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On Secondary Unitary Matrices

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Abstract

The concept of secondary unitary (s-unitary) matrices is introduced. Characterization of secondary unitary matrices and equivalent conditions are obtained.

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1. Introduction
Anna Lee[1] has initiated the study of secondary symmetric matrices. Also she has shown that for a complex matrix A, the usual transpose A^T and secondary transpose A^s are related as $A^s = VA^TV$ where 'V' is the permutation matrix with units in its secondary diagonal.

Also $\overline{A}^s$ denotes the conjugate secondary transpose of A i.e $\overline{A}^s = (c_{ij})$ where $c_{ij} = \overline{a_{n-j+1,\ n-i+1}}$ [2]. In this paper we introduce the concept of secondary unitary matrices (s-unitary).

1.1 Preliminaries and Notations
Let C_{nxn} be the space of nxn complex matrices of order n. For $A \in C_{nxn}$. Let A^T, $\overline{A}$, A^*, A^s, $\overline{A}^s$ denote transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose of a matrix A respectively.

For a complex matrix A, the Moore – penrose A^+ of A is the unique matrix X satisfying the following four penrose equations. [3]

(i) $AXA = A$
(ii) $XAX = X$
(iii) $(AX)^* = AX$
(iv) $(XA)^* = XA$

Let ‘k’ be the fixed product of disjoint transpositions in S_n, the set of all permutations on {1,2……n} and ‘K’ be the associated permutation matrix which satisfies the following properties.

$$\overline{K} = K^T = K^* = K,\ K^2=I,\ K^TK = KK^T = I.$$

Let ‘V’ be the associated permutation matrix whose elements on the secondary diagonal are 1, other elements are zero. Also ‘V’ satisfies the following properties.

$$V^T = \overline{V} = V^* = V \text{ and } V^2 = I$$

A matrix $A\in C_{nxn}$ is called unitary if $AA^* = A^*A = I$. [4]

2. s-Unitary Matrix

Definition 2.1

A matrix $A\in C_{nxn}$ is said to be s-Unitary if $A\overline{A}^s = \overline{A}^s A = I$ [5]
i.e. $AVA^*V = VA^*VA = I$
i.e. $VA^*V = A^{-1}$.

Example: 2.2

(i) $$A = \begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} -i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -i \end{bmatrix}$$

are s-unitary matrices.

Theorem: 2.3

Let $A\in C_{nxn}$. If A is s-unitary matrix then $\overline{A}$ is also s-unitary matrix.

Proof:

A is s-unitary $\Rightarrow VA^*V=A^{-1}$

$V(\overline{A})^*V = V\overline{(\overline{A})}^T = V\overline{(\overline{A}^T)}\ V$

$= \; V\overline{A^{*}}\,V$

$= \; \overline{VA^{*}V}$

$= \; \overline{A^{-1}}$

$= \; \overline{A}^{-1}$

$\therefore\ \overline{A}$ is s-unitary.

Theorem: 2.4

Let $A \in C_{nxn}$. If A is s-unitary then A^{T} is s-unitary.

Proof:

A is s-unitary $\Rightarrow A^{-1} = VA^{*}V$

$(A^{-1})^{T} \;=\; (VA^{*}V)^{T}$

$= \; V^{T}(A^{*})^{T}\,V^{T} \;=\; V(A^{*})^{T}\,V$

$(A^{-1})^{T} \;=\; V\,(A^{T})^{*}V$

$(A^{T})^{-1} \;=\; V(A^{T})^{*}V$

$\therefore\ A^{T}$ is s-Unitary.

Theorem: 2.5

Let $A \in C_{nxn}$. If A is s-unitary then A^{*} is s-unitary.

Proof:

A is s-unitary $\Rightarrow A^{-1} = VA^{*}V$

$(A^{-1})^{*} \;=\; (VA^{*}V)^{*}$

$= \; V^{*}(A^{*})^{*}\,V^{*}$

$(A^{*})^{-1} \;=\; V(A^{*})^{*}\,V$

$\therefore\ A^{*}$ is s-unitary.

Theorem 2.6

Let $A \in C_{nxn}$. If A is s-unitary then A^{-1} is s-unitary.

Proof:

A is s-unitary $\Rightarrow A^{-1} = \overline{A}^{s} = VA^{*}V$

$A^{-1} \;=\; \overline{A}^{s}$

$(A^{-1})^{-1} = \; (\overline{A}^{s})^{-1}$

$= \; \left(\overline{A}^{-1}\right)^{s}$

$(A^{-1})^{-1} \;=\; \overline{A^{-1}}^{s}$

$\therefore\ A^{-1}$ is s-unitary.

Theorem: 2.7

Let $A \in C_{nxn}$. If A is s-unitary then iA is s-unitary.

Proof:

A is s-unitary $\Rightarrow A^{-1} = VA^{*}V$

$$iA^{-1} = i(VA^{*}V)$$

$$-(iA)^{-1} = V(iA^{*})\,V = V(-\bar{i}\,\overline{A}^{T})\,V$$

$$(iA)^{-1} = V\left(\overline{iA}^{T}\right)$$

$$(iA)^{-1} = V(iA)^{*}\,V$$

$\therefore$ iA is s-unitary.

Theorem: 2.8

Let $A,B\in C_{nxn}$. If A and B are s-unitary matrices then AB is s-unitary matrix.

Proof:

A is s-unitary $\Rightarrow VA^{*}V = A^{-1}$

B is s-unitary $\Rightarrow VB^{*}V = B^{-1}$

$$\begin{aligned} V(AB)^{*}V &= V(B^{*}A^{*})\,V \\ &= (VB^{*}V)\,(VA^{*}V) \\ &= B^{-1}\,A^{-1} \\ &= (AB)^{-1} \\ \therefore V(AB)^{*}V &= (AB)^{-1} \end{aligned}$$

$\therefore$ AB is s-unitary matrix.

Theorem: 2.9

Let $A, B\in C_{nxn}$ and A, B are s-unitary matrices and $A\overline{B}^{s} = \overline{B}^{s}A$, $B\overline{A}^{s} = \overline{A}^{s}B$

If $A\overline{B}^{s} + B\overline{A}^{s} = -I$ then A+B is s-unitary

If $A\overline{B}^{s} + B\overline{A}^{s} = I$ then A-B is s-unitary

Proof:

A and B are s-unitary matrices

$\therefore A^{-1} = \overline{A}^{s}$, $B^{-1} = \overline{B}^{s}$

(i) We have to show $(A+B)\,\overline{(A+B)}^{s} = I$

$(A+B)\,\overline{(A+B)}^{s} = (A+B)\left(\overline{A}^{s} + \overline{B}^{s}\right)$

$= A\overline{A}^{s} + (A\overline{B}^{s} + B\overline{A}^{s}) + B\overline{B}^{s}$

$= I - I + I = I$

Ill'y we can prove $\overline{(A+B)}^{s}\,(A+B) = I$

$\therefore (A+B)\,\overline{(A+B)}^{s} = \overline{(A+B)}^{s}\,(A+B) = I$

$\therefore$(A+B) is s-unitary

ii. We have to show (A-B) $\overline{(A-B)}^{s}$ = I

(A-B) $\overline{(A-B)}^{s}$ = (A-B) ($\overline{A}^{s} - \overline{B}^{s}$).

= $A\overline{A}^{s} - A\overline{B}^{s} - B\overline{A}^{s} + B\overline{B}^{s}$

= $A\overline{A}^{s} - (A\overline{B}^{s} + B\overline{A}^{s}) + B\overline{B}^{s}$

= I – (I) + I = I

Ill'y $\overline{(A-B)}^{s}$ (A-B) = I

$\therefore$(A-B) $\overline{(A-B)}^{s}$ = $\overline{(A-B)}^{s}$ (A-B) = I

$\therefore$(A-B) is s-unitary.

Theorem 2.10

If A is s-unitary and VA=AV then VA is unitary.

Proof:

A is s-unitary

$\therefore$ VA* V=A^{-1}

V($A^{*}V^{*}$) = A^{-1}

V$(VA)^{*}$ = A^{-1}

AV$(VA)^{*}$ = AA^{-1}

(VA) $(VA)^{*}$ = I (1) ($\because$AV=VA)

VA^{*} V = A^{-1}

($V^{*}A^{*}$) V = A^{-1}

$(AV)^{*}$ VA = $A^{-1}A$

$(VA)^{*}$ VA = I (2) ($\because$AV=VA)

From (1) & (2) (VA) $(VA)^{*}$ = $(VA)^{*}$ (VA) = I

$\therefore$ VA is unitary.

Theorem 2.11

If A is s-unitary and VA=AV then AV is unitary.

Proof:

A is s-unitary $\Rightarrow$ VA*V = A^{-1}

($V^{*}A^{*}$) V = A^{-1}

$(AV)^{*}$ V = A^{-1}

$(AV)^{*}$ VA = $A^{-1}A$

$(AV)^{*}$(AV) = I(1) ($\because$VA=AV)

$VA^{*}V$ = A^{-1}

V($A^{*}V^{*}$) = A^{-1}

V$(VA)^{*}$ = A^{-1}

V $(AV)^{*}$ = A^{-1}

(AV) $(AV)^{*}$ = AA^{-1} = I (2)

From (1) and (2) (AV) (AV)* = (AV)* (AV) = I

$\therefore$ AV is s-unitary.

Thorem 2.12

Let $A \in C_{nxn}$ and A^+ be the Moore – Penrose of A. Then A is s-unitary iff A^+ is s-unitary.

Proof:

If A is s-unitary then $A\overline{A}^s = \overline{A}^s A = I$

$$\left(A\overline{A}^s\right)^+ = I^+$$

$$\left(\overline{A}^s\right)^+ A^+ = I$$

$$\left(\overline{A^+}^s\right) A^+ = I$$

Similarly we may prove $A^+ \left(\overline{A^+}^s\right) = I$

$$\therefore\ A^+ \left(\overline{A^+}^s\right) = \left(\overline{A^+}^s\right) A = I$$

A^+ is s-unitary.

Conversely

Assume that A+ is s-unitary.

$\therefore$ $\left(A^+\right)^+$ is s-unitary.

$\therefore$ A is s-unitary. (by using result in page 49 of [3])

Definition 2.13

A matrix $A \in C_{nxn}$ is said to be skew secondary unitary matrix if $A^{-1} = -\overline{A}^s$

Example 2.14

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$ is a skew-s-unitary matrix.

Theorem 2.15

If A is skew s-unitary matrix then iA is skew secondary unitary matrix.

Proof:

A is skew s-unitary $\Rightarrow A^{-1} = -\overline{A}^s$

$$iA^{-1} = -i\overline{A}^{s}$$

$$-(iA)^{-1} = i\overline{A}^{s}$$

$$(iA)^{-1} = -(\overline{iA}^{s})$$

$\therefore$ iA is skew s-unitary.

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