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Application of Harmonic Convexity to Multi-objective Non-linear Programming Problems

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Abstract

Here, in this paper we study H -convex (H –concave) function with respect to duality in multi-objective non-linear programming problems. We formulate the multi-objective programming problem with its dual and give weak, strong and converse duality theorems for this problem. Also we define multi-objective fractional programming problem and its dual and discuss convexity, pseudo-convexity and H –convexity in this context. Also we discuss symmetric duality and self duality for multi-objective programming problem.

Keywords: Harmonic Convexity, Multi-objective, Non-linear Programming Problem, Duality.

Introduction

Generalized convexity (concavity) plays an important role in the development of multi-objective non-linear programming problems. Generalized convex functions which are the many non-convex functions that share at least one of the valuable properties of convex functions and which are often more suitable for describing real world problems.[1] During the last decades, several necessary and sufficient conditions for generalized convexity (concavity) on subsets of R^n with non-linear interior have been proposed by several mathematicians. The term harmonic convexity was first introduced by Das. [2] The concept has been used by many mathematicians in many means. Some applications of harmonic convexity to optimization problems have already been discussed by Das, Roy and Jena. [3] The form of a dual problem of Mond-Weir type for multi-objective programming problems of generalized functions is defined and theorems of weak duality, direct duality and inverse duality are proven. [4] A study of various constraint qualification conditions for the existence of lagrange multipliers for convex minimization based on a new formula for the normal cone to the constraint set, on local metric regularity and a metric regularity property on bounded subsets has been done[5]. A consideration of an operation on subsets of a topological vector space which is closely related to what has been called the inverse addition by R.T. Rockafellar has also been done. [6] A new class of higher order functions for a multi-objective programming problem has been introduced which subsumes several known studied classes and formulated higher order Mond-Weir and Schaible type dual programs for a non-differentiable multi-objective fractional programming problem where the objective function and the constraints contain support functions of compact convex sets in R^n and studied weak and strong duality results[7]. Also a new class of generalized type I vector valued functions have been introduced and duality theorems are proved for multi-objective programming problem with inequality constraints.[8]

Here, in this paper we study H -convex (H –concave) function with respect to duality in multi-objective non-linear programming problems. The present study done in this paper will show that most of the results derived earlier are found to be a particular case of this study. Indeed H –convexity is the generalized version of convexity and different kind of convexities can be derived from H –convexity.

The paper formulation is as follows: The next section introduces some preliminaries used in this paper. After that in section 3. we formulate the multi-objective programming problem with its dual and give weak, strong and converse duality theorems for this problem. In section 4. we define multi-objective fractional programming problem and its dual and discuss convexity, pseudoconvexity and H –convexity in this context. In section 5 and 6, we discuss symmetric and self duality respectively. In section 7, conclusions are drawn.

Preliminaries

In this section, we define some terms, explain some properties, and give the notations used in this paper.

Definition 1. A positive function $f(x)$ defined on a convex set $S \subseteq R^n$ is said to be harmonic convex or H –convex (harmonic concave or H –concave) if its reciprocal is concave (convex) and conversely.

We use following notations in the paper. The vector norm $\|\cdot\|$ always denote the Euclidean norm denoted by $\|x\| = (x^T x)^{1/2}$ for a function $f: S \subseteq R^n \to R$ where S is an open set $\nabla f(x)$ denotes the gradient vector with partial derivative $\frac{\partial f}{\partial x_j}$, $j = 1, 2, \dots n$. The Hessian matrix $\nabla^2 f(x)$ is the $n \times n$ matrix of second order partial derivatives, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ $i, j = 1, 2, \dots n$.

Here in this paper, we use several geometric properties of convex functions to define convexity by inequalities. If we relate the arithmetic, geometric and harmonic inequalities, we can find an alternate and equivalent definition of H -convexity and H -concavity.

Definition 2. A positive function defined on a convex set $S\subseteq R^n$ is said to be harmonic convex (H -convex) on S if , for $x^1, x^2 \in S$ and $0 \le \lambda \le 1$,

$$\left(\frac{\lambda}{f(x^1)} + \frac{1-\lambda}{f(x^2)}\right]^{-1})$$

And harmonic concave (H -concave) if ,

$$\left(\frac{\lambda}{f(x^1)} + \frac{1-\lambda}{f(x^2)}\right]^{-1})$$

And “strictly H -convex” if strict inequality holds.

If we relate the inequality in definition 2. To geometric, arithmetic and harmonic inequalities, we have

$$f(\lambda x^1 + (1-\lambda)x^2) \le [(\lambda/f(x^1)) + (1-\lambda)/f(x^2)]^{(-1)} \le (f(x^1))^{\lambda}(f(x^2))^{(1-\lambda)} \le \lambda f(x^1) + (1-\lambda)f(x^2)$$

Thus if, f is H -convex, then it is logarithmic convex and also it is convex. But the converse is not true. Similarly by the inequality for harmonic concavity in definition 2, we have

$$\lambda f(x^1) + (1-\lambda)f(x^2) \ge (f(x^1))^{\lambda}(f(x^2))^{(1-\lambda)} \ge [(\lambda/f(x^1)) + (1-\lambda)/f(x^2)]^{(-1)} \le f(\lambda x^1 + (1-\lambda)x^2)$$

It is clear that if f is concave, then it is logarithmic concave and also H -concave but the converse is not true.

Property 1. Let $f(x)$ be positive differentiable function on some open set A containing the convex set $S\subseteq R^n$.

A necessary and sufficient condition for $f(x)$ to be H -convex on S is that, for each

$$\forall x^1, x^2 \in S, (x^2 - x^1)^T \nabla f(x^1) \le f(x^1)/f(x^2)\ (f(x^2) - f(x^1)) \tag{1}$$

And H -concave on S, if for each

$$x^1, x^2 \in S, (x^2 - x^1)^T \nabla f(x^1) \ge f(x^1)/f(x^2)\ (f(x^2) - f(x^1)) \tag{2}$$

It is to be noted that,

If we define $p(x^1, x^2) = \dfrac{f(x^1)}{f(x^2)}$, where $p: S \times S \to R$, then the inequalities given above in property 1. Reduce respectively to strong pseudoconvexity and strong pseudo concavity of Weir [9] and several authors.

Moreover, the setting $p(x^1, x^2) = f(x^1)/f(x^2) = 1, \forall x^1, x^2 \in S$, and p is defined as in 1. Then both the inequalities in property 1. reduce to the usual definition of convexity and concavity respectively.

Setting $h(x^1, x^2) = p(x^1, x^2)(x^2 - x^1)$, where $p(x^1, x^2) = \dfrac{f(x^1)}{f(x^2)}$ and p is defined as in 1. ,then f is invex function defined in [10].

If $(x^2 - x^1)^T \nabla f(x^1) \geqq 0$, then the inequality showing the necessary and sufficient condition in property 1. for H -convexity implies that $f(x^2) \geq f(x^1)$. Thus f is pseudoconvex function. Similarly for $(x^2 - x^1)^T \nabla f(x^1) \leqq 0$, the inequality showing the necessary and sufficient condition in property 1. for H -concavity implies that $f(x^2) \leqq f(x^1)$ and f is pseudoconcave function defined in [11].

From the above points it is clear that H -convexity includes convexity, pseudoconvexity, strongly pseudoconvexity and invexity. Also we claim that the class of H -convex (H -concave) functions is the weakest version of convexity (concavity). Moreover, a comprehensive development of the properties of H -convex (H -concave) functions is given in [12].

Problem Formulation and Duality

Let us consider the following multi-objective programming problem:
The primal problem (P) is defined as:

Minimize $f_1(x)$
Minimize $f_2(x)$
.
.
.
Minimize $f_n(x)$

Subject to $g(x) \leqq 0, x \geqq 0$

$g: S \subseteq R^n \to R^m$, $f_1: S \subseteq R^n \to R$, $f_2: S \subseteq R^n \to R$, $\dots f_n: S \subseteq R^n \to R$

All $f_1, f_2, \dots f_n$ and g are continuously differentiable functions and S is an open convex set.

The dual (D) of the above problem is defined as follows:

Maximize $L_1(x, u_1) = f_1(x) + u_1^T g(x)$
Maximize $L_2(x, u_2) = f_2(x) + u_2^T g(x)$
.
.
.
Maximize $L_n(x, u_n) = f_n(x) + u_n^T g(x)$

Subject to $\nabla f_1(x) + u_1^T g(x) = 0$
$\nabla f_2(x) + u_2^T g(x) = 0$
.
.
.

$$\nabla f_n(x) + u_n^T g(x) = 0$$

Here $u_1, u_2, \ldots u_n \geq 0$

Now we prove the following duality theorems for the above defined problem.

Theorem 1. (Weak Duality) Let $f_1, f_2, \ldots f_n$ be H -convex functions on S and each component of g is H -convex on S. If $\hat{x}$ be the feasible solution for the primal problem, and ($\bar{x}, \bar{u}_1$, $\bar{u}_2$,… $\bar{u}_n$) be the feasible solution for the dual problem , then $f_j(\hat{x}) \geq L_j(\bar{x}, \bar{u}_j)\ \forall\ j = 1,2, \ldots n$.

Proof: From H -convexity of $f_1, f_2, \ldots f_n$, we have on substituting $x_1 = \bar{x}$ and $x_2 = \hat{x}$ by inequality (1),

$$(\hat{x} - \bar{x})^T \nabla f_j(\bar{x}) \leq \frac{f_j(\bar{x})}{f_j(\hat{x})}(f_j(\hat{x}) - f_j(\bar{x}))$$

$$\Rightarrow \frac{f_j(\hat{x})}{f_j(\bar{x})}(\hat{x} - \bar{x})^T \nabla f_j(\bar{x}) \leq (f_j(\hat{x}) - f_j(\bar{x}))$$

$$\Rightarrow (f_j(\hat{x}) - f_j(\bar{x})) \geq \frac{f_j(\hat{x})}{f_j(\bar{x})}(\hat{x} - \bar{x})^T \nabla f_j(\bar{x}) \geq -(\hat{x} - \bar{x}) \sum_{i=1}^{m} \bar{u}_{ij} \Delta g_i(\bar{x}),\ \forall j = 1,2, \ldots n \tag{3}$$

This follows from the dual feasibility and positivity of $f_j(\hat{x}) / f_j(\bar{x})$. Also from H -convexity of $g(x)$,

$$g_i(\hat{x}) - g_i(\bar{x}) \geq (\hat{x} - \bar{x})^T \nabla g_i(\bar{x})$$

Since $\bar{u}_{ij} \geq 0$ and $\hat{x}$ is feasible for the primal problem (P), we obtain

$$\sum_{i=1}^{m} \bar{u}_{ij} (g_i(\hat{x}) - g_i(\bar{x})) \geq (\hat{x} - \bar{x})^T \sum_{i=1}^{m} \nabla \bar{u}_{ij}\ g_i(\bar{x})$$

Or $$-(\hat{x} - \bar{x})^T \sum_{i=1}^{m} \nabla \bar{u}_{ij}$$

$$g_i(\hat{x}) \geq -\sum_{i=1}^{m} \bar{u}_{ij} (g_i(\hat{x}) - g_i(\bar{x})) \geq \sum_{i=1}^{m} \bar{u}_{ij}\, g_i(\bar{x}) \tag{4}$$

From (3) and (4) , we obtain

$$f_j(\hat{x}) \geq f_j(\bar{x}) + \sum_{i=1}^{m} \bar{u}_{ij}\, g_i(\bar{x}) = L_j(\bar{x}, \bar{u}_j)$$

Theorem 2. (Strong Duality) If the primal constraint set satisfies Slater’s constraint qualification and if $\bar{x}$ is an optimal solution to the primal problem (P), then $\exists\ \overline{u_j}$ such that $(\bar{x}, \overline{u_j})$ is optimal solution to the dual problem (D) and corresponding extrema are equal.

Proof: The H -convexity (with respect to x) of f_j and g_i, $j = 1,2,...n$ and $i = 1,2,...m$ implies H -convexity of L_j (with respect to x for given u_j) $\forall u_j \geq 0$. Furthermore, the dual problem (D) has a feasible solution due to the constraint qualification and the existence of an optimal solution to the primal problem (P). Assume that $(\bar{x}, \bar{u_j})$ and $(\hat{x}, \bar{u_j})$ are arbitrary dual feasible solutions. The H -convexity of L_j implies that

$$L_j(\bar{x}, \bar{u_j}) - L_j(\hat{x}, \bar{u_j}) \geq (\bar{x} - \hat{x})^T \nabla_x L_j(\hat{x}, \bar{u_j}) = 0$$
$$L_j(\hat{x}, \bar{u_j}) - L_j(\bar{x}, \bar{u_j}) \geq (\hat{x} - \bar{x})^T \nabla_x L_j(\bar{x}, \bar{u_j}) = 0$$

Therefore $L_j(\bar{x}, \bar{u_j}) = L_j(\hat{x}, \bar{u_j})$, so that for any $u_j \geq 0, \forall \ \ j = 1,2,...n$, the solution $L_j(x, \bar{u}_j)$ does not depend on x as $(x, \bar{u}_j)$ is dual feasible. Hence $\exists \overline{u_j}$ such that $(\bar{x}, \overline{u_j})$ is dual feasible and furthermore

$$L_j(\bar{x}, \overline{u_j}) = \max_{u_j \geq 0} L_j(\bar{x}, u_j)$$
$$\geq \max\{L_j(\bar{x}, u_j): (\bar{x}, u_j) \; is \; dual \; feasible \forall \; j = 1,2,...n\}$$
$$= \max\{L_j(\bar{x}, u_j): (x, u_j) \; is \; dual \; feasible \forall \; j = 1,2,...n\}$$
$$\geq L_j(\bar{x}, \overline{u_j})$$

Consequently $(\bar{x}, \overline{u_j})$ solves the dual problem (D). Finally as the constraint qualification is satisfied , this implies that

$$L_j(\bar{x}, \overline{u_j}) = f_j(\bar{x}) + \sum_{i=1}^{m} \overline{u_j} \nabla g_i(\bar{x}) = f_j(\bar{x})$$

From the weak duality theorem, we get

$$f_j(\bar{x}) - L_j(\bar{x}, \overline{u_j}) \geq 0$$

Whereas the above strong duality theorem assures that the differences $f_j(\bar{x}) - L_j(\bar{x}, \overline{u_j})$, called the duality gap is actually zero.

Theorem 3. (Converse Duality) Let $(\bar{x}, \overline{u_j})$ solve the dual problem (D) and the $n \times n$ matrix $H(\bar{x}, \overline{u_j}) = \nabla^2 f_j(\bar{x}) + \nabla^2 \overline{u_j} \nabla g(\bar{x})$ be non-singular, then $\bar{x}$ solves the primal problem (D) and the corresponding objective function values are equal.

Proof: The proof of the theorem is obvious.

Fractional Programming and Duality

A unified approach for duality in fractional programming was presented by Schaible in 1976. [13].

Now we consider the following multi-objective fractional programming problem:

(MFP) Minimize $\varphi_1(x) = (f_1(x))/g(x)$

Minimize $\varphi_2(x) = (f_2(x))/g(x)$

.

.

.

Minimize $\phi_n(x) = (f_n(x))/g(x)$

Subject to $h(x) \leq 0, x \geq 0$

$g: S \subseteq R^n \to R^m$, $f_1: S \subseteq R^n \to R$, $f_2: S \subseteq R^n \to R$, $\ldots f_n: S \subseteq R^n \to R$, $h: S \subseteq R^n \to R^m$

Here we assume that $f_j, g, h, \forall j = 1,2,\ldots n$ are differentiable functions on S.

Some authors consider the above multi-objective fractional programming problem assuming that $f_j, -g, h, \forall j = 1,2,\ldots n$ are convex functions. These assumptions imply that f_j/g, $\forall j = 1,2,\ldots n$ are pseudoconvex. However, weaker conditions than convexity, pseudoconvexity and strongly pseudoconvexity can be replaced by H -convexity.

Theorem 4. Let f_j and $-g$ $\forall j = 1,2,\ldots n$ be H -convex with respect to proportional function p defined as $p_j(x^1, x^2) = \frac{f_j(x^1)}{f_j(x^2)}$, where $p_j: S \times S \to R$. Then $f_j/-g$, $\forall j = 1,2,\ldots n$ are H -convex on S.

Proof: Let $\phi_1(x) = (f_1(x))/g(x)$, $\phi_2(x) = (f_2(x))/g(x)$, $\ldots \phi_n(x) = (f_n(x))/g(x)$ be n functions.

Then $(y-x)^T \nabla\phi_j(x) = (y-x)^T [(g(x)\nabla f_j(x) - f_j(x)\nabla g(x))/(g(x))^2]$, $\forall j = 1,2,\ldots n$

Now from H -convexity of f_j and $-g$ $\forall j = 1,2,\ldots n$, with respect to p_j, we have

$$f_j(y) - f_j(x) \geq p_j(x,y)(y-x)^T \nabla f_j(x) \tag{5}$$

and
$$-(g(y) - g(x)) \geq -p_j(x,y)(y-x)^T \nabla g(x) \tag{6}$$

Thus here we get n inequalities denoted by (5) and n inequalities denoted by (6) $\forall j = 1,2,\ldots n$.

Now on multiplying each inequality of (5) by $g(x)$ and on multiplying j^{th} inequality of (6) by $f_j(x)$ and on adding and dividing both sides by $(g(x))^2$, we have

$$\left[\frac{g(x)f_j(y) - f_j(x)g(y)}{(g(x))^2}\right] \geq p_j(x,y)(y-x)^T \left[\frac{g(x)f_j(x) - f_j(x)g(x)}{(g(x))^2}\right]$$

$$= p_j(x,y)(y-x)^T \nabla\phi_j(x)$$

Or $g(y)/g(x)\ [\phi_j(y) - \phi_j(x)] \geq p_j(x,y)(y-x)^T \nabla\phi_j(x)$

This implies that $[\phi_j(y) - \phi_j(x)] \geq (p_j(x,y))/Q(x,y)\ (y-x)^T \nabla\phi_j(x)$

Where $Q(x,y) = \frac{g(y)}{g(x)}$

If $\frac{p_j(x,y)}{Q(x,y)} = M_j(x,y)$, then $[\varphi_j(y) - \varphi_j(x)] \geq M_j(x,y)(y-x)^T \nabla \varphi_j(x)$

Thus $\varphi_j(x), \forall j = 1,2,\dots n$ are H-convex with respect to proportionality function $M_j(x,y)$. It follows immediately that f_j/g, $\forall j = 1,2,\dots n$ are strongly pseudoconvex if f_j and g are H-convex.

The pseudoconvexity and convexity follows that:

The Duality results of the previous section may then be involved by the H-convexity on S, with respect to the proportionality functions.

Thus we state the following dual program:

(MFD) Maximize $\varphi_1(x) + \sum_{i=1}^{m} u_{i1} h_i(x)$

Maximize $\varphi_2(x) + \sum_{i=1}^{m} u_{i2} h_i(x)$

.

.

.

Maximize $\varphi_n(x) + \sum_{i=1}^{m} u_{in} h_i(x)$

Subject to $\nabla \varphi_j(x) + \sum_{i=1}^{m} u_{ij} \nabla h_i(x) = 0, \quad \forall j = 1,2,\dots n$

$u_{ij} \geq 0, \quad \forall i = 1,2,\dots m, \forall j = 1,2,\dots n$.

Symmetric Duality

We know that a pair of primal and dual programs are symmetric in the sense that the dual of the dual is the original primal problem, i.e when the dual problem is reset in the primal format; it's dual is the primal problem itself. Here, we give a different pair of symmetric dual multi-objective non-linear programming problems, assuming the H-convexity and establish the duality results.

(MPS) Minimize $f_1(x,y)$

Minimize $f_2(x,y)$

.

.

.

Minimize $f_n(x,y)$

Subject to $\nabla f_j(x,y) \leq 0$,$\forall j = 1,2,\dots n$ (7)

$y^T \nabla_y f_j(x,y) \geq 0$ (8)

$x \geq 0$ (9)

(MDS) Maximize $f_1(u_1, v)$

Maximize $f_2(u_2, v)$

.
.
.

Maximize $f_n(u_n, v)$

Subject to $\nabla_x f_j(u_j, v) \geq 0, \qquad \forall j = 1,2,\dots n$. (10)

$u_j^T \nabla \leq 0$ (11)

$v \geq 0$ (12)

We assume that $f_j, \ \forall j = 1,2,\dots n$ are twice differentiable real valued functions of x and y, where $x \in R^n$, $y \in R^m, \nabla_x f_j$ and $\nabla_y f_j$ denote the gradient vectors with respect to x and y respectively. $\nabla_{yy} f_j$ and $\nabla_{yx} f_j$ denote respectively the $m \times m$ matrices of second partial derivatives.

Theorem 5. Let $f_j(\cdot, y), \ \forall j = 1,2,\dots n$ be H-convex (for fixed y) and $f_j(x, \cdot), \forall j = 1,2,\dots n$ be H-concave (for fixed x). Let (x, y) be feasible for (MPS) and (u_j, v) be feasible for (MDS). Then $f_j(x, y) \geq f_j(u_j, v)$.

Proof: From (9), (10) and (11), we have

$$(x - u_j)^T \nabla_x f_j(u_j, v) \geq 0$$

Since $f_j(\cdot, y)$, $\forall j = 1,2,\dots n$ be H-convex, it follows that

$$f_j(x, v) - f_j((u_j, v) \geq \frac{f_j((u_j, v)}{f_j(x, v)}(x - u_j)^T \nabla_x f_j(u_j, v) \geq 0$$

Thus $f_j(x, v) \geq f_j((u_j, v), \qquad \forall j = 1,2,\dots n$ (13)

Likewise, from (7), (8) and (12), we have $(v - y)^T \nabla_y f_j(x, y) \leq 0$

Since $f_j(x, \cdot), \forall j = 1,2,\dots n$ be H-concave, it follows that

$$f_j(x, v) - f_j(x, y) \leq \frac{f_j(x, v)}{f_j(x, y)}(v - y)^T \nabla_y f_j(x, y) \leq 0$$

Thus $f_j(x, v) \leq f_j(x, y)$ (14)

Thus from (13) and (14), we have

$$f_j((u_j, v) \leq f_j(x, y)$$

Theorem 6. (Strong Duality) Let $f_j(\cdot, y), \forall j = 1,2,\dots n$ be H-convex (for fixed y) and $f_j(x, \cdot), \forall j = 1,2,\dots n$ be H-concave (for fixed x). Let (x_0, y_0) be a local or global optimal solution of the primal problem (MPS), $\nabla_{yy} f_j(x_0, y_0)$ be positive or negative definite and $\nabla_y f_j(x_0, y_0) \neq 0$, then (x_0, y_0) also gives the optimal solution of the dual problem.

Proof: Since (x_0, y_0) is an optimal solution to the primal problem, $\exists$ $\in R, r \in R^m, w \in R, s \in R^n$

Such that the Fritz John conditions are given as follows:

$$\tau\nabla_x f_j - (wy_0 - r)^T\nabla_{yx} - S = 0 \quad (15)$$
$$((- w)\nabla_y f_j - (wy_0 - r)^T\nabla_{yy} f_j = 0 \quad (16)$$
$$r^T\nabla_y f_j = 0 \quad (17)$$
$$wy_0{}^T\nabla_y f_j = 0 \quad (18)$$
$$s^T x_0 = 0 \quad (19)$$
$$((, r, w, s) \neq 0 \quad (20)$$
$$((, r, w, s) \geqq 0 \quad (21)$$

On multiplying (16) by $(wy_0 - r)^T$, and on applying (17) and (18), we get

$$(wy_0 - r)^T\nabla_{yy} f_j\big((wy_0 - r)\big) = 0$$

Since $\nabla_{yy} f_j$ is positive or negative definite, so $wy_0 = r$ (22)

Thus from (10), $(w - r)^T\nabla_y f_j = 0$, and since by assumption $\nabla_y f_j \neq 0,$ Then
$w = r$ (23)

If $(= 0, w = 0$ by (23), $r = 0,$ by (15) and (22) $\nabla_x f_j \geqq 0,$ and (15) and (19) give $\mathbf{x}_0^T\nabla_x f_j = 0.$

Thus (x_0, y_0) is feasible for (MDS) and the values of the objective function are equal and optimality follows from weak duality.

Self Duality

A mathematical programming problem is self-dual if dual of the dual is the primal problem itself. A function $f(x, y)$ is said to be skew-symmetric if $f(x, y) = -f(y, x) \ \forall\, x, y(\, dom. f$

Theorem 7. Let $f_j, \forall\, j = 1, 2, \ldots n$ be skew-symmetric functions. Then (MPS) is itself dual. If also (MPS) and (MDS) are dual programs, and (x_0, y_0) is a joint optimal solution , then so is (y_0, x_0) and $f_j(x_0, y_0) = f_j(y_0, x_0) = 0, \forall\, j = 1, 2, \ldots n$

Proof: (MDS) can be written as

Minimize $f_1(u_1, v)$

Minimize $f_2(u_2, v)$

.

.

.

Minimize $f_n(u_n, v)$

Subject to $\nabla_x f_j\big(u_j, v\big) \leq 0, \qquad \forall\, j = 1, 2, \ldots n$.

$u_j^T\nabla_x f_j\big(u_j, v\big) \geqq 0$

$v \geqq 0$

Since $f_j, \forall\, j = 1, 2, \ldots n$ are skew-symmetric (

$\nabla_x f_j\big(u_j, v\big) = -\nabla_y f_j\big(u_j, v\big)$

and thus the (MDS) becomes

Minimize $f_1(u_1, v)$

Minimize $f_2(u_2, v)$

.

.

.

Minimize $f_n(u_n, v)$

Subject to $\nabla_y f_j(u_j, v) \leq 0, \qquad \forall j = 1, 2, \ldots n$.

$$u_j^T \nabla_y f_j(u_j, v) \geq 0$$

$$u_j \geq 0$$

Which is just the (MPS) problem

Thus if (x_0, y_0) is the optimal solution for the dual problem $\Rightarrow (y_0, x_0)$ is the optimal solution for the primal problem and by symmetric duality also for (MDS) problem. Therefore, $f_j(x_0, y_0) = f_j(y_0, x_0) = -f_j(x_0, y_0) = 0$

Conclusion

This paper has introduced the application of H -convexity in multi-objective non-linear programming problems. In the last few years, quite large number of researchers have used H -convexity in different names for single objective non-linear programming problems and have obtained duality results. This paper have pointed out that H -convexity is the generalized version of convexity and different convexities can be derived from H -convexity.

References

[1] Cambini, A. and Martein, L., 2009, Generalized Convexity and Optimization: Theory and Applications, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag.

[2] Das, C., 1975, “Mathematical Programming in Complex space”, Ph.D Dissertation, Sambalpur University.

[3] Das, C., Roy, K.L., and Jena, K.N., 2003, “Harmonic Convexity and Application to Optimization Problems,” The Mathematics Education, Vol. 37, No. 2, pp. 58–64.

[4] Pang, L., Wang, W. and Xia, Z., 2006, “Duality Theorems on Multi-objective Programming of Generalized Functions,” Acta Mathematicae Applicatae Sinica (English Series), Vol. 22, No. 1, pp. 49-58.

[5] Tiba, D. and Zalinescu, C., 2004, “On the Necessity of Some Constraint Qualification Conditions in Convex Programming, Journal of Convex Analysis, Vol. 11, No. 1, pp. 95-110.

[6] Penot, J.P. and Zalinescu, C., 2000, “Harmonic Sum and Duality, Journal of Convex Analysis, Vol. 7, No. 1, pp. 95-113.

[7] Suneja, S.K, Srivastava, M.K. and Bhatia, M., 2008, "Higher Order Duality in Multi-Objective Fractional Programming with Support Functions", Journal of Mathematical Analysis and Applications, Vol. 347, Issue 1,pp. 8-17.

[8] Aghezzaf, B. and Hachimi, M., 2000, "Generalized Invexity and Duality in Multi-Objective Programming Problems", Journal of Global Optimization, Vol. 18, Issue 1, pp. 91-101.

[9] Weir, T., 1990, "On Strong Pseudo-convexity in Non-linear Programming and Duality," Opsearch, Vol. 27, No. 2, pp. 117-121.

[10] Nanda, S., 1988, " Invex Generalization of Some Duality Results, Opsearch, Vol. 25, No. 2, pp. 105-111.

[11] Mond, B. and Weir, T., 1981, "Generalized Concavity and Duality, in : Generalized Concavity in Optimization and Economics," (eds) S. Schaible and W.T. Ziemba, Academic Press, New York, pp. 263-279.

[12] Das, C., 1981, "Generalized Convexity and Optimization Problems", Mathematicki Vesnik, 5(18) (15), pp. 349-359.

[13] Schaible, S., 1976, "Duality in Fractional Programming, A Unified Approach", Operations Research, 24, pp. 452-461.