International Journal of Computational Science and Mathematics.
ISSN 0974-3189 Volume 2, Number 3 (2010), pp. 237--245

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# A Crack Problem for Initially Stressed Neo-Hookean Solids

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**Abstract**

The paper deals with the integral transform techniques, which is found to be very convenient for solving indentation and crack problems. The problem has been considered within the framework of incremental deformation theory for Neo-Hookean solids. Using the Hankel's transformation, the components of incremental displacement and incremental stress are found out and their variations are studied graphically.

**Keywords:** Hankel's transformation; Incremental deformation theory; Integral transform technique; Bessel function; Stresses; Neo-Hookean solids.

## Introduction

Various elastic bodies are found to posses initial stress, which exists in the body by process of preparation or by the action of body forces e.g. a sheet of metal rolled up into a cylinder and the edges welded together. If such a body is further subjected to deforming forces then apart from the initial finite deformation it will have incremental deformation also. Trefftz [ 1 ], Neuber [ 2 ], Green [ 3 ], [ 4 ] and Biot [ 5 ], [ 6 ] have discussed and given basic equations of such incremental deformation theory. The derivation of basic equations generally comes from the theory of finite deformation making case of tensor calculus. But Biot [ 5 ], [ 6 ] as developed his theory using cartesian concepts and elementary mathematical method. The major contribution towards the development of the subject in this direction is due to Sneddon [ 7 ]. Lowengrub and Sneddon [ 8 ] have written a monograph on crack problems in the classical theory of elasticity. Crack and Punch problems for transversely isotropic bodies have been solved by Eilliot [ 9 ], using Hankel's transform. Later on Kurashige

[ 10 ] has discussed a two-dimensional crack problem for an initially stressed Neo-Hookean solid. Using integral transform technique, Ali [ 11 ] has discussed the case of opening of a crack of prescribed shape in an initially stressed body. Hara et.al. [ 12 ], Inove et.al. [ 13 ] and Sakamoto et.al. [ 14 ] have discussed some contact problems. Recently, Sarkar et.al. [ 15 ] have discussed the problem of four coplanar Griffith cracks moving in an infinitely long elastic strip under anti-plane shear stress. An analysis based on finite element approach by Anifantis [16] and initiation and propagation of surface cracks by Dag and Erdogan [17] has been carried out. The present problem has been solved with the help of the theory of Bessel's function.

In this paper, attempts have been made to obtain a basic equation to be satisfied by a displacement function and to express the components of incremental displacement, stress and strain by it for an axisymmetric problem under incremental deformation theory of Neo-Hookean solid and obtain the solution of a crack problem for initially stressed body. The medium is supposed to be isotropic, homogeneous and incompressible.

## Basic formulae

We have adopted the fundamental equations of incremental deformation theory constructed by Biot [ 5 ] [ 6 ]. In rectangular cartesian coordinates $x_i$ and time t, the equation of motion for incremental deformation theory and the expressions of incremental boundary forces per unit area are

$$\frac{\partial s_{ij}}{\partial x_j} + S_{jk}\frac{\partial w_{ik}}{\partial x_j} + S_{ik}\frac{\partial w_{jk}}{\partial x_j} - e_{jk}\frac{\partial S_{ik}}{\partial x_j} = \rho\,\frac{\partial^2 u_i}{\partial t^2} \tag{1}$$

$$\Delta f_i = (s_{ij} + S_{jk}w_{ik} + S_{ij}e - S_{ik}e_{jk})\,n_j\ . \tag{2}$$

In equations (1) and (2), the usual convention for summation over repeated indices is applied. The second, third and fourth terms of left hand side in equation (1) represent the effect of initial stress.

The stress – strain relations are:

$$S_{11} - S_{22} = \mu_0(\lambda_1{}^2 - \lambda_2{}^2)\,, \tag{3a}$$

$$S_{22} - S_{33} = \mu_0(\lambda_2{}^2 - \lambda_3{}^2)\,, \tag{3b}$$

$$S_{33} - S_{11} = \mu_0(\lambda_3{}^2 - \lambda_1{}^2)\,. \tag{3c}$$

The half plane is assumed to be deformed under the impact of an axisymmetric body. The cylindrical polar coordinates $(r,\theta,z)$ of a point in the initially deformed body are connected with rectangular coordinates by relations

$$r = \sqrt{x_1^{\ 2} + x_2^{\ 2}}, \quad \theta = \tan^{-1}(x_2/x_1), \quad z = x_3 \tag{4}$$

It is assumed that the only no-zero components of initial stress are $S_{rr}$, $S_{qq}$ and $S_{zz}$, which are uniform throughout the body and the body is in the state of symmetrical incremental strain with respect to z-axis. The equation of (1) reduce, in the cylindrical coordinates to:

$$\frac{\partial s_{rr}}{\partial r} + \frac{s_{rr} - s_{\theta\theta}}{r} + \frac{\partial s_{rz}}{\partial z} - (S_{rr} - S_{zz}) \frac{\partial w_{rz}}{\partial z} = \rho \frac{\partial^2 u_r}{\partial t^2}, \tag{5a}$$

$$\frac{\partial s_{zz}}{\partial z} + \frac{1}{r}\frac{\partial}{\partial r}(r\, s_{zr}) - (S_{rr} - S_{zz}) \frac{1}{r}\frac{\partial}{\partial r}(r\, w_{rz}) = \rho \frac{\partial^2 u_z}{\partial t^2}. \tag{5b}$$

The incremental displacement $u_r$ and $u_z$ in terms of potential function $\Phi$ (r,z) are given by

$$u_r = -\frac{\partial^2 \Phi}{\partial r \partial z}, \quad u_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \Phi}{\partial r}\right). \tag{6}$$

The function $\Phi$ is given by the simple partial differential equation

$$\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2}\right\}\left\{k^2 \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{\partial^2 \Phi}{\partial z^2} - \frac{\rho}{\mu_0}\frac{\partial^2 \Phi}{\partial t^2}\right\} = 0,$$

where, $k = \lambda_r / \lambda_z$. (8)

## Formulation of the Problem

The semi-infinite medium $z \geq 0$ is supposed to be initially deformed and the components $s_{zz}$, in addition to $s_{\theta\theta}$ is also to be zero, so that

$$S_{rr} = \mu_0 (\lambda_r^2 - \lambda_z^2) = -P. \tag{9}$$

Further, without loss of generality the length of crack of is taken to be unity in the plane z = 0, the center of crack lying on the z –axis. The distribution of stress in the neighbourhood of the crack is studied with the help of Hankel transform.

The Hankel transformation and its inversion transform of order zero are given following relations:

$$\bar{\Phi}(\xi) = \int_0^\infty \Phi(r)\ r J_0(r\,\xi)\, dr, \tag{10a}$$

$$\Phi(r) = \int_0^\infty \bar{\Phi}(\xi)\ \xi\, J_0(r\,\xi)\, d\xi. \tag{10b}$$

From which the following formulae for differentiated function are obtained

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) = -\int_0^\infty \overline{\Phi}\,\xi^3 J_0(r\,\xi)\,d\xi\,, \tag{11a}$$

$$\frac{\partial}{\partial r}\Phi = -\int_0^\infty \overline{\Phi}\,\xi^2 J_1(r\,\xi)\,d\xi\,,$$

where $J_0$ and $J_1$ are Bessel functions of order 0 and 1, respectively.

Using these formulae, the incremental displacement $u_r$ and $u_z$ , the incremental rotation $w_{rz}$ and the components of incremental stress $s_{zz}$ , $s_{zr}$ and $s_{rr}$ are expressed in Hankel inversion as follows,

$$u_r = \int_0^\infty \frac{\partial\overline{\Phi}}{\partial z}\,\xi^2 J_1(r\,\xi)\,d\xi \tag{12a}$$

$$u_z = -\int_0^\infty \overline{\Phi}\,\xi^3 J_0(r\,\xi)\;d\xi\,. \tag{12b}$$

$$w_{rz} = -\frac{1}{2}\int_0^\infty \left[\xi^4\overline{\Phi} - \xi^2\frac{\partial^2\overline{\Phi}}{\partial z^2}\right] J_1(r\,\xi)\;d\xi\,, \tag{13}$$

$$s_{zz} = s - \mu_0\,\lambda_z^2\,(2+k^2)\int_0^\infty \xi^3\,\frac{\partial\overline{\Phi}}{\partial z}\,J_0(r\,\xi)\;d\xi\,, \tag{14a}$$

$$s_{zr} = \frac{1}{2}\,\mu_0\,\lambda_z^2\,(1+k^2)\int_0^\infty \left(\xi^4\overline{\Phi} + \xi^2\frac{\partial^2\overline{\Phi}}{\partial z^2}\right) J_1(r\,\xi)\,d\xi\,, \tag{14b}$$

$$s_{rr} = \mu_0\lambda_z^2\left\{\int_0^\infty\left(k^2\xi^3\frac{\partial\overline{\Phi}}{\partial z} + \xi\frac{\partial^3\overline{\Phi}}{\partial z^3}\right)J_0(r\,\xi)\;d\xi - \frac{2k^2}{r}\int_0^\infty \xi^2\,\frac{\partial\overline{\Phi}}{\partial z}\,J_1(r\,\xi)\,d\xi\right\}, \tag{14c}$$

where

$$s = \frac{1}{2}\,(s_{rr} + s_{\theta\theta}) = \mu_0\;\lambda_z^2\int_0^\infty \xi\;\frac{\partial^3\overline{\Phi}}{\partial z^3}\;J_0(r\,\xi)\;d\xi \tag{14d}$$

Equation (7), by Hankel's transform, reduces to the ordinary differential equation,

$$(D^2 - \xi^2)\;(D^2\overline{\Phi} - k^2\xi^2\overline{\Phi}) = 0, \tag{15}$$

where $D = \dfrac{d}{dz}$.

If the pressure p(r) is supposed to be applied to the plane z = 0, the boundary conditions for the semi-infinite body are presented as follows:

$$\left.\begin{aligned} s_{rz} &= 0 && (0 \le r \le \infty) \\ s_{zz} &= -p(r) && (0 \le r \le \infty) \end{aligned}\right\} \qquad (16)$$

$$\left.\begin{aligned} u_z &= 0 && (1 \le r \le \infty) \\ s_{zz} &= -r && (0 \le r \le 1) \end{aligned}\right\} \qquad (17)$$

## Solution of the problem

The solution of the ordinary differential equation (15) for a half space z ≥ 0 is given by

$$\overline{\Phi} = A(\xi)\, e^{\xi z} + B(\xi)\, e^{k\xi z}, \qquad (18)$$

where A ($\xi$) and B ($\xi$) are the integral constants.

Now applying the boundary conditions (16) to the equation (18), we get

$$A(\xi) = \left[\frac{\left(1+k^2\right)^2}{\mu_0 \lambda_z^2 \left\{\left(1+k^2\right)^2 - 4k\right\}}\right] \frac{\overline{p}(\xi)}{\xi^3}, \qquad (19a)$$

$$B(\xi) = \left[\frac{-2}{\mu_0 \lambda_z^2 \left\{\left(1+k^2\right)^2 - 4k\right\}}\right] \frac{\overline{p}(\xi)}{\xi^3}. \qquad (19b)$$

The boundary conditions (17) applied to equation (18) gives the following dual integral equations:

$$\int_0^\infty \xi\, \overline{p}(\xi) J_0(r\,\xi)\, d\xi = r, \qquad (r < 1) \qquad (20a)$$

$$\int_0^\infty \xi^2\, \overline{p}(r\xi) J_0(r\,\xi)\, d\xi = 0, \qquad (r > a) \qquad (20b)$$

where $\overline{p(\xi)}$ is an unknown function. The above dual integral equations are a special case of the pair of equations considered by Titchmarsh [18] and therefore, the equations (20) is given as follows

$$\overline{p(\xi)} = \frac{1}{2}\left[\frac{2\cos\xi}{\xi^3} + \frac{2\sin\xi}{\xi^2} - \frac{\cos\xi}{\xi} - \frac{2}{\xi^3}\right]. \tag{21}$$

Thus $\overline{p(\xi)}$ being determined, the constants $A(\xi), B(\xi)$ and $\overline{p(\xi)}$ are known. Now the function $\overline{\Phi}$ can easily be determined from the equation (18). Hence the problem is completely solved.

Thus the non-vanishing components of stresses and displacements in terms of Hankel inversion are as follows:

$$u_z = \frac{-1}{\mu_0 \lambda_z^2 \{(1+k^2)^2 - 4k\}} \int_0^\infty \left\{ (1+k^2)\, e^{\xi z} - 2e^{k\xi z} \right\} R_1\, J_0(r\,\xi)\, d\xi\,, \tag{22a}$$

$$u_r = \frac{1}{\mu_0 \lambda_z^2 \{(1+k^2)^2 - 4k\}} \int_0^\infty \left\{ (1+k^2)\, e^{\xi z} - 2ke^{k\xi z} \right\} R_1\, J_0(r\,\xi)\, d\xi\,, \tag{22b}$$

$$s_{zz} = \frac{1}{\{(1+k^2)^2 \text{-} 4k\}} \int_0^\infty \left\{ (1+k^2)^2\, e^{\xi z} - 4ke^{k\xi z} \right\} R_2\, J_0(r\,\xi)\, d\xi, \tag{23a}$$

$$s_{zr} = \frac{-1}{\{(1+k^2)^2 \text{-} 4k\}} \int_0^\infty (1+k^2)^2\, (e^{\xi z} - e^{k\xi z})\, R_2\, J_1(r\,\xi)\, d\xi, \tag{23b}$$

$$s_{rr} = \frac{1}{\{(1+k^2)^2 \text{-} 4k\}} \Bigg[ \int_0^\infty \left\{ (1+k^2)^2 e^{\xi z} - 4k^3 \right\} R_2\, J_0(r\,\xi)\, d\xi$$

$$\frac{-2k^2}{r} \int_0^\infty \left\{ (1+k^2) e^{\xi z} - 2ke^{k\xi z} \right\} R_1\, J_1(r\,\xi)\, d\xi \Bigg]\,, \tag{23c}$$

and

$$s_{\theta\theta} = \frac{1}{\{(1+k^2)^2 \text{-} 4k\}} \Big[ \left(1 - k^4\right) \int_0^\infty e^{\xi z}\, R_2\, J_0(r\xi)\, d\xi\; + \tag{23c}$$

$$\frac{2k^2}{r} \int_0^\infty \left\{ (1+k^2) e^{\xi z} - 2ke^{k\xi z} \right\} R_1\, J_1(r\,\xi)\, d\xi\,. \tag{23d}$$

where

$$R_1 = \left\{ \frac{\cos\xi}{\xi^3} + \frac{\sin\xi}{\xi^2} - \frac{\cos\xi}{2\xi} - \frac{1}{\xi^3} \right\},$$

$$R_2 = \left\{ \frac{\cos\xi}{\xi^2} + \frac{\sin\xi}{\xi} - \frac{\cos\xi}{2} - \frac{1}{\xi^2} \right\}.$$

## Numerical Results

For a crack problem, variations of incremental stress and displacement component $s_{zz}$, $s_{rr}$ and $u_r$ with various parameters are plotted in Fig. 1 to 4. For a non- initially stressed body, P = 0 and which is given by k =1. If the initial stress is high, $P/\mu_0$ tends to unity. For different values of $P/\mu_0$ between 0.1 and 0.4, variations of stresses and displacement have been shown. Attention has been paid to investigate the influences of the initial stress.

Fig. 1 shows the variation of normal component of incremental stress $s_{zz}$ along $z/a$ =0.1. There is a sharp rise and fall in the neighbourhood of the edge of crack. It shows that as the crack comes in contact with elastic body, it produces larger stresses and explains the discontinuity of stress near r = *a*.

Fig.2 shows the distribution of normal component of incremental stress $s_{zz}$ along z-axis. The normal component of the incremental stress has a peak at a point from the surface of the crack and it decreases monotonically as the value of z increases for high initial stress. For no initial stress the peak is higher.

Fig.3 shows the variation of radial component of incremental stress $s_{rr}$ with r. It shows that there is a little influence on the variation of incremental stress in the neighbourhood of the edge of the crack.

Fig.4 shows the distribution of normal component of incremental displacement $u_z$ with r along $z/a$ =0.1. As the initial stress decreases, incremental stress has a high peak. The incremental stress decreases near the edge of the crack.

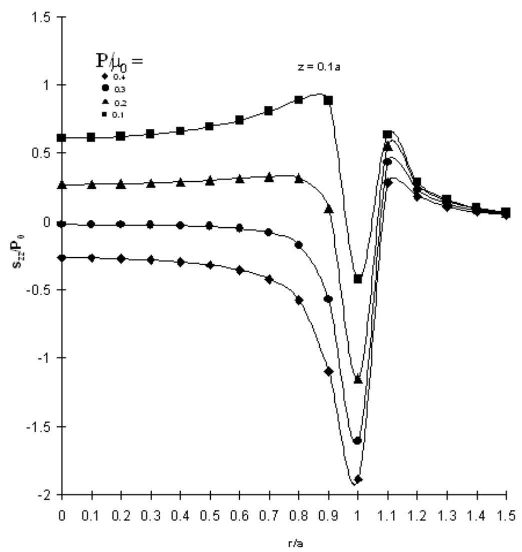



**Figure 1:** Variation of normal component of incremental stress $S_{zz}$ with r.

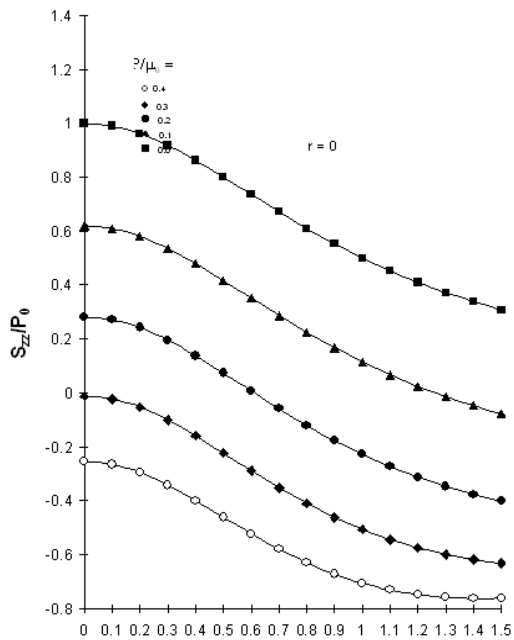
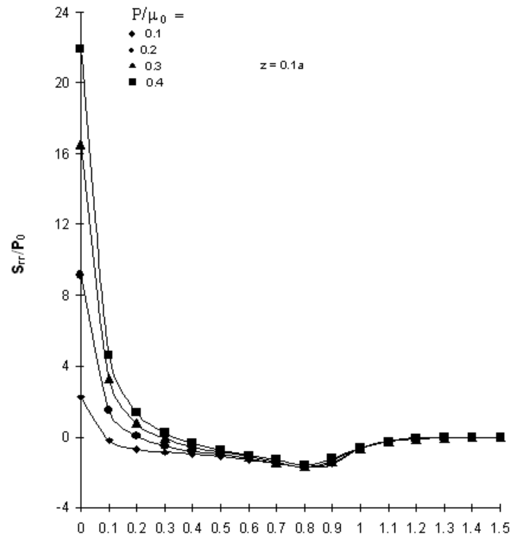



**Figure 2:** Variation of radial component of incremental stress $S_{zz}$ with r.



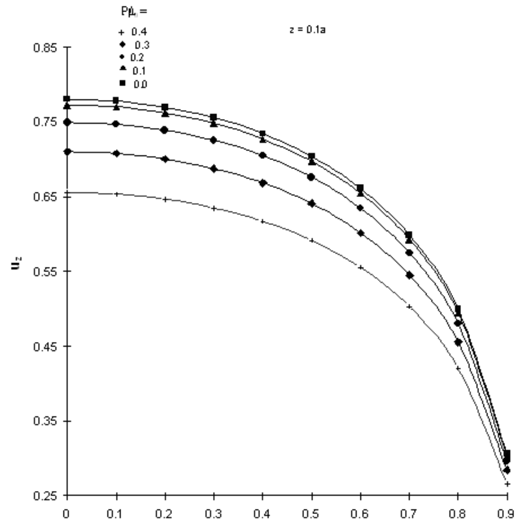




**Figure 3:** Variation of radial component of incremental stress $S_{zz}$ with r.

**Figure 4:** Variation of normal component of incremental displacement $u_z$ with r.

## Nomenclature

| | |
|---|---|
| $x_i$ | cartesian coordinates |
| $S_{ij}$ | initial stress, corresponding to initial finite deformation, referred to $x_i$ |
| $n_i$ | components of unit normal to boundary surface |
| $\rho$ | density in a finite deformation |
| $u_i$ | incremental displacement (infinitesimal) |
| W | elastic potential per unit volume |
| $e_{ij}$ | incremental strain |
| $\lambda_i$ | extension ratio |
| $\phi$ | displacement function |
| $w_{ij}$ | incremental rotation |
| $\mu_0$ | shear modulus in an unstrained state |
| e | incremental volume expansion |
| P | initial all-around compressive stress |
| $s_{ij}$ | incremental stress referred to axes which are incrementally displaced with the medium |
| $\Delta f_i$ | incremental boundary force per unit initial area |
| A ($\xi$), $B_2$ ($\xi$) | integral constants |

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