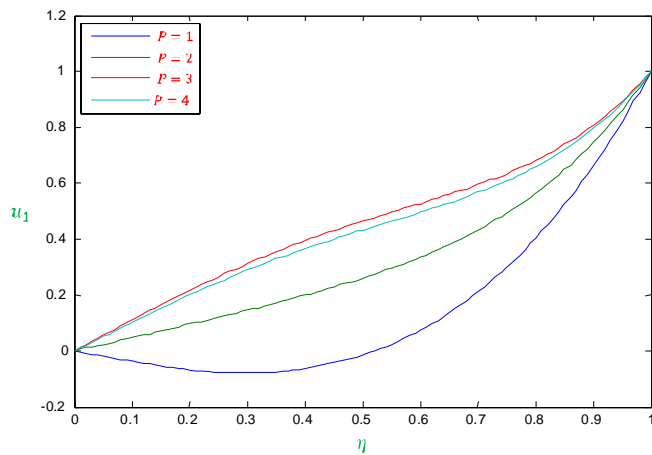


Figure 5.1: primary velocity profile for $\tau = 0.5, K = 0.1$

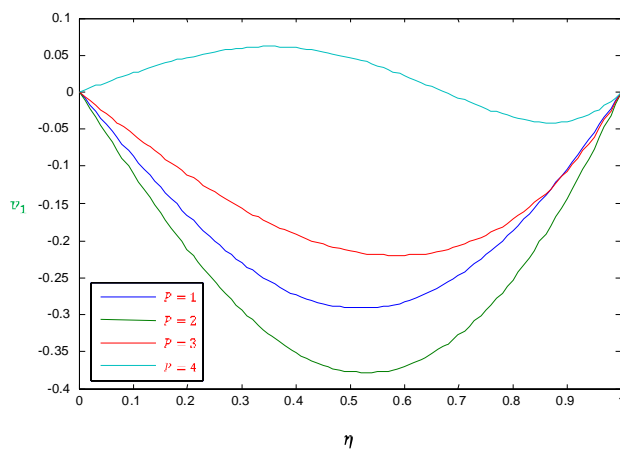


Figure 5.2: secondary velocity profile for $\tau = 0.5, K = 0.1$

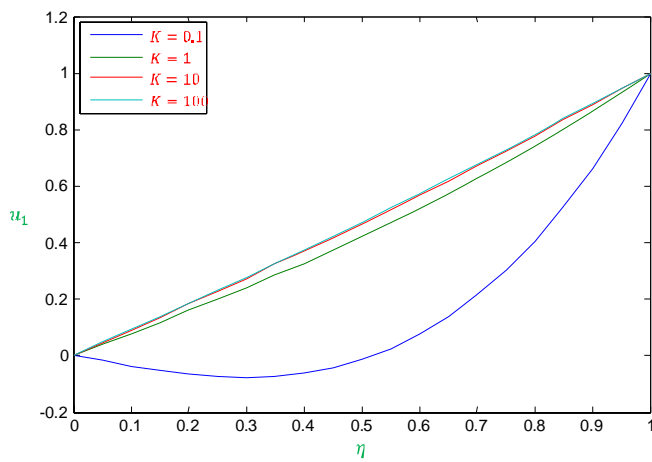


Figure 5.3: primary velocity profile for $\tau = 0.5, P = 1$

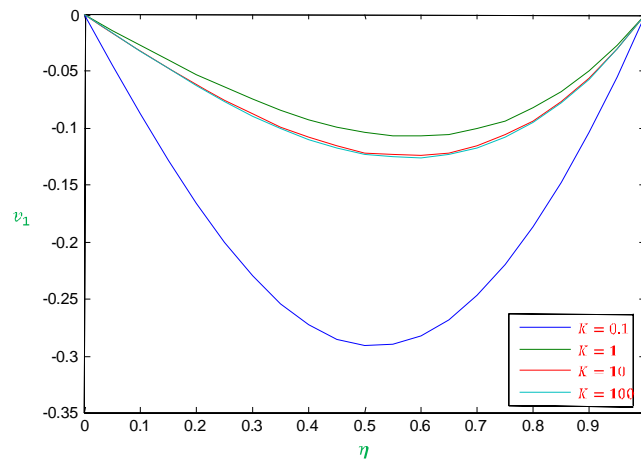


Figure 5.4: secondary velocity profile for $\tau = 0.5, P = 1$

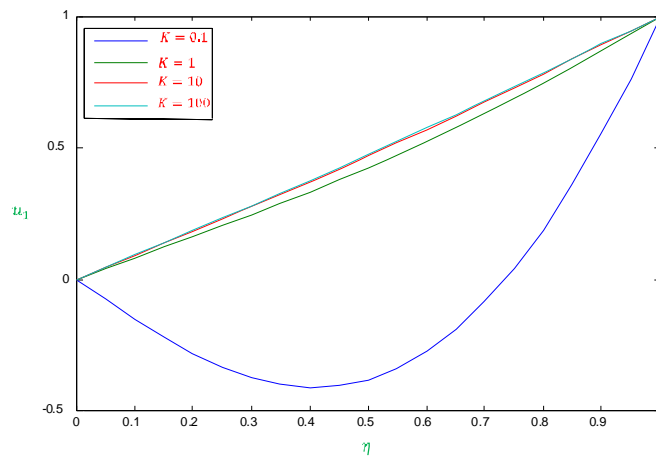


Figure 5.5: primary velocity profile for $\tau = 10, P = 1$

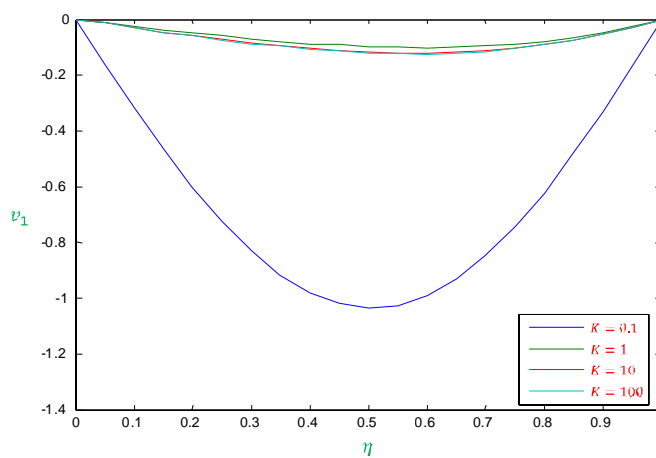


Figure 5.6: secondary velocity profile for $\tau = 10, P = 1$

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