International Journal of Computational Science and Mathematics.
ISSN 0974-3189 Volume 2, Number 3 (2010), pp. 277--283

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Transient Solution of An M/M/1/N Queue Subject to Uniformly Distributed Catastrophic Intensity with Restoration

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Abstract

In this paper we study the effect of varying catastrophic intensity with restoration to destroy a finite number of units in a queuing system. The model considered is a simple finite capacity Markovian queue with capacity N. The catastrophic intensity is uniformly distributed. Times dependent as well as steady state solution are obtained. Further, some particular cases of the queuing model are also derived and discussed.

Keywords: Transient Solution, Catastrophes, Steady State Solution, Catastrophic Intensity, Probability Generating Function, Restoration Time.

Introduction

From the very beginning, the M/M/1 queue has been the object of systematic and through investigation. In recent years the attention has been focused on certain extension that includes the effect of catastrophes, in particulars, birth and death models. A large number of research papers have been published on population processes under the influence of catastrophes; for instance, Swift[12], Kyriakidis [11], Brockwell [1,2], Brockwell et al.[3], among other have discussed berth and death models with catastrophes. In this connection, a reference may be made to the paper by A. Di. Crescenzo et al [6]. In this paper , the author have recognized the role played by the notion of catastrophes in various area of science and technology.In computer system, if a job is infected, this job may transmit virus which may be transferred to the other processor. The infected job may be modeled by catastrophes. Hence computer network with virus may be modeled by queuing networks with catastrophes [7]. Queuing model continue to be one of the most important area of computer networksand have played a vital role in performance evaluation of computer systems.Queuing model with environmental and catastrophic effects has been studied by Jain and Kanethia [9]. Jain and Kanethia proved that the change in the environment affects the state of the queuing system. A system will require some sort of time to function in a normal way if it suffered from catastrophes, which is taken as restoration time, this concept is given by Jain and Kumar [8]. Jain and Kumar obtained the transient solution of the correlated queues with special effect of catastrophes and restoration.The concept of varying catastrophic intensity is given by Jain and Bura [10]. The number of customers in a queuing system is instantly reset to zero or not depends upon the intensity of catastrophe.Queuing model with varying catastrophic intensity are applied in various field of biological sciences and agriculture. In agriculture, if a crop is infected with some disease then for the treatment of such type of disease we use some chemicals. The destruction of the number of bacteria present in the crops depends upon the intensity of the chemicals used, that is, the application of the chemical may destroy all or a part there of. The use of the chemicals is like the occurrence of catastrophe. Therefore the infected crops with use of chemical are modeled by the birth and death queue with varying catastrophic intensity. If the catastrophe destroys all the customers in a queuing system then the system will require some time to function in a normal way, which is taken as restoration time. The queuing model with varying catastrophic intensity is practically very important therefore we consider here an M/M/1/N queue with varying catastrophic intensity with restoration.The catastrophic intensity may follow any distribution but most proper of them is considered to be Uniform distribution. Therefore we undertake an M/M/1/N queuing model with uniformly distributed catastrophic intensity with restoration.

Transient Solution

We consider an M/M/1/N queuing system with FIFO discipline subject to varying catastrophic intensity at the service station with restoration time. By λ and μ we denote the ‘arrival rate and service rate, respectively. When the system is not empty, the catastrophes occur according to a Poisson Process with rate ξ.It depends upon the intensity of the catastrophe that it destroys all the customers or not. If it destroys all the customers, then the system will require some sort of time to function in a normal way, which is taken as restoration time. The restoration time is independently, identically and exponentially distributed with parameter η.

Define

$P_0(t)$ = the probability that there are zero customer in the system at time t without the occurrence of catastrophes.

$Q_0(t)$ = the probability that there are zero customer in the system at time t with the occurrence of catastrophes with varying intensity.

$P_n(t)$ = the probability that there are n customers in the system at time t.

The differential difference equations governing the system are:

$$P_0^{/}(t) = -\lambda P_0(t) + \mu P_1(t) + \eta Q_0(t) \tag{1}$$

$$Q_0^{/}(t) = -\eta Q_0(t) + \frac{\xi}{N}\sum_{n=1}^{N}\sum_{j=n}^{N} P_n(t) \tag{2}$$

$$P_n^{/}(t) = -(\lambda+\mu+\xi)P_n(t) + \lambda P_{(n-1)}(t) + \mu P_{(n+1)}(t) + \frac{\xi}{N}\sum_{j=1}^{N-n} P_{(n+j)}(t) \tag{3}$$

, n=1, 2, 3… N-1

$$P_N^{/}(t) = -(\mu+\xi)P_N(t) + \lambda P_{(N-1)}(t) \tag{4}$$

Taking, Lap lace Transform of equation (1), (2),(3) and (4) w.r.t.'t', we have

$$sP_0^*(s) = 1 - \lambda P_0^*(s) + \mu P_1^*(s) + \eta Q_0^*(s) \tag{5}$$

$$sQ_0^*(s) = -\eta Q_0^*(s) + \frac{\xi}{N}\sum_{n=1}^{N}\sum_{j=n}^{N} P_n^*(s) \tag{6}$$

$$sP_n^*(s) = -(\lambda+\mu+\xi)P_n^*(s) + \lambda P_{(n-1)}^*(s) + \mu P_{(n+1)}^*(s) + \frac{\xi}{N}\sum_{j=1}^{N-n} P_{(n+j)}^*(s) \tag{7}$$

$$sP_N^*(s) = -(\mu+\xi)P_N^*(s) + \lambda P_{(N-1)}^*(s) \tag{8}$$

Where

$$P_n^*(s) = \int_0^\infty e^{-st} P_n(t)\,dt \qquad \text{and} \quad P_n(0) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Define Probability Generating Function

$$P^*(z,s) = \sum_{n=0}^{N} P_n^*(s) z^n \tag{9}$$

Multiplying equation (5), (6),(7) and (8) by the suitable power of z, summing over all n and using (9), we have on simplification:

$$P^*(z,s) = \frac{\left(z^{N-1}\right)\left\{1 + \left(\mu - \frac{\mu}{z} + \frac{\xi}{N}\sum_{j=1}^{N-1}\left(1 - \frac{1}{z^j}\right)\right)P_0^*(s) + (\lambda - \lambda z)P_N^*(s)z^N + \frac{\xi}{Ns} + \frac{\xi}{N}\sum_{n=1}^{N-2}\sum_{j=1}^{N-(n+1)} P_n^*(s)\left(1 - \frac{1}{z^j}\right)\right\}}{\left\{-\lambda z^N + (s+\lambda+\mu+\xi)z^{N-1} - \left(\mu + \frac{\xi}{N}\right)z^{N-2} - \frac{\xi}{N}z^{N-3} - \frac{\xi}{N}z^{N-4} - \ldots\ldots\ldots - \frac{\xi}{N}z - \frac{\xi}{N}\right\}} \tag{10}$$

The denominator in (10) has N zero, these zero must vanish the numerator giving rise to a set of N equations solving these N equation we can determine all the N unknown occurring in the numerator .Hence $P^{*}(z,s)$ is completely determined.

Particular Case

If $\xi = 0$ then

$$P^{*}(z,s) = \frac{z\left(1+\left(\mu-\frac{\mu}{z}\right)P_0^{*}(s)+(\lambda-\lambda z)P_N^{*}(s)\, z^{N}\right)}{\left(-\lambda z^{2}+(s+\lambda+\mu)z-\mu\right)}$$

The result tallies with that of simple M/M/1/N Queuing model.

Steady State Solution

Steady state equations governing the system are:

$$0 = -(\lambda+\xi)P_0 + \mu P_1 + \eta Q_0 \tag{11}$$

$$0 = -\eta Q_0 + \frac{\xi}{N}\sum_{n=1}^{N}\sum_{j=n}^{N} P_n \tag{12}$$

$$0 = -(\lambda+\mu+\xi)P_n + \lambda P_{(n-1)} + \mu P_{(n+1)} + \frac{\xi}{N}\sum_{j=1}^{N-n} P_{(n+j)} \tag{13}$$

, n=1, 2, 3… N-1

$$0 = -(\mu+\xi)P_N + \lambda P_{(N-1)} \tag{14}$$

Solving this set of equations recursively, we have

$$P_n = \rho^{-N}\left\{\rho^{n} + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\ \sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]} \coprod_{(i-1)} \prod_{i}^{\wedge} \eta^{l_0}\rho^{L_i+n}\, Q_i\, O_i\right\} P_N \tag{15}$$

Where

$$\left(\frac{\lambda}{\mu+\xi}\right) = \rho \ , \ \left(\frac{\xi}{\mu+\xi}\right) = \eta \ , \ [k] \ \rightarrow \text{An integral function.}$$

$$\prod_{j}^{i} = 1 \ \text{ and } \ \sum_{j}^{i} = 0 \ \text{for } i < j$$

$$\coprod_{(i-1)} = \prod_{j=1}^{i-1}\left(\sum_{l_j=(i-j)}^{l_{j-1}-1}\right), \quad \hat{\coprod_i} = \prod_{m=0}^{i-1}\left(\sum_{k_{(m+1)}=k_m+1}^{[A_m]}\right), \; k_0 = 0,$$

$$A_m = \frac{N-n-(i-m)l_0+\sum_{a=1}^{i-m-1} l_a - k_m\, l_{(i-m)} - \sum_{b=1}^{m-1}\left(k_{(m-b)}-k_{(m-(b-1))}\right)l_{(i+b-m)}}{l_0 - l_{(i-m)}}$$

$$L_i = \sum_{j=1}^{i}\left(l_{(i-j)}-l_{(i-(j-1))}\right)k_j \quad , \; l_j = \begin{cases} 0 & \text{if } \; j=i \\ l_j & \text{if } \; 1 \le j < i \end{cases}$$

$$Q_i = \prod_{j=1}^{i}\binom{N - n - L_i - l_{(i-(j-1))}}{l_{(i-j)} - l_{(i-(j-1))}} \quad \text{and} \quad O_i = \prod_{j=1}^{i}\left(\frac{N-k_j}{N}\right)^{l_{(i-j)}-l_{(i-(j-1))}}$$

Using normalization condition, we have

$$P_N = \left[\rho^{-N}\sum_{n=0}^{N}\left\{\rho^n + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\;\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\coprod_{(i-1)}\hat{\coprod_i}\,\eta^{l_0}\rho^{L_i+n}\,Q_i\,O_i\right\}\right]^{-1} \tag{16}$$

Using (16) in (15), than we get

$$P_n = \frac{\rho^n + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\;\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\coprod_{(i-1)}\hat{\coprod_i}\,\eta^{l_0}\rho^{L_i+n}\,Q_i\,O_i}{\sum_{n=0}^{N}\left\{\rho^n + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\;\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\coprod_{(i-1)}\hat{\coprod_i}\,\eta^{l_0}\rho^{L_i+n}\,Q_i\,O_i\right\}} \tag{17}$$

And also from (12), we get

$$Q_0 = \frac{\xi}{N\eta}\sum_{n=1}^{N}\sum_{j=n}^{N}\left[\frac{\rho^n + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\;\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\coprod_{(i-1)}\hat{\coprod_i}\,\eta^{l_0}\rho^{L_i+n}\,Q_i\,O_i}{\sum_{n=0}^{N}\left\{\rho^n + \sum_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]}\;\sum_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]}\coprod_{(i-1)}\hat{\coprod_i}\,\eta^{l_0}\rho^{L_i+n}\,Q_i\,O_i\right\}}\right] \tag{18}$$

Measure of Effectiveness

The steady state probability distribution for the system size allows us to calculate what are commonly called measures of effectiveness. Two of immediate interest are the expected number of customers in the system and the expected number of customers in the queue.

To derive the forgoing measures, let L_s represent the expected number in the system and L_q represent the expected number in the queue. Thus

$$L_s = \sum_{n=1}^{N} n \left\{ \frac{\rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]} \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]} \coprod_{(i-1)} \overset{\wedge}{\coprod_{i}} \eta^{l_0} \rho^{L_i+n} Q_i O_i}{\sum\limits_{n=0}^{N} \left\{ \rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]} \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]} \coprod_{(i-1)} \overset{\wedge}{\coprod_{i}} \eta^{l_0} \rho^{L_i+n} Q_i O_i \right\}} \right\}$$

and

$$L_q = L_s - \left[1 - \frac{\left\{ 1 + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8N}}{2}\right]} \sum\limits_{l_0=i}^{\left[\frac{2N-(i(i-1))}{4}\right]} \coprod_{(i-1)} \overset{\wedge}{\coprod_{i}} \eta^{l_0} \rho^{L_i} Q_i O_i \right\}}{\sum\limits_{n=0}^{N} \left\{ \rho^n + \sum\limits_{i=1}^{\left[\frac{-3+\sqrt{9+8(N-n)}}{2}\right]} \sum\limits_{l_0=i}^{\left[\frac{2(N-n)-(i(i-1))}{4}\right]} \coprod_{(i-1)} \overset{\wedge}{\coprod_{i}} \eta^{l_0} \rho^{L_i+n} Q_i O_i \right\}} \right]$$

Acknowledgement

Gulab Singh Bura is grateful to Kurukshetra University, Kurukshetra for granting him the University Research Scholarship (URS) for carrying out this research work.

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