On Secondary Unitary Matrices

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Abstract

The concept of secondary unitary (s-unitary) matrices is introduced. Characterization of secondary unitary matrices and equivalent conditions are obtained.

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1. Introduction

Anna Lee[1] has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose Aᵀ and secondary transpose Aˢ are related as Aˢ = VAᵀV where ‘V’ is the permutation matrix with units in its secondary diagonal.

Also Aˢ denotes the conjugate secondary transpose of A i.e. Aˢ = (cᵢⱼ) where cᵢⱼ = aₙ−j₁, n−i₁ [2]. In this paper we introduce the concept of secondary unitary matrices (s-unitary).

1.1 Preliminaries and Notations

Let Cⁿˣⁿ be the space of nxn complex matrices of order n. For A∈ Cⁿˣⁿ. Let Aᵀ, A*, Aᵀ*, A* denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively.
For a complex matrix $A$, the Moore–Penrose $A^+$ of $A$ is the unique matrix $X$ satisfying the following four penrose equations. [3]

(i) $AXA = A$
(ii) $XAX = X$
(iii) $(AX)^* = AX$
(iv) $(XA)^* =XA$

Let ‘$k$’ be the fixed product of disjoint transpositions in $S_n$, the set of all permutations on $\{1,2,\ldots,n\}$ and ‘$K$’ be the associated permutation matrix which satisfies the following properties.


Let ‘$V$’ be the associated permutation matrix whose elements on the secondary diagonal are 1, other elements are zero. Also ‘$V$’ satisfies the following properties.

$V^T = V = V^* = V$ and $V^2 = I$.

A matrix $A \in \mathbb{C}^{n \times n}$ is called unitary if $AA^* = A^*A = I$. [4]

2. s-Unitary Matrix

Definition 2.1

A matrix $A \in \mathbb{C}^{n \times n}$ is said to be s-Unitary if $A^sA = A^*A = I$ [5]

i.e. $AVA^*V = VA^*VA = I$

i.e. $VA^*V = A^{-1}$.

Example: 2.2

(i) $A = \begin{bmatrix}
i & 1 \\
\sqrt{2} & \sqrt{2} \\
i & \sqrt{2} \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -i
\end{bmatrix}$

(ii) $A = \begin{bmatrix}
i & 1 \\
\sqrt{2} & \sqrt{2} \\
i & \sqrt{2} \\
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -i
\end{bmatrix}$

are s-unitary matrices.

Theorem: 2.3

Let $A \in \mathbb{C}^{n \times n}$. If $A$ is s-unitary matrix then $A^s$ is also s-unitary matrix.

Proof:

$A$ is s-unitary $\Rightarrow VA^*V = A^{-1}$

$V(A^s)^*V = V(A^*V)^T = V(A^T)^TV$
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= \sqrt{A^*AV}
= \sqrt{VA^*V}
= A^{-1}
= \overline{A}^{-1}
∴ \overline{A} is s-unitary.

**Theorem: 2.4**
Let \( A \in \mathbb{C}_{n \times n} \). If \( A \) is s-unitary then \( A^T \) is s-unitary.

**Proof:**
\( A \) is s-unitary \( \Rightarrow A^{-1} = VA^*V \)
\( (A^{-1})^T = (VA^*V)^T \)
\( = V(A^*)^TV^T = V(A^*)^TV \)
\( (A^{-1})^T = V(A^*)^*V \)
\( (A^T)^{-1} = V(A^T)^*V \)
∴ \( A^T \) is s-Unitary.

**Theorem: 2.5**
Let \( A \in \mathbb{C}_{n \times n} \). If \( A \) is s-unitary then \( A^* \) is s-unitary.

**Proof:**
\( A \) is s-unitary \( \Rightarrow A^{-1} = VA^*V \)
\( (A^{-1})^* = (VA^*V)^* \)
\( = V(A^*)^*V^* \)
\( (A^*)^{-1} = V(A^*)^*V \)
∴ \( A^* \) is s-unitary.

**Theorem 2.6**
Let \( A \in \mathbb{C}_{n \times n} \). If \( A \) is s-unitary then \( A^{-1} \) is s-unitary.

**Proof:**
\( A \) is s-unitary \( \Rightarrow A^{-1} = \overline{A}^t = VA^*V \)
\( A^{-1} = \overline{A}^t \)
\( (A^{-1})^{-1} = (\overline{A}^t)^{-1} \)
\( = (\overline{A}^{-1})^t \)
\( (A^{-1})^{-1} = \overline{A}^{-1} \)
∴ \( A^{-1} \) is s-unitary.

**Theorem: 2.7**
Let \( A \in \mathbb{C}_{n \times n} \). If \( A \) is s-unitary then \( iA \) is s-unitary.
Proof:

A is s-unitary \( \Rightarrow A^{-1} = VA^*V \)
\[
iA^{-1} = \text{i}(VA^*V)
\]
\[-(iA)^{-1} = V(iA^*)V = V(-i\overline{A}^T)V\]
\[(iA)^{-1} = V\left(\overline{iA}^T\right)\]
\[(iA)^{-1} = V(iA)^*V\]
\[\therefore iA \text{ is s-unitary.}\]

Theorem: 2.8
Let \( A, B \in \mathbb{C}_{n \times n} \). If \( A \) and \( B \) are s-unitary matrices then \( AB \) is s-unitary matrix.

Proof:

\[
A \text{ is s-unitary } \Rightarrow VA^*V = A^{-1}
\]
\[
B \text{ is s-unitary } \Rightarrow VB^*V = B^{-1}
\]
\[
V(AB)^*V = V(B^*A^*)V
\]
\[= (VB^*V)(VA^*V) = B^{-1}A^{-1} = (AB)^{-1}\]
\[\therefore V(AB)^*V = (AB)^{-1}\]
\[\therefore AB \text{ is s-unitary matrix.}\]

Theorem: 2.9
Let \( A, B \in \mathbb{C}_{n \times n} \) and \( A, B \) are s-unitary matrices and \( AB^s = B^sA, BA^s = A^sB \)
If \( AB^s + BA^s = -I \) then \( A + B \) is s-unitary
If \( AB^s + BA^s = I \) then \( A - B \) is s-unitary

Proof:

\( A \) and \( B \) are s-unitary matrices
\[\therefore A^{-1} = A^s, B^{-1} = B^s\]
(i) We have to show \((A+B)(\overline{A+B})^s = I\)
\[(A+B)(\overline{A+B})^s = (A+B)(\overline{A^s+B^s})\]
\[= AA^s + (\overline{A^s+B^s}) + BB^s = I - I + I = I\]
Ill’y we can prove \((\overline{A+B})^s(A+B) = I\)
\[\therefore (A+B) \text{ is s-unitary}\]
ii. We have to show \((A-B) (A-B)^s = I\)

\[
(A-B) (A-B)^s = (A-B) (A-B)^s.
\]

\[
= A\bar{A} - AB\bar{A} - B\bar{A} + BB^s
\]

\[
= A\bar{A}^s - (AB\bar{A} + B\bar{A}^s) + B\bar{B}^s
\]

\[
= I - (I) + I = I
\]

Ill’y \((A-B) (A-B)^s = I\)

\[
\therefore (A-B) (A-B)^s = (A-B)^s (A-B) = I
\]

\[
\therefore (A-B) \text{ is s-unitary.}
\]

**Theorem 2.10**

If A is s-unitary and \(VA=AV\) then VA is unitary.

**Proof:**

A is s-unitary

\[
\therefore VA^* V = A^{-1}
\]

\[
V(A^* V^*) = A^{-1}
\]

\[
V(VA)^s = A^{-1}
\]

\[
AV(VA)^s = AA^{-1}
\]

\[
(VA) (VA)^s = I (1) \quad (\because AV=VA)
\]

\[
VA^* V = A^{-1}
\]

\[
(V^* A^s) V = A^{-1}
\]

\[
(\bar{V} A)^s VA = A^{-1}A
\]

\[
(\bar{V} A)^s VA = I (2) \quad (\because AV=VA)
\]

From (1) & (2) \((VA) (VA)^s = (VA)^s (VA) = I\)

\[
\therefore VA \text{ is unitary.}
\]

**Theorem 2.11**

If A is s-unitary and \(VA=AV\) then AV is unitary.

**Proof:**

A is s-unitary \(\Rightarrow VA^* V = A^{-1}\)

\[
(V^* A^s) V = A^{-1}
\]

\[
(AV)^s V = A^{-1}
\]

\[
(AV)^* VA = A^{-1}A
\]

\[
(AV)^s (AV) = I (1) \quad (\because VA=AV)
\]

\[
VA^* V = A^{-1}
\]

\[
V(A^* V^*) = A^{-1}
\]

\[
V(VA)^s = A^{-1}
\]

\[
V (AV)^s = A^{-1}
\]

\[
(\bar{AV}) (AV)^s = AA^{-1} = I (2)
\]
From (1) and (2) $\text{AV} (\text{AV})^\ast = (\text{AV})^\ast (\text{AV}) = I$
\[ \therefore \text{AV is s-unitary.} \]

**Theorem 2.12**
Let $A \in \mathbb{C}^{n \times n}$ and $A^+$ be the Moore – Penrose of A. Then A is s-unitary iff $A^+$ is s-unitary.

**Proof:**
If A is s-unitary then 
\[ s \begin{bmatrix} A & A \end{bmatrix}^t = \begin{bmatrix} A & A \end{bmatrix} \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = I \]
\[ \begin{bmatrix} s \begin{bmatrix} A & A \end{bmatrix} \end{bmatrix}^t = I^+ \]
\[ \begin{bmatrix} s \begin{bmatrix} A^t & A^t \end{bmatrix} \end{bmatrix}^t A^t = I \]
\[ \begin{bmatrix} s \begin{bmatrix} A^t & A^t \end{bmatrix} \end{bmatrix}^t A^t = I \]

Similarly we may prove $A^+ \begin{bmatrix} s \begin{bmatrix} A^t & A^t \end{bmatrix} \end{bmatrix}^t = I$
\[ \therefore A^+ \begin{bmatrix} s \begin{bmatrix} A^t & A^t \end{bmatrix} \end{bmatrix}^t = \begin{bmatrix} s \begin{bmatrix} A^t & A^t \end{bmatrix} \end{bmatrix} A = I \]

$A^+$ is s-unitary.

Conversely 
Assume that $A^+$ is s-unitary.
\[ \therefore \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \begin{bmatrix} A^t & A^t \end{bmatrix}^t A^t = I \]
\[ \therefore A^\ast \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} = I \]

**Definition 2.13**
A matrix $A \in \mathbb{C}^{n \times n}$ is said to be skew secondary unitary matrix if $A^{-1} = -A^\ast$

**Example 2.14**
\[ A = \begin{bmatrix} 1 & i \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \]
is a skew-s-unitary matrix.

**Theorem 2.15**
If $A$ is skew s-unitary matrix then $iA$ is skew secondary unitary matrix.

**Proof:**
$A$ is skew s-unitary $\Rightarrow A^{-1} = -A^\ast$
iA⁻¹ = -i\bar{A}'
-(iA)⁻¹ = i\bar{A}'
(iA)⁻¹ = -(i\bar{A}')
∴ iA is skew s-unitary.

References
