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Evaluation of Special Type of Real Definite Integrals by Developing MATLAB Code

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Abstract

Real definite integrals like $\int_0^{2\pi} \frac{cos(2t)}{5+4\,cos\,(t)}\,dt$ and $\int_0^{\infty} \frac{dx}{1+x^2}$ may be evaluated using in built command "int" available in MATLAB. However, to resolve real integrals in the form of $\int_0^{\infty} \frac{sin(x)}{x(1-x^2)}\,dx$ whose pole(s) are lying on the real axis, do not have specialized command like 'int'. These types of integrals need to be evaluated through contour integration techniques, or through other methods. In the present article we have developed a command 'cint' using MATLAB. To show the efficiency of this code 'cint' we have given some examples.

Keywords: Contour integration, Real definite integrals, Matlab code for contour integrtion

Mathematics Subjects Classification: 68N15; 30E20; 30E99

Introduction

In early literatures, Cauchy (1789-1857) used his integral theorem as a tool for evaluating various definite integrals of functions of a real variable, especially improper integrals. The application of Cauchy-Gourst theorem played a key role in the history of mathematics [6].

Now, to develop the code 'cint' the following theorems have been used.

Theorem A [7]: If $f(z)$ has a pole z_0 of order 'k' then the residue of $f(z)$ at z_0 is denoted by $Res\,(f(z);\ z_0)$ and is given by $Res\,(f(z);\ z_0) = \frac{1}{k!}\ lim_{z\to z_0}\left[\frac{d^{k-1}}{dz^{k-1}}\{(z- z_0)^k\ f(z)\ \}\right]$

Theorem B: Cauchy-Residue theorem [1]: Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, z_3, \ldots., z_n$ inside C then

$$\int_C f(z)dz = 2\pi i \left\{ \sum_{j=1}^{n} Res\left(f(z);\ z_j\right) \right\}$$

Theorem C [5]: Let $f(z) = \frac{P(z)}{Q(z)}$ where P and Q are polynomials with real coefficients of degree 'm and n' respectively, where $n \geq m + 2$. If P and Q has simple zeros at the points $t_1, t_2, \ldots, t_l$ on the x-axis, then

$$P.V \int_{-\infty}^{\infty} f(x)dx = P.V \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{j=1}^{k} Res(f, z_j) + \pi i \sum_{j=1}^{l} Res(f, t_j),$$

where $z_1, z_2, \ldots, z_k$ are the poles of 'f' that lie in the upper half-plane.

Theorem D [2]: Let f be a meromorphic in $\mathbb{C}$ with finitely many singularities in the closure of the upper-half space H. Suppose $lim_{z\to\infty} f(z) = 0$. Then for any $a > 0$, we have

$$PV \int_{-\infty}^{\infty} f(x)e^{iax}\ dx = 2\pi i \left[\sum_{w\epsilon H} R_w \left\{f(z)e^{iaz}\right\}\right] + \pi i\left[\sum_{w\epsilon \mathbb{R}} R_w\left\{f(z)e^{iaz}\right\}\right]$$

The following code 'mult2' has been used in the executed MATLAB code 'cint'.

MATLAB code: (Unique elements and its order of multiplicity in a given array: mult2)

```
%a=input('enter an array:');
function [ua rc]=mult2(a)
ua=sort(unique(a));
lua=length(ua);
al=length(a);
rc=[];
for i=1:lua
   r=0;
   for j=1:al
      b=ua(i)-a(j);
      if b==0
         r=r+1;
      end
   end
         rc=[rc r];
end
```

MATLAB code: cint

```
syms x z
n1=input('enter the numerator of integrand in terms of x:');
d1=input('enter the denominator of integrand in terms of x:');
l=input('enter the int limits as -inf, 0, inf in an array:');
%[n1,d1]=numden(f)% if you use this command there is -ve sign difference in
%the answer. so better use n1, d1 seperately%
drp1=sym2poly(d1);
in_n1=inline(char(n1));
h1=n1;
h1d=diff(h1);
h1d2=diff(h1d);
a11=simplify(h1d2/h1);
a1=sqrt(abs(a11));
la1=length(a1);
c1=h1;
if la1~=0
c1=[cos(a1*x) sin(a1*x) c1];
n10=simplify(n1/c1(1));
n11=simplify(n1/c1(2));
c_2=isreal(n10);
c_3=isreal(n11);
n2=inline(char(n1));
nr=n2(z);
else
   c_2=0;
   c_3=0;
   nr=h1;
end
d2=inline(char(d1));
dr=d2(z);
drp=((1/drp1(1))*drp1);
[r,p,k]=residue(1,drp); % root command will not give the desired answer always, hence residue command has been used%
```

```
lp=length(p);
a_ip=angle(p);
aip=rad2deg(a_ip);
p1=[];
if c_2==1 || c_3==1
    nr=exp(i*a1*z);
else
    nr=nr;
end
for k1=1:lp
    if 0<aip(k1) & aip(k1)<180
        p1=[p1 p(k1)];
    end
end
[p2,m1]=mult2(p1);% unique elements of p1 with multiplicity wrt to p1
lp2=length(p2);
sum=0;
if lp2~=0
   for k2=1:lp2
       a=[p2(k2)];
       s=[];
       for k3=1:lp
           if p(k3)~=a
              s=simplify([s p(k3)]);
           end
       end
       ls=length(s);
       ps=1;
       if ls~=0
           for k4=1:ls
               ps=(ps*(z-s(k4)));
           end
       end
```

```
        nd1=simplify(nr/ps); %multiplying of given fn with pole fn factor
        dn1=simplify((1/factorial(m1(k2)-1))*diff(nd1,(m1(k2)-1)));
        dnin1=inline(char(dn1));
        rq1=dnin1(p2(k2));
        sum=sum+(2*pi*i*rq1);
    end
end
rp=[];
for k=1:lp
    r1=isreal(p(k));
    if r1==1
        rp=[rp p(k)];
    end
end
lrp=length(rp);
[rp1,m2]=mult2(rp);
lf=0;
if lrp~=0
for j1=1:lrp  % for Jordan's in equality %
    nf1=simplify((z-rp1(j1))^(m2(j1))*(nr/dr));
    nf2=simplify((1/(factorial(m2(j1)-1)))*diff(nf1,m2(j1)-1));
    nf2in=inline(char(nf2));
    nf=nf2in(rp1(j1));
    th=i*nf*(0-pi);
    lf=lf+th;
end
end
rf=((1/drp1(1))*sum)-lf;
r_rf=real(rf);
i_rf=imag(rf);
g1=0;
if c_2==1
    g1=g1+r_rf;
```

```
else if c_3==1
        g1=g1+i_rf;
    else if c_2==0 & c_3==0
    g1=g1+rf;
        end
    end
end
if l(1)==0
    g=g1/2;
disp('the integral value in [0,inf] is:');disp(g)
else if l(1)==-inf
disp('the integral value in [-inf,inf] is:');disp(g1)
    end
end
```

Now, we are going to show in the following examples 1 to 3 both ‘int’ and ‘cint’ codes can be used.

Examples

1. Prove that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

   Solution: let $\int_C f(z)\,dz$ where $f(z) = \frac{1}{1+z^2}$ and C is the contour consisting of a large semi-circle Γ of radius R along with the part of real axis from $x = -R$ to $x = +R$

   $\therefore \int_C f(z)\,dz = \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_\Gamma \frac{1}{1+z^2} dz = 2\pi i \sum R^+$

   Where $\sum R^+$=sum of the residues of the poles within C

   Now $lim_{z\to\infty} z\, f(z) = lim_{z\to\infty} \frac{z}{1+z^2} = 0$

   Also, $lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

   $\therefore \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = 2\pi i \sum R^+ = 2\pi i \left(\frac{1}{2i}\right) = \pi$

   $\Longrightarrow \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

2. Prove that $\int_{-\infty}^{\infty} \frac{sin(x)}{x^2+4x+5} dx = \frac{-\pi}{e} sin\,(2)$

   Solution: let $h(z) = e^{iz} f(z) = e^{iz}.\frac{1}{z^2+4z+5}$ where $lim_{z\to\infty} f(z) = lim_{z\to\infty} \frac{1}{z^2+4z+5} = 0$

   Proceed as above example-1, we may get $\int_{-\infty}^{\infty} h(x)dx = \frac{\pi}{e}\{cos(2) - i\, sin\,(2)\}$

Equating the imaginary part on both sides, $\int_{-\infty}^{\infty} \frac{sin(x)}{x^2+4x+5} dx = \frac{-\pi}{e} sin\,(2)$

3. Prove that $\int_0^{\infty} \frac{sin(\pi x)}{x(1-x^2)} dx = \pi$

   Solution: let $\int_C\ f(z)\,dz$ where $f(z) = \frac{e^{i\pi z}}{z(1-z^2)}$

   The given function has poles at $z = -1, z = 0\ and\ z = 1$ on the real axis. We choose the contour C to be a large circle $|z| = R$ indented at $z = -1, z = 0\ and\ z = 1$ and by small semi circles $\gamma_1, \gamma_2\ and\ \gamma_3$ respectively of radii $r_1, r_2\ and\ r_3$.

   The given function has no singularity within C and as such by Cauchy's residue theorem

   $\int_C\ f(z)dz = 2\pi i\ \sum R_C +\ \pi i\ \sum R_R$

   Where $\sum R_C$ =Sum of the residues of the poles within C

   $\sum R_R$ =Sum of the residues of the poles on the real axis

   $\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x(1-x^2)} dx = 2\pi i(0) + i\pi\left(\frac{1}{2} + 1 + \frac{1}{2}\right) = 2i\pi$

   Equating the imaginary part on both sides, $\int_{-\infty}^{\infty} \frac{sin(\pi x)}{x(1-x^2)} dx = 2\pi$

   $\Longrightarrow \int_0^{\infty} \frac{sin(\pi x)}{x(1-x^2)} dx = \pi$

We have seen how the residue theorem is useful to evaluate the real definite integrals of the forms [4]:

1 $\int_0^{2\pi} F(cos\theta, sin\theta)d\theta$

2 $\int_{-\infty}^{\infty} f(x)\,dx$

3 $\int_{-\infty}^{\infty} f(x)\,cos\,(ax)\,dx$ or $\int_{-\infty}^{\infty} f(x)\,sin(ax)\,dx$ where F in (1) and f in (2) and (3) are rational functions. For the rational function $f(x) = \frac{P(x)}{Q(x)}$ in (2) and (3), we will assume that the polynomials P and Q have no common factors.

The same command 'int' cannot give the solution in the following examples-4 to 8, while the solution can be obtained by contour integration. Hence, a new code has been developed by incorporating contour integration, and thereby developing a new command 'cint' to evaluate real integrals in short span of time. To further substantiate the validity of the developed code, different real integrals are evaluated in this study, and the responses are recorded using the code which show accuracy and effectiveness.

4. Evaluate $\int_0^{\infty} \frac{sin(x)}{x(1-x^2)}\ dx$ [ans: $\frac{\pi}{2}\{1 - cos(1)\}$]
5. Evaluate $\int_0^{\infty} \frac{cos(x)}{1-x^2}\ dx$ [ans: $\frac{\pi}{2} sin\,(1)$]
6. Evaluate $\int_0^{\infty} \frac{x^4}{x^6-1}\ dx$ [ans: $\frac{\pi}{6}\sqrt{3}$]
7. Evaluate $\int_{-\infty}^{\infty} \frac{1}{x^3-1}\ dx$ [ans: $\frac{-\pi}{\sqrt{3}}$]
8. Evaluate $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^2\,(x-2)}\ dx$ [ans: $\frac{-7\pi}{25}$]

Ex-1: syms x >> int(1/(1+x^2),0,inf) ans = 1/2*pi	Or	cint enter the numerator of integrand in terms of x:1 enter the denominator of integrand in terms of x:1+x^2 enter the int limits as -inf, 0, inf in an array:[0 inf] the integral value in [0,inf] is: 1.5708

Ex-2: syms x >> simplify(int(sin(x)/(x^2+4*x+5),-inf,inf)) ans = -pi*sin(2)*(-sinh(1)+cosh(1))
Or
Cint enter the numerator of integrand in terms of x:sin(x) enter the denominator of integrand in terms of x:x^2+4*x+5 enter the int limits as -inf, 0, inf in an array:[-inf inf] the integral value in [-inf,inf] is: -1.0509

Ex-3: syms x >> (int(sin(pi*x)/(x*(1-x^2)),0,inf)) ans = Pi	Or	Cint enter the numerator of integrand in terms of x:sin(pi*x) enter the denominator of integrand in terms of x:x*(1-x^2) enter the int limits as -inf, 0, inf in an array:[0 inf] the integral value in [0,inf] is: 3.1416

Ex-4: cint enter the numerator of integrand in terms of x:sin(x) enter the denominator of integrand in terms of x:x*(1-x^2) enter the int limits as -inf, 0, inf in an array:[0 inf] the integral value in [0,inf] is:	Ex-5: cint enter the numerator of integrand in terms of x:cos(x) enter the denominator of integrand in terms of x: 1-x^2 enter the int limits as -inf, 0, inf in an array:[0 inf]

0.7221	*the integral value in [0,inf] is:* *1.3218*
Ex-6: *cint* *enter the numerator of integrand in terms of x:x^4* *enter the denominator of integrand in terms of x:x^6-1* *enter the int limits as -inf, 0, inf in an array:[0 inf]* *the integral value in [0,inf] is:* *0.9069 + 0.0000i*	*Ex-7:* *cint* *enter the numerator of integrand in terms of x:1* *enter the denominator of integrand in terms of x:x^3-1* *enter the int limits as -inf, 0, inf in an array:[-inf inf]* *the integral value in [-inf,inf] is:* *-1.8138 + 0.0000i*
Ex-8: *cint* *enter the numerator of integrand in terms of x:1* *enter the denominator of integrand in terms of x:(x^2+1)^2*(x-2)* *enter the int limits as -inf, 0, inf in an array:[-inf inf]* *the integral value in [-inf ,inf] is:* *-0.8796 - 0.0000i*	

Type-1 integrals $\{\int_0^{2\pi} F(cos\theta, sin\theta)d\theta\}$, cannot be evaluated by using ‘cint’ command as it is not considered in developing the ‘cint’ code. The reason being MATLAB has an inbuilt command ‘int’ to evaluate these kinds of integrals.

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