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Dynamical Behavior in a Three Species Discrete Model of Prey-Predator Interactions

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Abstract

This paper aims at determining the equilibria of 3-D discrete time predator–prey system. Stability conditions of the equilibrium points are obtained. The phase portraits are generated for different sets of parameter values. Also bifurcation diagram is provided for selective range of growth parameter in predator population. Numerical simulations exhibit rich dynamics of the discrete model and chaotic dynamics in predatorpopulations.

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1. INTRODUCTION

The main purpose of this paper is to investigate some dynamical behavior of three species interactions using a discrete dynamical model. Mathematical population theory dates back to Malthus which dealt with the study of single species population [8]. The original model was then modified with limited resources. Ecology is old diverse discipline. Population ecology is a branch of ecology which uses mathematical equations to describe interactions among species. Many of these equations appear both as discrete models described by difference equations and continuous time models described by differential equations. Predator –prey interaction is one of the basic interspecies relations in ecology. The study of interacting populations consisting of two or three species in an ecosystem is of practical interest. In recent years many researchers investigated the dynamical properties of three species food chain [1, 3, 4, 5, 7, 12]. Alfred Lotka (1925) and VitoVolterra (1926) formulated the first model of predator – prey interactions [6, 11]. This model helped both theoretical and mathematical biologists in understanding the dynamical nature of the interactions. The basic model had been improved by many mathematicians.

2. MODEL DESCRIPTION AND EQUILIBRIUM POINTS

In recent years more and more attention is being paid to discrete time population models because discrete time models are appropriate when populations have non overlapping generations. When the model is described by difference equations, the behavior of the orbit is determined by iterating the function over and over. They are more suitable for numerical simulations [1, 2, 9, 10, 13, 14]. Linear iterative systems are not adequate in modeling interacting processes in biology. Hence we investigate the dynamics of interspecies interactions by the following system of difference equations. For three-species predator – prey interactions two possibilities arise: two preys – one predator system and two predators – one prey system. In the following system of difference equations we model the interactions among two preys and one predator.

$$\begin{aligned} x(n+1) &= (1+a)\,x(n) - b\,x(n)\,z(n) \\ y(n+1) &= r\,y(n)(1-y(n)) - c\,y(n)\,z(n) \\ z(n+1) &= (1-d)\,z(n) + e\,x(n)\,z(n) + f\,y(n)\,z(n), \quad x(n), y(n), z(n) \geq 0 \end{aligned} \tag{1}$$

where $r,\ a,\ b,\ c,\ d,\ e,\ f > 0$. The system (1) has the five equilibria $E_0 = (0,0,0)$, $E_1 = \left(0, \left(1-\frac{1}{r}\right), 0\right)$, $E_2 = \left(0, \frac{d}{f}, \frac{1}{c}\left(r-1-\frac{dr}{f}\right)\right)$, $E_3 = \left(\frac{d}{e}, 0, \frac{a}{b}\right)$ and $E_4 = \left(\frac{br(d-f)+f(b+ac)}{ber}, 1-\frac{b+ac}{br}, \frac{a}{b}\right)$.

3. STABILITY ANALYSIS OF EQUILIBRIUM POINTS

The recent decades have witnessed development of theory of nonlinear dynamical systems having applications in various fields including population biology. Due to the complexity in analyzing nonlinear equations, a comprehensive theory of dynamics of nonlinear systems still awaits development. An important technique for analyzing nonlinear systems qualitatively is the analysis of the behavior of the solutions near equilibrium points using linearization. One of the most urgent problems demanding the attention of researchers is the stability of ecosystems. The local stability analysis of the model can be carried out by computing the Jacobian matrix corresponding to each equilibrium point [15]. The Jacobian Matrix J for the system (1) is

$$J(x,y,z) = \begin{pmatrix} 1+a-bz & 0 & -bx \\ 0 & r-2ry-cz & -cy \\ ez & fz & 1-d+ex+fy \end{pmatrix} \tag{2}$$

The following Lemma [15] is needed to discuss the stability of the equilibrium points of (1).

Lemma 1. Let

$$p(\lambda)=\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \quad (3)$$

be the characteristic equation for a matrix defined by (2). Then the following statements are true:

1. If every root of equation (3) has absolute value less than one, then the equilibrium point of the system (1) is locally asymptotically stable and equilibrium point is called a sink.
2. If at least one of the roots of equation (3) has absolute value greater than one, then the equilibrium point of the system (1) is unstable and equilibrium point is called a saddle.
3. If every root of equation (3) has absolute value greater than one, then the equilibrium point of the system (1) is a source.
4. The equilibrium point of system (1) is called hyperbolic if no root of equation (3) has absolute value equal to one. If there exists a root of equation (3) with absolute value equal to one, then the equilibrium point is called non-hyperbolic.

For the system (1), we have the following propositions.

Proposition 2. The equilibrium point E_0 is a
(a) sink if $r < 1$, $-2 < a < 0$ and $0 < d < 2$.
(b) source if $a > 0$, $r > 1$ and $d > 2$.
(c) saddle if $a > 0$, $r < 1$ and $0 < d < 2$.
(d) non hyperbolic if either $r = 1$ or $d = 2$.

Proof. From (2), Jacobian matrix for E_0 is given by

$$J(E_0)=\begin{pmatrix} 1+a & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & 1-d \end{pmatrix}$$

The eigen values of the matrix $J(E_0)$ are $\lambda_1=1+a, \lambda_2=r$ and $\lambda_3=1-d$. In view of Lemma 1, we see that, E_0 is a sink if $r< 1$, $-2 <a< 0$ and $0 <d< 2$; E_0 is a source if $a> 0$, $r> 1$ and $d> 2$; E_0 is a saddle if $a> 0$, $r< 1$ and $0 <d< 2$; and also E_0 is non hyperbolic if either $r = 1$ or $d = 2$.

Proposition 3. The equilibrium point E_1 is a
(a) sink if $1<r<3, -2<a<0$ and $r<\frac{f}{f-d}$.

(b) source if $r>3, a>0$ and $r>\frac{f}{f-d}$.

(c) saddle if $1<r<3, a>0$ and $r<\frac{f}{f-d}$.

Proof. From (2), Jacobian matrix for E_1 is given by

$$J(E_1)=\begin{pmatrix} 1+a & 0 & 0 \\ 0 & 2-r & \frac{c(1-r)}{r} \\ 0 & 0 & 1-d+\frac{f(r-1)}{r} \end{pmatrix}$$

The eigen values of the matrix $J(E_1)$ are $\lambda_1=1+a, \lambda_2=2-r$ and $\lambda_3=1-d+\frac{f(r-1)}{r}$. By Lemma 1, we obtain, E_1 is a sink if $1<r<3, -2<a<0$ and $r<\frac{f}{f-d}$; E_1 is a source if $r>3, a>0$ and $r>\frac{f}{f-d}$; and finally E_1 is a saddle if $1<r<3, a>0$ and $r<\frac{f}{f-d}$.

Proposition 4. The equilibrium point E_2 is a

(a) sink if $r>\frac{f(b+ac)}{b(f-d)}$ and $r>\frac{f}{f-d}$;

(b) source if $r<\frac{f(b+ac)}{b(f-d)}$ and $r<\frac{f}{f-d}$;

(c) saddle if $\frac{f}{f-d}<r<\frac{f(b+ac)}{b(f-d)}$.

Proof. From (2), Jacobian matrix for the positive interior equilibrium point E_2 is given by

$$J(E_2)=\begin{pmatrix} 1+a+\frac{b}{c}\left(1-r+\frac{dr}{f}\right) & 0 & 0 \\ 0 & 1-\frac{dr}{f} & -\frac{dc}{f} \\ \frac{e}{c}\left(r-1-\frac{dr}{f}\right) & \frac{f}{c}\left(r-1-\frac{dr}{f}\right) & 1 \end{pmatrix}$$

Hence, the eigen values are $\lambda_1=1+a+\frac{b}{c}\left(1-r+\frac{dr}{f}\right)$ and

$\lambda_{2,3} = 1 - \frac{dr}{2f} \pm \frac{1}{2f}\sqrt{4d^2 fr + d^2r^2 - 4df^2r + 4df^2}$ .By Lemma 1, we conclude that E_2 is a sink if $r > \frac{f(b+ac)}{b(f-d)}$ and $r > \frac{f}{f-d}$; E_2 is a source if $r < \frac{f(b+ac)}{b(f-d)}$ and $r < \frac{f}{f-d}$; and furthermore E_2 is a saddle if $\frac{f}{f-d} < r < \frac{f(b+ac)}{b(f-d)}$.

The Jacobian matrix for the equilibrium point E_3 is given by,

$$J(E_3) = \begin{pmatrix} 1 & 0 & -\frac{bd}{e} \\ 0 & r - \frac{ac}{b} & 0 \\ \frac{ae}{b} & \frac{af}{b} & 1 \end{pmatrix}$$

The eigen values are $\lambda_1 = r - \frac{ac}{b}$ and $\lambda_{2,3} = 1 \pm i\sqrt{ad}$ (complex eigen values). The Jacobianmatrix for the interior equilibrium point E_4 is given by,

$$J(E_4) = \begin{pmatrix} 1 & 0 & -\frac{br(d-f)+f(b+ac)}{re} \\ 0 & 2 - r + \frac{ac}{b} & \frac{c(b+ac)}{br} - c \\ \frac{ae}{b} & \frac{af}{b} & 1 \end{pmatrix}$$

4. NUMERICAL STUDY

Numerical simulations help us to analyze the phase diagrams of dynamical systems depending up on parameters. Numerical study of nonlinear discrete dynamical systems gives an insight in to dynamical characteristics. In this section, we present the time plots for *x(n), y(n), z(n),* phase portraits and bifurcation diagrams for the system (1). Dynamic behaviors of the system (1) about the equilibrium points under different sets of parameter values are presented.

Example 1. We shall consider the values r = 1.85,a = 0.45,b = 0.99,c = 0.99,d =0.39,e = 0.23,f = 2.3. The equilibrium point $E_2 = (0, 0.17, 0.54)$ and the eigen values are $\lambda_1 = 0.9140$ and $\lambda_{2,3} = 0.8431 \pm i0.4295$ (complex eigen values). Also $|\lambda_{1,2,3}| < 1$, which isStable, see figure-1 (a, b)

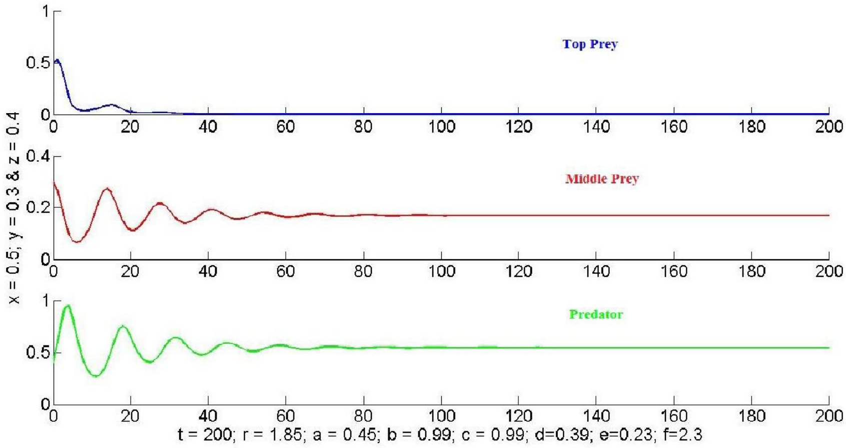

Figure 1(a). Stability at E2

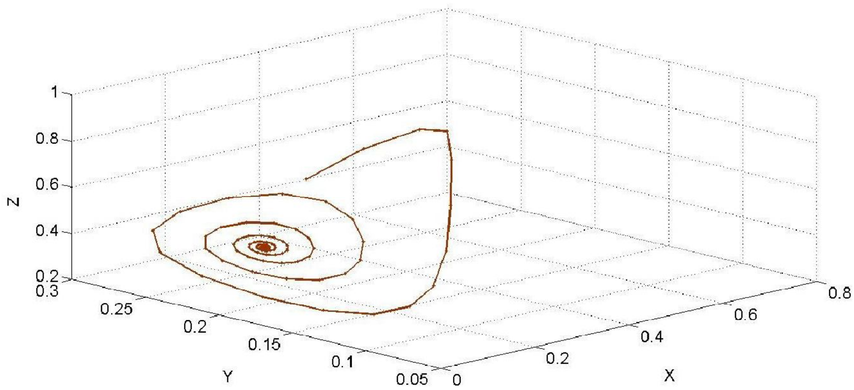

Figure 1(b). Phase Portrait of the Equilibrium point E_2

Example 2. We shall consider the values r = 0.99,a = 0.00095,b = 0.085,c = 0.25,d =0.005,e = 0.09, f= 0.05. The equilibrium point $E_3 = (0.05, 0, 0.01)$ and the eigen values are $\lambda_1 = 0.9872$ and $\lambda_{2,3} = 1 \pm i\,0.0022$ (complex eigen values). Here $|\lambda_1| = 0.9872 < 1$ and $|\lambda_{2,3}| > 1$, which is unstable, see figure-2 (a, b)

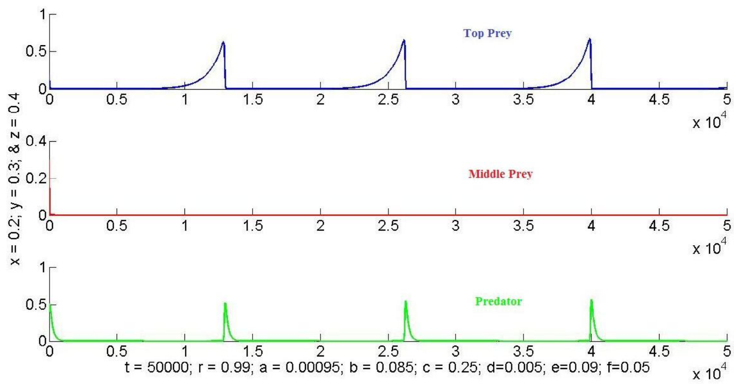

Figure 2(a). Unstability at E_3

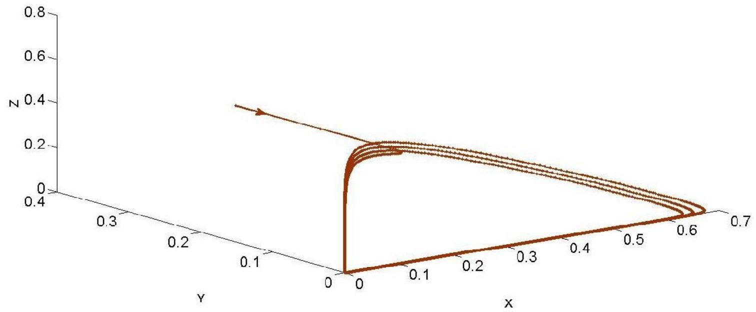

Figure 2(b). Phase Portrait of the Equilibrium point E_3

Example 3. We shall consider the values r = 1.65,a = 0.45,b = 0.93,c = 0.8,d =0.45,e = 0.15,f = 2.25. The equilibrium point $E_4 = (0.61, 0.16, 0.48)$ and the eigen values are $\lambda_1 = 0.9352$ and $\lambda_{2,3} = 0.9009 \pm i\,0.3967$ (complex eigen values). Also $|\lambda_1| = 0.9352 < 1$ and $|\lambda_{2,3}| = 0.9844 < 1$, which is Stable see figure-3 (a, b)

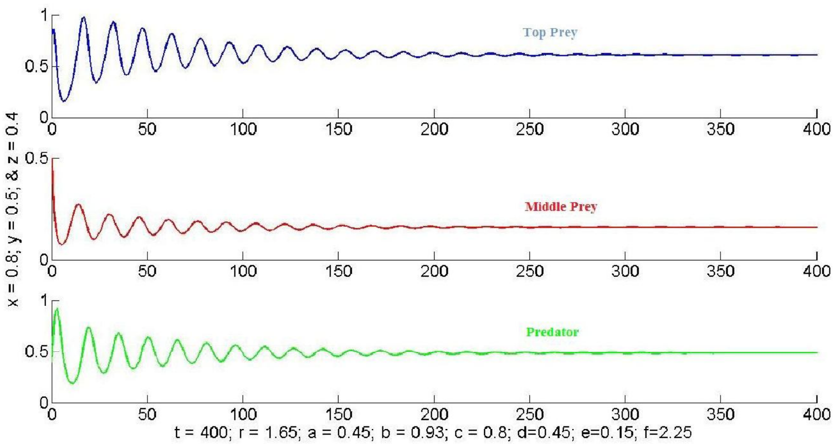

Figure 3(a). Stability at E4

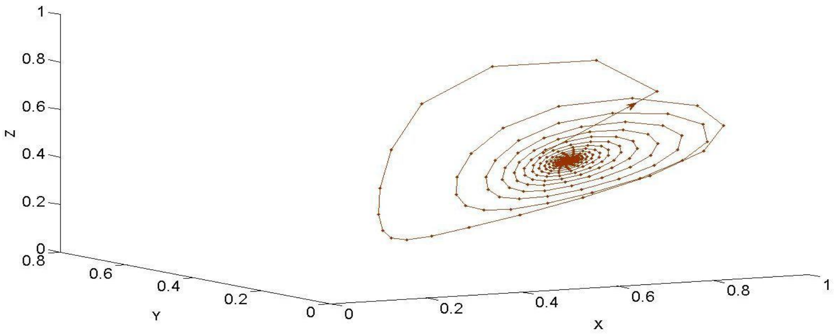

Figure 3(b). Phase Portrait of the Equilibrium point E_4

5. LIMIT CYCLES AND BIFURCATION

Existence of limit cycles for selective set of parameters is established through phase planes in figure-4. Bifurcation diagrams provide information about abrupt changes in the dynamics, see figure-5. The parametric values at which these changes occur are called bifurcation points. They provide information about the dependence of the dynamics on a certain parameter. If the qualitative change occurs in a neighborhood of an equilibrium point or periodic solution, it is called a local bifurcation. Any other qualitative change that occurs is considered as a global bifurcation.

Example 4. We shall consider the values $r = 1.95, a = 0.008, b = 0.099, c = 0.75, d = 0.0305, e = 0.235, f = 0.0075$ and the initial conditions are $x(0) = 0.2, y(0) = 0.3$ and $z(0) = 0.4$. The trajectory in the phase plane finally settles as a limit cycle.

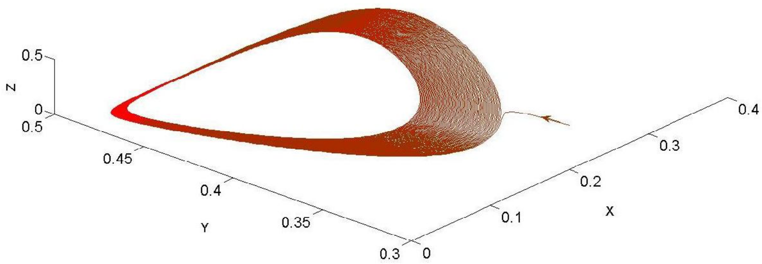

Figure 4. Limit Cycle

In this paper, ecological model with interspecies interactions in three species food chainwith two preys and a predator is proposed and some dynamical behaviors are investigated.Dynamical behavior of three species food chain model is investigated at equilibrium points. Numerical study shows the rich and interesting complicated dynamics of the model. Also the limit cycle and bifurcation diagram are provided.

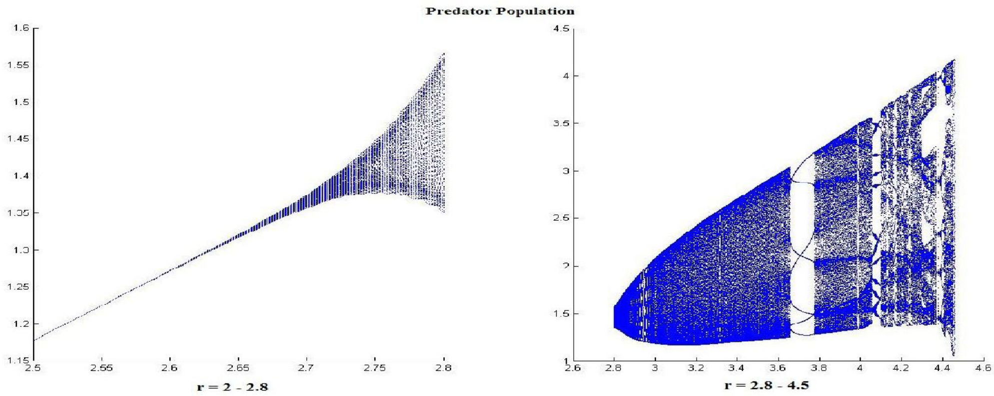

Figure 5. Bifurcations diagram for Predator Population

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