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Exceedance Measure of a Class of Random Algebraic Polynomials

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Abstract

The present paper provides an asymptotic estimate for the expected area cut off by the x-axis and above a level K by a curve representing an algebraic polynomial with independent random coefficients. The idea of exceedance measure for stochastic processes for n large, as given by Cramer & Leadbetter is adopted here to obtain the result.The method shown here can be used to obtain the result for similar random polynomials.

Keywords and phrases: Random Algebraic Polynomial, Exceedance Measure, Average Number of Real Zeros

Introduction

Let us consider the random algebraic polynomial as

$$\mathrm{P}(x) = \sum_{j=0}^{n} a_j \binom{n}{j}^{\frac{1}{2}} x^j \tag{1}$$

where $a_j (j = 0,1,2,............n)$ be a sequence of independent random variables. Considering a_j to be normally distributed with mean 0 and variance 1, Edelman & Kostland $[2]$ were the first to obtain the average number of real zeros of the above polynomials. Subsequently Farahmand $[3]$ obtained the average number of K- level crossings of the said polynomial under the similar conditions.Later on Farahmand $[4]$ derived an asymptotic estimate for the expected area of the curve representing the above polynomial cut off by the x- axis, where the coefficients $a_j\left(j = 0,1,2,\ldots\ldots\ldots\ldots n\right)$ are assumed to be independently normally distributed random variables with mean o and variance σ^2. He has used the notion of exceedence measure for stochastic processes as developed by Cramer and Leadbetter $[1]$. In this paper we have shown the expected area cut off by the x-axis and above the level K of the said polynomial, where the coefficients are normally distributed random variables with mean $\mu\binom{n}{j}^{\frac{1}{2}}$ and variance σ^2. This is a more generalization of the theorem due to Farahmand. Hence we have the theorem.

Theorem 1. If the coefficients of P(x) are independently normally distributed random variables with mean $\mu\binom{n}{j}^{\frac{1}{2}}$ and variance σ^2, then for sufficiently large n, the mathematical expectation of the area cut off by this curve, above the level K = 0 in the interval $\left(0,|T|\right)$ is asymptotic to ;

$$\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\frac{(1+T)^{n+1}}{n+1} \quad \text{, For } |T|\leq 1$$

and $$\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\left(\frac{T^{n+1}-2^{n+1}}{n+1}\right)\text{, For } |T|>1$$

And for K > 0 is asymptotic to

$$\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right]\frac{(1+T)^{n+1}}{n+1}\text{, For } |T|\leq 1$$

and $$\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right]\left(\frac{T^{n+1}-2^{n+1}}{n+1}\right)\text{, For } |T|>1$$

Derivation of the Formula

First of all we need to obtain the estimates for the mean of P(x) and the variance of P(x). From the definition of (1) we have,

$$\text{E }\left(P(x)\right)=\mu\left(1+c_1x+c_2x^2+\ldots\ldots\ldots\ldots\ldots+c_nx^n\right)=\mu(1+x)^n = \mu' \text{ (Say)} \qquad (2)$$

$$\text{V}\left(P(x)\right) = \sigma^2\left(1+c_1x^2+c_2x^4+\ldots\ldots\ldots\ldots c_nx^{2n}\right)=\sigma^2\left(1+x^2\right)^n = \sigma'^2 \text{ (Say)} \qquad (3)$$

Consider a random variable Z_k^K which takes account of the excursions of P(x) above the level K in various ways.For any non-negative integer k and $\alpha \le x \le \beta$, let

$$\eta_k^K(x) = \begin{cases} \{P(x)\}^k & if\, P(x) > K \\ 0 & Otherwise \end{cases}$$

and $Z_k^K(\alpha,\beta) = \int_\alpha^\beta \eta_k^K(x)dx$

Now from Fubini's theorem, $E\left[Z_k^K(\alpha,\beta)\right]$ is given by

$$E\left[Z_k^K(\alpha,\beta)\right] = \int_\alpha^\beta E\left(\eta_k^K(x)\right)dx$$

$$= \int_\alpha^\beta dx \frac{1}{\sqrt{2\pi}} \int_K^\infty \left(\frac{u^k}{\sigma'}\right) e^{-\frac{1}{2}\left(\frac{u-\mu'}{\sigma'}\right)^2} du \quad \text{Put} z = \frac{u-\mu'}{\sigma'}$$

$$\Rightarrow u = \mu' + \sigma' z \Rightarrow du = \sigma' dz$$

If u = $\infty, z = \infty$

and if u = K, z = $\left(\frac{K-\mu'}{\sigma'}\right)$

then, $E\left[Z_k^K(\alpha,\beta)\right]$ becomes,

$$E\left[Z_k^K(\alpha,\beta)\right] = \int_\alpha^\beta dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{K-\mu'}{\sigma'}\right)}^\infty \frac{\left(\mu' + \sigma' z\right)^k}{\sigma'} e^{-\frac{1}{2}z^2} \sigma' dz$$

$$= \int_\alpha^\beta dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{K-\mu'}{\sigma'}\right)}^\infty \left(\mu' + \sigma' z\right)^k e^{-\frac{1}{2}z^2} dz \qquad (4)$$

We are interested for the calculation of $E\left[Z_1^K(\alpha,\beta)\right]$ which gives a first power rectification of P(x) followed by an integration.

Proof of the Theorem

(i) x-axis crossing

For k = 1 and K = 0 we have,

$E\left[Z_k^K(\alpha,\beta)\right] = E\left[Z_1^0(\alpha,\beta)\right]$

$$= \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{-\mu'}{\sigma'}\right)}^{\infty} \left(\mu' + \sigma' z\right)^k e^{-\frac{1}{2}z^2} dz$$

$$= \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{-\mu'}{\sigma'}\right)}^{\infty} \mu' e^{-\frac{1}{2}z^2} dz + \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{-\mu'}{\sigma'}\right)}^{\infty} \sigma' z\, e^{-\frac{1}{2}z^2} dz$$

$= I_{11} + I_{12}$ (5)

Where $I_{11} = \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{-\mu'}{\sigma'}\right)}^{\infty} \mu' e^{-\frac{1}{2}z^2} dz$

and $I_{12} = \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{-\mu'}{\sigma'}\right)}^{\infty} \sigma' z\, e^{-\frac{1}{2}z^2} dz$

Now $I_{11} = \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{-\mu'}{\sigma'}\right)}^{\infty} \mu' e^{-\frac{1}{2}z^2} dz$ Consider $\frac{1}{2}z^2 = t \Rightarrow zdz = dt \Rightarrow dz = \frac{dt}{\sqrt{2t}}$

If z = $\left(\frac{-\mu'}{\sigma'}\right)$, t = $\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2$ and if z = ∞, t = ∞

Since, $\Gamma(a, x) = \int_{x}^{\infty} e^{-t} t^{a-1} dt$ then, $\Gamma\left(\frac{1}{2}, x\right) = \int_{x}^{\infty} e^{-t} t^{-\frac{1}{2}} dt$

$$= \frac{\mu}{2\sqrt{\pi}} \Gamma\left[\left(\frac{1}{2}\right), \frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right] \int_{\alpha}^{\beta} (1+x)^n dx \quad (6)$$

(referring 2)

then, $I_{12} = \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{-\mu'}{\sigma'}\right)}^{\infty} \sigma' z\, e^{-\frac{1}{2}z^2} dz$ Consider $\frac{1}{2}z^2 = p \Rightarrow zdz = dp$

If z = ∞, p = ∞

and, if z $=\left(\frac{-\mu'}{\sigma'}\right)$ p $=\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2$

$$=\int_\alpha^\beta dx\frac{\sigma'}{\sqrt{2\pi}}\int_{\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2}^{\infty} e^{-t}\,dt \;=\int_\alpha^\beta dx\frac{\sigma'}{\sqrt{2\pi}}\left[e^{-t}\right]_{\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2}^{\infty} = \int_\alpha^\beta dx\frac{\sigma'}{\sqrt{2\pi}}\,e^{-\left(\frac{\mu'}{\sqrt{2}\sigma'}\right)^2}$$

$$=\frac{\sigma}{\sqrt{2\pi}}\int_\alpha^\beta\left(1+x^2\right)^{\frac{n}{2}} e^{\frac{-1\mu^2}{2\sigma^2}\left(\frac{(1+x)^2}{1+x^2}\right)^n}dx \tag{7}$$

(referring 3)

Putting the values of (6) and (7) in (5) we have,

$$\mathrm{E}\left[Z_1^0(\alpha,\beta)\right]=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\int_\alpha^\beta(1+x)^n\,dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_\alpha^\beta\left(1+x^2\right)^{\frac{n}{2}} e^{\frac{-1\mu^2}{2\sigma^2}\left(\frac{(1+x)^2}{1+x^2}\right)^n}dx \tag{8}$$

As in the first part of integration, the power of binomial expression is n, so there is no effect whether ‘n’ is odd or even, but for second part the power of the binomial expression is a fraction, let $\frac{n}{2}=n^*$

$$\mathrm{E}\left[Z_1^0(\alpha,\beta)\right]=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\int_\alpha^\beta\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_\alpha^\beta\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*} e^{\frac{-1\mu^2}{2\sigma^2}\left(\frac{(1+x)^2}{1+x^2}\right)^n}dx \tag{9}$$

Now to estimate the value of $\mathrm{E}\left[Z_1^0(\alpha,\beta)\right]$, Considering the interval (α,β) from $(0,T)$, $(1-\varepsilon,1)$, $(1,1+\varepsilon)$ and $(1+\varepsilon,T)$, we have,

$$\left[Z_1^0(0,T)\right]=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\int_0^T\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_0^T\left(\sum_{j=0}^{n*}\binom{n^*}{j}x^{2j}\right)^{n^*}e^{\frac{-1\mu^2}{2\sigma^2}\left(\frac{(1+x)^2}{1+x^2}\right)^n}dx$$

(for T< 1-ε)

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu^{'}}{\sigma^{'}}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\left[\frac{x^{j+1}}{j+1}\right]_0^T$$

(Second term vanishes as exponential part vanishes for sufficiently large n)

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu^{'}}{\sigma^{'}}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\frac{T^{j+1}}{j+1}$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu^{'}}{\sigma^{'}}\right)^2\right]\frac{(1+T)^{n+1}}{n+1} \tag{10}$$

Next, for $(1-\varepsilon)<x<1$

$$\mathrm{E}\left[Z_1^0(1-\varepsilon,1)\right]=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu^{'}}{\sigma^{'}}\right)^2\right]\int_{1-\varepsilon}^{1}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_{1-\varepsilon}^{1}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*}e^{\frac{-1\mu^2}{2\sigma^2}\left(\frac{(1+x)^2}{1+x^2}\right)^n}dx$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu^{'}}{\sigma^{'}}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\left[\frac{x^{j+1}}{j+1}\right]_{1-\varepsilon}^1$$

(The second term vanishes as exponential part becomes infinite for sufficiently large n)

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu^{'}}{\sigma^{'}}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\left[\frac{1^{j+1}}{j+1}-\frac{(1-\varepsilon)^{j+1}}{j+1}\right]$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu^{'}}{\sigma^{'}}\right)^2\right]\left[\frac{2^{n+1}-1}{n+1}-\frac{(2-\varepsilon)^{n+1}}{n+1}\right]$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\left[\frac{2^{n+1}-(2-\varepsilon)^{n+1}}{n+1}\right] \tag{11}$$

Hence from (10) and (11) we have,

$$\mathrm{E}\left[Z_1^0(0,1)\right]=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\frac{(1+T)^{n+1}}{n+1} \tag{12}$$

(for sufficiently large n & choosing $\varepsilon=\frac{\log n}{n}$)

Next, for $1<x<1+\varepsilon$

$$\mathrm{E}\left[Z_1^0(1,1+\varepsilon)\right]=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\int_1^{1+\varepsilon}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_1^{1+\varepsilon}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*}e^{\frac{-1\mu^2}{2\sigma^2}\left(\frac{(1+x)^2}{1+x^2}\right)^n}dx$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\left[\frac{x^{j+1}}{j+1}\right]_1^{1+\varepsilon}$$

(The second term vanishes as exponential part becomes infinite for sufficiently large n)

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\left[\frac{(1+\varepsilon)^{j+1}}{j+1}-\frac{1^{j+1}}{j+1}\right]$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\left[\frac{(2+\varepsilon)^{n+1}-1}{n+1}-\frac{2^{n+1}-1}{n+1}\right]$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\left[\frac{(2+\varepsilon)^{n+1}-2^{n+1}}{n+1}\right] \tag{13}$$

Lastly, for $1+\varepsilon<x<T$,

$$\mathrm{E}\left[Z_1^0(1+\varepsilon,T)\right]=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\int_{1+\varepsilon}^{T}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_{1+\varepsilon}^{T}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*} e^{\frac{-1\mu^2}{2\sigma^2}\left(\frac{(1+x)^2}{1+x^2}\right)^n} dx$$

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\left[\frac{x^{j+1}}{j+1}\right]_{1+\varepsilon}^{T}$$

(The second term vanishes as exponential part becomes infinite for sufficiently large n)

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\sum_{j=0}^{n}\binom{n}{j}\left[\frac{T^{j+1}}{j+1}-\frac{(2+\varepsilon)^{j+1}}{j+1}\right]$$

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\left[\frac{T^{n+1}-1}{n+1}-\frac{(2+\varepsilon)^{n+1}-1}{n+1}\right]$$

$$=\frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\left[\frac{T^{n+1}-(2+\varepsilon)^{n+1}}{n+1}\right] \tag{14}$$

Hence, from (13) and (14)

$$\mathrm{E}\left[Z_1^0(1,T)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\mu'}{\sigma'}\right)^2\right]\left(\frac{T^{n+1}-2^{n+1}}{n+1}\right) \tag{15}$$

K-Level Crossing

From (4), we have

$$\mathrm{E}\left[Z_k^K(\alpha,\beta)\right] = \int_{\alpha}^{\beta} dx\,\frac{1}{\sqrt{2\pi}}\int_{\left(\frac{K-\mu'}{\sigma'}\right)}^{\infty}\frac{(\mu'+\sigma' z)^k}{\sigma'}e^{-\frac{1}{2}z^2}\sigma' dz$$

$$= \int_{\alpha}^{\beta} dx\,\frac{1}{\sqrt{2\pi}}\int_{\left(\frac{K-\mu'}{\sigma'}\right)}^{\infty}\mu' e^{-\frac{1}{2}z^2}dz + \int_{\alpha}^{\beta} dx\,\frac{1}{\sqrt{2\pi}}\int_{\left(\frac{K-\mu'}{\sigma'}\right)}^{\infty}\sigma' z e^{-\frac{1}{2}z^2}dz$$

$= I_{21} + I_{22}$

Where,

$$I_{21} = \int_{\alpha}^{\beta} dx \, \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{K-\mu'}{\sigma'}\right)}^{\infty} \mu' e^{-\frac{1}{2}z^2} dz$$

and $$I_{22} = \int_{\alpha}^{\beta} dx \, \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{K-\mu'}{\sigma'}\right)}^{\infty} \sigma' z e^{-\frac{1}{2}z^2} dz$$

Now,

$$I_{21} = \int_{\alpha}^{\beta} dx \, \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{K-\mu'}{\sigma'}\right)}^{\infty} \mu' e^{-\frac{1}{2}z^2} dz$$

$$= \frac{\mu}{2\sqrt{\pi}} \Gamma\left[\left(\frac{1}{2}\right), \frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right] \int_{\alpha}^{\beta} (1+x)^n dx \qquad (16)$$

(referring 6)

and $$I_{22} = \int_{\alpha}^{\beta} dx \frac{1}{\sqrt{2\pi}} \int_{\left(\frac{K-\mu'}{\sigma'}\right)}^{\infty} \sigma' z e^{-\frac{1}{2}Z^2} dz$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \left(1+x^2\right)^{\frac{n}{2}} e^{\frac{-1}{2}\left(\frac{K-\mu(1+x)^n}{\sigma\left(1+x^2\right)^{\frac{n}{2}}}\right)^2} dx$$

(referring 7)

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{\alpha}^{\beta} \left(1+x^2\right)^{\frac{n}{2}} e^{\frac{-1}{2}\left[\frac{K^2}{\sigma^2\left(1+x^2\right)^n} + \frac{\mu^2(1+x)^{2n}}{\sigma^2\left(1+x^2\right)^n} - \frac{2K\mu(1+x)^n}{\sigma^2\left(1+x^2\right)^n}\right]} dx \qquad (17)$$

Putting the value of (16) and (17), we have,

$$E\left[Z_1^K(\alpha,\beta)\right] = \frac{\mu}{2\sqrt{\pi}} \Gamma\left[\left(\frac{1}{2}\right), \frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right] \int_{\alpha}^{\beta} (1+x)^n dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_{\alpha}^{\beta}\left(1+x^2\right)^{\frac{n}{2}} e^{\frac{-1}{2}\left[\frac{K^2}{\sigma^2\left(1+x^2\right)^n}+\frac{\mu^2(1+x)^{2n}}{\sigma^2\left(1+x^2\right)^n}-\frac{2K\mu(1+x)^n}{\sigma^2\left(1+x^2\right)^n}\right]} dx$$

As in the first part of integration, the power of binomial expression is n, so there is no effect whether 'n' is odd or even, but for second part the power of the binomial expression is a fraction, let $\frac{n}{2}=n^*$. Then,

$$\mathrm{E}\left[Z_1^K(\alpha,\beta)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu'\right)}{\sigma'}\right)^2\right]\int_{\alpha}^{\beta}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_{\alpha}^{\beta}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*} e^{\frac{-1}{2}\left[\frac{K^2}{\sigma^2\left(1+x^2\right)^n}+\frac{\mu^2(1+x)^{2n}}{\sigma^2\left(1+x^2\right)^n}-\frac{2K\mu(1+x)^n}{\sigma^2\left(1+x^2\right)^n}\right]} dx$$

Now to estimate the value of $\mathrm{E}\left[Z_1^K(\alpha,\beta)\right]$, Considering the interval (α,β) from $(0,T)$, $(1-\varepsilon,1)$, $(1,1+\varepsilon)$ and $(1+\varepsilon,T)$, we have,

$$\left[Z_1^K(0,T)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu'\right)}{\sigma'}\right)^2\right]\int_{0}^{T}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_{0}^{T}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*} e^{\frac{-1}{2}\left[\frac{K^2}{\sigma^2\left(1+x^2\right)^n}+\frac{\mu^2(1+x)^{2n}}{\sigma^2\left(1+x^2\right)^n}-\frac{2K\mu(1+x)^n}{\sigma^2\left(1+x^2\right)^n}\right]} dx$$

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu'\right)}{\sigma'}\right)^2\right]\int_{0}^{T}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

(Second term vanishes as exponential part vanishes for sufficiently large n)

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu'\right)}{\sigma'}\right)^2\right]\frac{(1+T)^{n+1}}{n+1} \tag{18}$$

(referring 10)

Next for, $1-\varepsilon<x<1$

$$\left[Z_1^K(1-\varepsilon,1)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu'\right)}{\sigma'}\right)^2\right]\int_{1-\varepsilon}^{1}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_{1-\varepsilon}^{1}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*} e^{\frac{-1}{2}\left[\frac{K^2}{\sigma^2\left(1+x^2\right)^n}+\frac{\mu^2(1+x)^{2n}}{\sigma^2\left(1+x^2\right)^n}-\frac{2K\mu(1+x)^n}{\sigma^2\left(1+x^2\right)^n}\right]}dx$$

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu^{'}\right)}{\sigma^{'}}\right)^2\right]\int_{1-\varepsilon}^{1}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

(Second term vanishes as exponential part vanishes for sufficiently large n)

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu^{'}\right)}{\sigma^{'}}\right)^2\right]\left[\frac{2^{n+1}-(2-\varepsilon)^{n+1}}{n+1}\right] \tag{19}$$

(referring 11)

From (18) and (19) we have,

$$\left[Z_1^K(0,1)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu^{'}\right)}{\sigma^{'}}\right)^2\right]\frac{(1+T)^{n+1}}{n+1} \tag{20}$$

(for sufficiently large n & choosing $\varepsilon = \frac{\log n}{n}$)

Now for $1 < x < 1+\varepsilon$

$$\mathrm{E}\left[Z_1^K(1,1+\varepsilon)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu^{'}\right)}{\sigma^{'}}\right)^2\right]\int_{1}^{1+\varepsilon}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\frac{\sigma}{\sqrt{2\pi}}\int_{1}^{1+\varepsilon}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*} e^{\frac{-1}{2}\left[\frac{K^2}{\sigma^2\left(1+x^2\right)^n}+\frac{\mu^2(1+x)^{2n}}{\sigma^2\left(1+x^2\right)^n}-\frac{2K\mu(1+x)^n}{\sigma^2\left(1+x^2\right)^n}\right]}dx$$

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu^{'}\right)}{\sigma^{'}}\right)^2\right]\int_{1}^{1+\varepsilon}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

(Second term vanishes as exponential part vanishes for sufficiently large n)

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{\left(K-\mu^{'}\right)}{\sigma^{'}}\right)^2\right]\left[\frac{(2+\varepsilon)^{n+1}-2^{n+1}}{n+1}\right] \tag{21}$$

(referring 13)

Lastly, for $1+\varepsilon < x < T$,

$$\mathrm{E}\left[Z_1^K(1+\varepsilon,T)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right]\int_1^{1+\varepsilon}\left(\sum_{j=0}^{n}\binom{n}{j}x^j\right)dx$$

$$+\ \frac{\sigma}{\sqrt{2\pi}}\int_1^{1+\varepsilon}\left(\sum_{j=0}^{n^*}\binom{n^*}{j}x^{2j}\right)^{n^*} e^{\frac{-1}{2}\left[\frac{K^2}{\sigma^2(1+x^2)^n}+\frac{\mu^2(1+x)^{2n}}{\sigma^2(1+x^2)^n}-\frac{2K\mu(1+x)^n}{\sigma^2(1+x^2)^n}\right]}dx$$

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right]\int_{1+\varepsilon}^{T}\frac{(x^n-1)}{(x-1)}dx$$

(Second term vanishes as exponential part vanishes for sufficiently large n)

$$= \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right]\left[\frac{T^{n+1}-(2+\varepsilon)^{n+1}}{n+1}\right] \tag{22}$$

(referring 14)

From (20) and (21) we have

$$\mathrm{E}\left[Z_1^K(1,T)\right] = \frac{\mu}{2\sqrt{\pi}}\Gamma\left[\left(\frac{1}{2}\right),\frac{1}{2}\left(\frac{(K-\mu')}{\sigma'}\right)^2\right]\left(\frac{T^{n+1}-2^{n+1}}{n+1}\right) \tag{23}$$

(referring 13)

Which proves the theorem.

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