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On Orthogonal Generalized Derivations of Semiprime Gamma Rings

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Abstract

The main purpose of this paper is to study and investigate some results concerning orthogonal generalized derivations on a semiprime Γ-rings, which are related parallel to those earlier obtained on a semiprime rings.

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Introduction

The notion of a Γ-ring was first introduced by Nobusawa [7], more general than a ring. The class of Γ-rings contains not only all rings but also Hestenes ternary rings [6]. W. E. Barnes [3] weakened slightly the conditions in the definition of Γ-ring in the sense of Nobusawa. After these two authors, many mathematicians made works on Γ-ring in the sense of Barnes and Nobusawa, which are parallel to the results in the ring theory.

The gamma ring is defined by Barnes in [3] as follows

A Γ-ring is a pair (M, Γ) where M and Γ are additive abelian groups for which there exists a map from $M X \Gamma X M \to M$ (the image of (x, α, y) was denoted by $x\alpha y$) for all $x, y\ z \in M$ and $\alpha \in \Gamma$ satisfying the following conditions:

i $(x + y)\alpha z = x\alpha z + y\alpha z$,
$x(\alpha + \beta)y = x\alpha y + x\beta y$,
$x\alpha(y + z) = x\alpha y + x\alpha z$,

ii $(x\alpha y)\beta z = x\alpha(y\beta z)$, A Γ-ring M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for $x \in M$. M is called a prime Γ- ring if for any two elements $x, y \in M$, $x\Gamma M\Gamma y = 0$ implies either $x = 0$ or $y = 0$, and M is called a semiprime if $x\Gamma M\Gamma x = 0$ with $x \in M$ implies $x = 0$. Note that every prime Γ-ring is semiprime.

Note that the notion of derivation of Γ-ring has been introduced by M. Sapanci and A.Nakajima in [8], where the concept of generalized derivation of a Γ-ring has been introduced by Y.Ceven and M.A.Oztürk in [5]. Let M be a Γ-ring and let $d: M \to M$ be additive map. Then d is called a derivation if

$d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Furthmore, an additive map $G: M \to M$ is called a generalized derivation if there exists a derivation $d: M \to M$ such that $G(x\alpha y) = G(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Two mappings f and g of a Γ-ring M are said to be orthogonal on M if

$f(x)\Gamma M\Gamma g(y) = 0 = g(y)\Gamma M\Gamma f(x)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

In [4], Bers□ar and Vukman introduced the notion of orthogonality for a pair of derivations (d, g) of a semiprime ring. In [2] M. Ashraf and M. Jamal introduced the notion of orthogonality for a pair of derivations (d, g) of a semiprime Γ-ring and give several necessary and sufficient conditions for d and g to be orthogonal. In [1] N. Argac□, A. Nakajima and E. Albas□ extended the results of [4] to orthogonality for a pair of generalized derivations (D, d) and (G, g), and gave some necessary and sufficient conditions for (D, d) and (G, g) to be orthogonal.

In this paper, we study the concept of orthogonal generalized derivations in Γ-ring, and obtain some results parallel to those obtained by [1].

Throughout this paper, the condition $x\alpha y\beta z = x\beta y\alpha z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ will be represented by (*).

Preliminaries

For proving the main results, we have needed some important lemmas. So we start as follows

Definition 2.1: Let M be a Γ-ring. Two generalized derivations D and G of M associated with two derivations d and g of M, respectively are said to be orthogonal if

$D(x)\Gamma M\Gamma G(y) = 0 = G(y)\Gamma M\Gamma D(x)$ for all $x, y \in M$.

Example 2.2: Let (M_1, Γ_1) and (M_2, Γ_2) be prime gamma-rings. Let M be the direct product of M_1 & M_2 and Γ be the direct product of Γ_1 & Γ_2. Then it can be easily verified that (M, Γ) is a semiprime gamma-ring which is a direct sum of (M_1, Γ_1) and (M_2, Γ_2).

Let d_1 and g_2 be a nonzero derivations of M_1 and M_2, respectively and $M = M_1 \oplus M_2$. Then the maps d and g on the Γ-ring M which are defined by

$d((x, y)) = (d_1(x), 0)$ and $g(x, y) = (0, g_2(x))$ for all $x, y \in M$

are derivations of M. Moreover, if (D_1, d_1) and (G_2, g_2) are generalized derivations of M_1 and M_2 respectively. Defining

$D((x, y)) = (D_1(x), 0)$ and $G(x, y) = (0, G_2(x))$ for all $x, y \in M$,

We see that (D, d) and (G, g) are generalized derivations of M such that (D, d) and (G, g) are orthogonal.

Lemma 2.3. (2, Lemma 2.2]): Let M be a 2-torsion free semiprime Γ-ring and x, y the elements of M. Then the following conditions are equivalent:

(i) $x\alpha M\beta y = (0)$ for all $\alpha, \beta \in \Gamma$.
(ii) $y\alpha M\beta x = (0)$ for all $\alpha, \beta \in \Gamma$.
(iii) $x\alpha M\beta y + y\alpha M\beta x = (0)$ for all $\alpha, \beta \in \Gamma$.

If one of these conditions are fulfilled then $x\gamma y = y\gamma x = 0$ for all $\gamma \in \Gamma$.

Lemma 2.4. ([2, Lemma 2.4]): Let M be a 2-torsion free semiprime Γ-ring and let d and g be derivations of M. Derivations d and g are orthogonal if and only if $d(x)\alpha g(y) + g(x)\alpha d(y) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Main Results

To prove the main result we need the following lemma

Lemma 3.1: Let M be a 2-torsion free semiprime Γ-ring. Suppose that D and G are a generalized derivations of M associated with derivations d and g resp. of M. If D and G are orthogonal, then for all $x, y \in M$ and $\alpha \in \Gamma$, the following relations are holds $D(x)\alpha G(y) = G(x)\alpha D(y) = 0$, hence $D(x)\alpha G(y) + G(x)\alpha D(y) = 0$.

(ii) d and G are orthogonal and $d(x)\alpha G(y) = G(y)\alpha d(x) = 0$.
(iv) g and D are orthogonal and $g(x)\alpha D(y) = D(y)\alpha g(x) = 0$.
(iv) d and g are orthogonal and this implies $d(x)\alpha g(y) = 0$.
(v) $dG = Gd = 0$ and $gD = Dg = 0$.
(vi) $DG = GD = 0$.

Proof. (i) Since D and G are orthogonal generalized derivation of M. Then we have

$D(x)\alpha m\beta G(y) = 0 = G(y)\alpha m\beta D(x)$ for all $x, y, m \in M$ and $\alpha, \beta \in \Gamma$.

Hence by Lemma (2.3) we get

$D(x)\alpha G(y) = 0 = G(x)\alpha D(y)$ and $D(x)\alpha G(y) + G(x)\alpha D(y) = 0$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

(ii) Since D and G are orthogonal generalized derivations of M. Then we have

$D(x)\alpha m\beta G(y) = 0$ for for all $x, y, m \in M$ and $\alpha, \beta \in \Gamma$. (1)

By Lemma (2.3) we have

$D(x)\alpha G(y) = 0$ for for all $x, y \in M$ and $\alpha \in \Gamma$. (2)

Replacing x by $z\gamma x$ in (2), we obtain
$D(z)\gamma x\alpha G(y) + z\gamma d(x)\alpha G(y) = 0$, for all $x, y, z\in M$ and $\alpha, \gamma\in \Gamma$ (3)

Using (1) in (3) we get
$z\gamma d(x)\alpha G(y) = 0$ for all $x, y, z\in M$, and $\alpha, \gamma\in \Gamma$.

By semiprimeness of M it follows that $d(x)\alpha G(y) = 0$. Thus d and G are orthogonal. Replacing x by $x\beta m$ we obtain
$0 = d(x)\beta m\alpha G(y) + x\beta d(m)\alpha G(y) = d(x)\beta m\alpha G(y)$ for all $x, y, m\in M$ and $\alpha, \beta\in \Gamma$.

Therefore by Lemma (2.3) we obtain $G(y)\alpha d(x) = 0$ for all $x, y\in M$ and $\alpha\in \Gamma$.
That is $d(x)\alpha G(y) = G(y)\alpha d(x) = 0$

(iii) The proof is similar to (ii).
(iv) Since D and G are orthogonal generalized derivations of M. Then we have
$D(x)\alpha m\beta G(y) = 0 = G(y)\alpha m\beta D(x)$ for all $x, y,m \in M$ and $\alpha, \beta\in \Gamma$. (4)

By Lemma(2.3) we get
$D(x)\alpha G(y) = 0$ for all $x, y\in M$ and $\alpha\in \Gamma$. (5)

Replacing x by $x\gamma z$ in (5) and using (4), we get
$x\gamma d(z)\alpha G(y) = 0$ for all $x, y, z\in M$ and $\alpha, \gamma \in \Gamma$. (6)

Now replacing y by $s\gamma y$ in (5) and using (4), we get
$x\gamma d(z)\alpha s\gamma g(y) = 0$, for all $x, y, z, s\in M$ and $\alpha, \gamma \in \Gamma$. (7)

By semiprimeness we obtain $d(z)\alpha s\gamma g(y) = 0$ for all $y, z \in M$ and $\alpha, \beta, \gamma \in \Gamma$.
Then d and g are orthogonal and by Lemma (2.3) we have $d(z)\alpha g(y)= 0$, In particular $d(x)\alpha g(y)=0$.

(v) Using (ii) since d and G are orthogonal, then we have
$d(x)\alpha z\beta G(y) = 0$, for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$. (8)

Hence
$0 = G(d(x)\alpha z\beta G(y))$
$= G(d(x))\alpha z\beta G(y) + d(x)\alpha g(z)\beta G(y) + d(x)\alpha z\beta g(G(y))$ for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$.
Since d and g are orthogonal, we get
$G(d(x))\alpha z\beta G(y) = 0$ for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$. (9)

Replacing y by $d(x)$ in (9) and using semiprimeness of M we obtain $G(d(x)) = 0$, for all $x \in M$.
Similarly we can get $dG = 0$, $gD=0$, and $Dg = 0$.

(iv) Since D and G are orthogonal we have

$D(x)\alpha m\beta G(y) = 0$, for all $x, y, m\in M$ and $\alpha, \beta\in \Gamma$.

Hence
$0 = G(D(x)\alpha m\beta G(y))$
$= G(D(x))\alpha m\beta G(y) + D(x)\alpha g(m\beta G(y))$, for all $x, y,m\in M$ and $\alpha,\beta\in \Gamma$.

By (iii) D and g are orthogonal, we get
$G(D(x))\alpha m\beta G(y) = 0$, for all $x, y,m\in M$ and $\alpha, \beta\in \Gamma$.

Replacing y by D(x) and using semiprimeness of M we obtain $G(D(x)) = 0$, for all $x \in M$.
Similarly we show that $DG =0$.

Remark 3.2: If M is a 2-torsion free prime or semiprime Γ-ring, and (D, d),(G, g) are generalized derivations of M. If (DG, dg) is a generalized derivation, then (D, d) and (G, g) are not orthogonal.

Example 3.3: Let a and b be two nonzero elements of M such that $a\alpha b = 0$, $D(x) = a\alpha x$, and $G(x) = b\beta x$ for all $x \in M$ and $\alpha, \beta \in \Gamma$. Then (D, 0) and (G, 0) are nonzero generalized derivations such that (DG, 0) is generalized derivation.

Now we show that D and G are not orthogonal. If (D, 0) and (G, 0) are orthogonal, then by Lemma 3.1 (i) we have $D(x)\gamma G(y) = 0$, for all $x, y\in M$ and $\gamma\in\Gamma$. Then we have

$$a\alpha x\gamma b\beta y = 0, \text{ for all } x, y\in M \text{ and } \alpha, \beta, \gamma\in \Gamma. \qquad (10)$$

If M is prime Γ-ring then the relation (10) implies that $a = 0$ or $b = 0$ and this is contradiction, and if M is semiprime Γ-ring, then taking $a = b$ and then the relation (10) implies that $a = 0$ and this is also contradiction. Hence D and G are not orthogonal. Then there exists non-orthogonal generalized derivation (D, d) and (G, g) such that (DG, dg) is a generalized derivation

We now to prove our main result

Theorem 3.4: Let M be a 2-torsion free semiprime Γ-ring, and (D, d), (G, g) are generalized derivations of M. Then the following conditions are equivalent:
D and G are orthogonal.
(ii) For all $x, y \in M$, the following relations hold:
(a) $D(x)\alpha G(y) + G(x)\alpha D(y) = 0$,
(b) $d(x)\alpha G(y) + g(x)\alpha D(y) = 0$.
(iii) $D(x)\alpha G(y) = d(x)\alpha G(y) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.
(iv) $D(x)\alpha G(y) = 0$ for all $x, y \in M$, $\alpha \in \Gamma$ and $dG = dg = 0$.
(v) DG is a generalized derivation of M associated with derivation dg of M and $D(x)\alpha G(y) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Proof: (i) → (ii) are proved by Lemma 3.1.

(ii) → (iii) We have $D(x)\alpha G(y) + G(x)\alpha D(y) = 0$, for all $x, y \in M$ and $\alpha \in \Gamma$.
Replacing x by $y\beta x$, we get

$D(y)\beta x\alpha G(y)+y\beta d(x)\alpha G(y)+G(y)\beta x\alpha D(y)+y\beta g(x)\alpha D(y)=0$, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

By (ii)(b) we obtain

$D(y)\beta x\alpha G(y) + G(y)\beta x\alpha D(y) = 0$, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$.

By Lemma 2.3 and Lemma (2.2) we get

$$D(x)\beta x\alpha G(y) = 0, \text{ for all } x, y\in M \text{ and } \alpha, \beta\in \Gamma \tag{11}$$

and so is

$$D(x)\alpha G(y) = 0, \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma. \tag{12}$$

Replacing x by $z\beta x$ in (12), we get

$$D(z)\beta x\alpha G(y) + z\beta d(x)\alpha G(y) = 0, \text{ for all } x, y\in M \text{ and } \alpha, \beta\in \Gamma \tag{13}$$

By using (12) in (13) we obtain

$$z\beta d(x)\alpha G(y) = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma. \tag{14}$$

By semiprimeness, we obtain $d(x)\alpha G(y) = 0$.
(iii) → (iv) we have $D(x)\alpha G(y) = d(x)\alpha G(y) = 0$, for all $x, y \in M$ and $\alpha\in\Gamma$.
Replacing x by $x\beta z$, and using (iii) we have

$$d(x)\beta z\alpha G(y) = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma. \tag{15}$$

From (13)

$0 = d(d(x)\beta z\alpha G(y))$

$= d(d(x)\beta z\alpha G(y)+d(x)\beta d(z)\alpha G(y)+d(x)\beta z\alpha d(G(y))$ for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$.

Therefore, by using (15) and (iii), we obtain

$d(x)\beta z\alpha d(G(y)) = 0$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Replacing x by G(y),and using semiprimeness of M we get $d(G(y)) = 0$ for all $y \in M$.
Also by (iii) we have $d(x)\alpha G(y) = 0$. Replacing y by $y\beta z$, and using (iii) we get

$$d(x)\alpha y\beta g(z) = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma. \tag{16}$$

Now we have $d(d(x)\alpha y\beta g(z)) = 0$,
Hence it follows that

$0 = d(d(x))\alpha y\beta g(z) + d(x)\alpha d(y\beta g(z))$

$= d(d(x))\alpha y\beta g(z) + d(x)\alpha d(y)\beta g(z)) + d(x)\alpha y\beta d(g(z))$, for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$.

Therefore by using (16) and (iii) we obtain

$d(x)\alpha y\beta d(g(z)) = 0$, for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$.

Replacing x by g(z), and using semiprimeness of M we get $d(g(z)) = 0$, for all $z \in$

M.

(iv) → (v) We have $D(x)\alpha G(y) = 0$. for all $x, y\in M$ and $\alpha\in \Gamma$.

Replacing y by $y\beta z$, and using (iv) we get

$$D(x)\alpha y\beta g(z) = 0, \text{ for all } x, y, z \in M \text{ and } \alpha, \beta \in \Gamma.$$

By Lemma 2.3 we obtain

$$D(x)\alpha g(z) = 0, \text{ for all } x, z \in M \text{ and } \alpha\in \Gamma \quad (17)$$

By (iv) we have $dG(x) = dg(x) = 0$, for all $x \in M$.

Replacing x by $x\alpha y$, we get

$$dG(x\alpha y) = dg(x\alpha y) = 0$$

$$d(G(x)\alpha y + x\alpha g(y)) = d(g(x)\alpha y + x\alpha g(y)) = 0$$

$$d(G(x))\alpha y+G(x)\alpha d(y)+d(x)\alpha g(y)+x\alpha d(g(y))=d(g(x))\alpha y+g(x)\alpha d(y)+d(x)\alpha g(y)+x\alpha d(g(y))$$

Hence it follows that

$$G(x)\alpha d(y) = g(x)\alpha d(y) = 0, \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma. \quad (18)$$

Now we take

$$DG(x\alpha y) = DG(x)\alpha y+G(x)\alpha d(y)+D(x)\alpha g(y)+x\alpha dg(y), \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma.$$

Using (17) and (18) we get

$$DG(x\alpha y) = DG(x)\alpha y + x\alpha dg(y), \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma.$$

(iii)→(i) since $D(x)\alpha G(y) = d(x)\alpha G(y) = 0$, for all $x, y\in M$ and $\alpha\in \Gamma$. Replacing x by $x\beta z$ we get

$0 = D(x)\beta z\alpha G(y) + x\beta d(z)\alpha G(y) = D(x)\beta z\alpha G(y)$, this implies that D and G are orthogonal.

(v) → (i) By hypothesis we have

$$DG(x\alpha y) = DG(x)\alpha y + x\alpha dg(y), \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma. \quad (19)$$

In other hand we have

$$DG(x\alpha y) = DG(x)\alpha y+G(x)\alpha d(y)+D(x)\alpha g(y)+x\alpha dg(y), \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma. \quad (20)$$

Comparing (19) and (20) we get

$$G(x)\alpha d(y)+D(x)\alpha g(y) = 0, \text{ for all } x, y\in M \text{ and } \alpha\in \Gamma. \quad (21)$$

Since $D(x)\alpha G(y) = 0$, for all $x, y\in M$ and $\alpha \in \Gamma$, replacing x by $y\beta z$ we get

$$0 = D(x)\beta G(y)\beta z + D(x)\alpha y\beta g(z) = D(x)\alpha y\beta g(z) , \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma$$

By Lemma (2.3) we obtain $g(z)\alpha D(x) = 0$, for all $x, z\in M$ and $\alpha\in \Gamma$. Replacing z by $y\beta z$, we get

$$g(y)\beta z\alpha D(x) = 0 \text{ for all } x, y, z\in M \text{ and } \alpha, \beta\in \Gamma.$$

By Lemma (2.3) we obtain $D(x)\alpha g(y) = 0$.

Now The relation (21) we get $G(x)\alpha d(y) = 0$ for all $x, y\in M$ and $\alpha\in \Gamma$. Hence by Lemma(2.3) we have $d(y)\alpha G(x) = 0$ for all $x, y\in M$ and $\alpha\in \Gamma$, therefore by (iii) we get D and G are orthogonal.

Theorem 3.5: Let M be a 2-torsion free semiprime Γ-ring satisfying (*), and (D, d) and (G, g) be generalized derivations of M. Then the following conditions are equivalent:

DG is a generalized derivation associated with derivation dg of M.

(ii) GD is a generalized derivation associated with derivation gd of M.

(iii) D and g are orthogonal, and G and d are orthogonal.

Proof. (i) → (iii) since DG is a generalized derivation associated with derivation dg of M, that is

$$DG(x\alpha y) = DG(x)\alpha y + x\alpha dg(y), \text{ for all } x, y \in M \text{ and } \alpha \in\Gamma \tag{22}$$

In other hand we have

$$DG(x\alpha y) = DG(x)\alpha y + G(x)\alpha d(y) + D(x)\alpha g(y) + x\alpha dg(y), \text{ for all } x, y\in M \text{ and } \alpha\in \Gamma. \tag{23}$$

Comparing (22) and (23) we get

$$G(x)\alpha d(y) + D(x)\alpha g(y) = 0, \text{ for all } x, y\in M \text{ and } \alpha\in \Gamma. \tag{24}$$

Since dg is derivation then we have

$$dg(x\alpha y) = dg(x)\alpha y + x\alpha dg(y), \text{ for all } x, y \in M \text{ and } \alpha\in \Gamma \tag{25}$$

In other hand we have

$$dg(x\alpha y) = d(g(x))\alpha y+g(x)\alpha d(y)+d(x)\alpha g(y)+x\alpha d(g(y)), \text{ for all } x, y\in M \text{ and } \alpha\in \Gamma \tag{26}$$

Comparing (25) and (26) we get

$$g(x)\alpha d(y) + d(x)\alpha g(y) = 0, \text{ for all } x, y\in M \text{ and } \alpha\in \Gamma. \tag{27}$$

By Lemma 2.4 we get d and g are orthogonal. Now replacing x by $z\beta x$ in (24), we obtain

$G(z)\beta x\alpha d(y)+z\beta g(x)\alpha d(y)+D(z)\beta x\alpha g(y)+z\beta d(x)\alpha g(y) = 0$, for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$.

Using (27), we get

$$G(z)\beta x\alpha d(y) + D(z)\beta x\alpha g(y), \text{ for all } x, y, z \in M \text{ and } \alpha, \beta \in \Gamma. \tag{28}$$

Substituting $g(r)\gamma x$ for x in (28), we obtain

$G(z)\beta g(r)\gamma x\alpha d(y) + D(z)\beta g(r)\gamma x\alpha g(y) = 0$, for all $x, y, z, r\in M$ and $\alpha, \beta, \gamma\in \Gamma$.

By orthogonality d and g of M we get

$D(z)\beta g(r)\gamma x\alpha g(y) = 0$, for all $x, y, z, r \in M$ and $\alpha, \beta, \gamma \in \Gamma$.

In particular if $r = y$, and putting x by $x\beta D(z)$, we obtain
$D(z)\beta g(y)\gamma x\beta D(z)\alpha g(y) = 0$, for all $x, y, z\in M$ and $\alpha, \beta, \gamma\in \Gamma$.

Using (*) and semiprimeness of M we obtain
$D(z)\beta g(y) = 0$, for all $y,z \in M$ and $\beta \in \Gamma$. (29)

Replacing y by $y\alpha x$ in(29) and using (29), we get $D(z)\beta y\alpha g(x) = 0$, for all $x, y, z\in M$ and $\alpha, \beta\in \Gamma$
That is D and g are orthogonal.
Substituting (29) in (24) we get G and d are orthogonal.
Conversely (iii) → (i) since D and g are orthogonal, we get $D(x)\alpha m\beta g(y) = 0$, for all $x, y, m \in M$ and $\alpha, \beta \in \Gamma$. By Lemma 2.3 we get
$D(x)\alpha g(y) = 0$, for all $x, y\in M$ and $\alpha\in \Gamma$. (30)

Since G and d are orthogonal, we get $G(x)\alpha m\beta d(y) = 0$, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. By Lemma 2.3 we get
$G(x)\alpha d(y) = 0$, for all $x, y\in M$ and $\alpha\in \Gamma$. (31)

Hence
$DG(x\alpha y) = D(G(x)\alpha y + x\alpha g(y))$
$= D(G(x))\alpha y+G(x)\alpha d(y)+D(x)\alpha g(y)+x\alpha d(g(y))$, for all $x, y\in M$ and $\alpha\in \Gamma$.

Using (30) and (31) we get the result.
(ii) ↔ (iii) proved by the similar way.

Proposition 3.6: Let M be a 2-torsion free semiprime Γ-ring, and (D, d) is a generalized derivation of M. If $D(x)\alpha D(y) = 0$, for all $x, y \in M$ and $\alpha \in \Gamma$, then we have $D = d = 0$.

Proof. By the hypothesis we have
$D(x)\alpha D(y) = 0$, for all $x, y \in M$ and $\alpha \in \Gamma$. (32)

Replacing y by $y\beta z$, we get
$0 = D(x)\alpha(D(y)\beta z + D(x)\alpha y\beta d(z) = D(x)\alpha y\beta d(z)$, for all $x, y, z\in M$ and $\alpha, \beta \in \Gamma$

By Lemma 2.3 we obtain
$d(z)\alpha D(x) = 0$, for all $x, z \in M$ and $\alpha \in \Gamma$. (33)

Replacing x by $x\gamma z$ in (33) and using (33), we obtain
$0 = d(z)\alpha D(x)\gamma z + d(z)\alpha x\gamma d(z) = d(z)\alpha x\gamma d(z)$, for all $x, z \in M$ and $\alpha,\gamma \in \Gamma$.

By semiprimeness we obtain $d(z) = 0$, for all $z\in M$. Now replacing x by $y\beta x$ in (32) and using (33), we get
$D(y)\beta x\alpha D(y) + y\beta d(x)\alpha D(y) = D(y)\beta x\alpha D(y) = 0$, for all $x, y\in M$ and $\alpha, \beta\in \Gamma$.

By semiprimeness we obtain $D(y) = 0$, for all $y \in M$. That is $D = d = 0$.

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