

Saigo-Maeda Fractional Differential Operators of the Multivariable H-Function

Kantesh Gupta and Meena Kumari Gurjar

*Department of Mathematics,
Malaviya National Institute of Technology, Jaipur-302017, Rajasthan (India)
E-mail: kanteshgupta1@gmail.com, meenanetj@gmail.com*

Abstract

In this paper, we study and develop the generalized fractional differential operators involving Appell's function $F_3(\cdot)$ [1, p. 224, Eq. 5.7.1 (8)] introduced by Saigo and Maeda [2, p. 393] to the multivariable H-function. First, we establish two theorems that give the images of the multivariable H-function in Saigo-Maeda operators. On account of general nature of these operators, a large number of new and known theorems involving Saigo, Riemann-Liouville, Erdélyi-Kober fractional differential operators and several special functions notably generalized wright hypergeometric function, Mittag-Leffler function, generalized lauricella function, Bessel functions follow as special cases of our main findings. The important results obtained by Gupta [3], Kilbas [4], Kilbas and Saigo [5], Kilbas and Sebastian [6], Saxena, Ram and Suthar [7] and Saxena and Saigo [8] follow as special cases of our results.

Keywords: Saigo-Maeda fractional differential operators, Appell function, Gauss hypergeometric function, Multivariable H-function, Bessel function, Mittag-Leffler function.

AMS Subject Classification: 26A33, 33C05, 33C10, 33C60, 33C65, 33E12.

1. Introduction:

The fractional differential operator involving various special functions, have been found significant importance and applications in various sub-field of application mathematical analysis. Since last five decades, a number of workers like Kiryakova [9], Srivastava et al. [10], Saxena et al. [11, 12], Saigo [13], Kilbas [4], Kilbas and

Sebastian [6], Samko et al. [14], Miller and Ross [15], and Gupta et al.[3] etc. have studied in depth, the properties, applications and different extensions of various hypergeometric operators of fractional differentiation.

Recently, Kumar and Daiya [16] have applied this fractional differentiation of the product of a general class of Polynomial and $\bar{H}$ -function. Saxena, Ram and Kumar [12] have also derived the generalized fractional differentiation of the Aleph function associated with the Appell function F_3 .

Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, $\gamma > 0$ and $x > 0$, then the generalized fractional differentiation operators [2] involving Appell function F_3 as a kernel are defined by the following equations:

$$\begin{aligned} \left(D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} f \right)(x) &= \left(I_{0+}^{-\alpha', -\alpha, -\beta', -\beta, -\gamma} f \right)(x) \\ &= \left(\frac{d}{dx} \right)^n \left(I_{0+}^{-\alpha', -\alpha, -\beta' + n, -\beta, -\gamma + n} f \right)(x), \quad (\Re(\gamma) > 0; n = [\Re(\gamma)] + 1) \\ &= \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dx} \right)^n \left(x^\alpha \right)_0^x (x-t)^{n-\gamma-1} t^\alpha F_3 \left(-\alpha', -\alpha, -\beta' + n, -\beta, n-\gamma, 1-\frac{t}{x}, 1-\frac{x}{t} \right) f(t) dt \end{aligned} \quad (1)$$

and

$$\begin{aligned} \left(D_-^{\alpha, \alpha', \beta, \beta', \gamma} f \right)(x) &= \left(I_-^{-\alpha', -\alpha, -\beta', -\beta, -\gamma} f \right)(x) \\ &= \left(-\frac{d}{dx} \right)^n \left(I_-^{-\alpha', -\alpha, -\beta', -\beta + n, -\gamma + n} f \right)(x), \quad (\Re(\gamma) > 0; n = [\Re(\gamma)] + 1) \\ &= \frac{1}{\Gamma(n-\gamma)} \left(-\frac{d}{dx} \right)^n \left(x^\alpha \right)_x^\infty (t-x)^{n-\gamma-1} t^\alpha F_3 \left(-\alpha', -\alpha, -\beta', n-\beta, n-\gamma, 1-\frac{x}{t}, 1-\frac{t}{x} \right) f(t) dt \end{aligned} \quad (2)$$

where $I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma}$ and $I_-^{\alpha, \alpha', \beta, \beta', \gamma}$ are Saigo- Maeda fractional integral operators.

F_3 the Appell-hypergeometric function of two variables is defined as

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; z, \xi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{z^m \xi^n}{m! n!}, \quad (|z| < 1, |\xi| < 1) \quad (3)$$

where $(z)_m$ and $(z)_n$ are the Pochhammer symbol defined by $z \in C$ and $m, n \in N_0 = N \cup \{0\}$, $N = \{1, 2, 3, \dots\}$ by $(z)_0 = 1, (z)_m = z(z+1)\dots(z+m-1)$.

The above series (3) is absolutely convergent for $(|z| < 1, |\xi| < 1)$ and $(|z| = 1, |\xi| = 1)$, where $(z, \xi \neq 1)$.

These operators reduce to Saigo fractional derivative operators [2; see also 17] as

$$\left(D_{0+}^{0, \alpha', \beta, \beta', \gamma} f \right)(x) = \left(D_{0+}^{\gamma, \alpha' - \gamma, \beta' - \gamma} f \right)(x), \quad (\Re(\gamma) > 0); \quad (4)$$

and

$$\left(D_-^{0, \alpha', \beta, \beta', \gamma} f \right) (x) = \left(D_-^{\gamma, \alpha' - \gamma, \beta' - \gamma} f \right) (x), \quad (\Re(\gamma) > 0). \quad (5)$$

Further, we also have [2, p. 394, Eqns. (4.18) and (4.19)]

$$\left(I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\sigma-1} \right) (x) = \Gamma \left[\begin{matrix} \sigma, \sigma + \gamma - \alpha - \alpha' - \beta, \sigma + \beta' - \alpha' \\ \sigma + \gamma - \alpha - \alpha', \sigma + \gamma - \alpha' - \beta, \sigma + \beta' \end{matrix} \right] x^{\sigma - \alpha - \alpha' + \gamma - 1} \quad (6)$$

where $\Re(\gamma) > 0, \Re(\sigma) > \max[0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')]$ and

$$\left(I_{-}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\sigma-1} \right) (x) = \Gamma \left[\begin{matrix} 1 + \alpha + \alpha' - \gamma - \sigma, 1 + \alpha + \beta' - \gamma - \sigma, 1 - \beta - \sigma \\ 1 - \sigma, 1 + \alpha + \alpha' + \beta' - \gamma - \sigma, 1 + \alpha - \beta - \sigma \end{matrix} \right] x^{\sigma - \alpha - \alpha' + \gamma - 1} \quad (7)$$

where $\Re(\gamma) > 0, \Re(\sigma) < 1 + \min[\Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)]$.

Here, we have used the symbol $\Gamma \left[\begin{matrix} \dots \\ \dots \end{matrix} \right]$ representing the fraction of many Gamma functions.

In this paper, the multivariable H-function will be defined and represented in the following manner [10, p. 251-252, Eqs. (C.1- C.3)]:

$$\begin{aligned} H[z_1, \dots, z_r] &= H_{P, Q: P_1, Q_1; \dots; P_r, Q_r}^{0, N: M_1, N_1; \dots; M_r, N_r} \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\xi_1, \dots, \xi_r) \prod_{i=1}^r \left\{ \theta_i(\xi_i) z_i^{\xi_i} \right\} d\xi_1 \dots d\xi_r \quad (8) \end{aligned}$$

where $\omega = \sqrt{(-1)}$

$$\theta_i(\xi_i) = \frac{\prod_{j=1}^{M_i} \Gamma(d_j^{(i)} - \delta_j^{(i)} \xi_i) \prod_{j=1}^{N_i} \Gamma(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i)}{\prod_{j=M_i+1}^{Q_i} \Gamma(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i) \prod_{j=N_i+1}^{P_i} \Gamma(c_j^{(i)} - \gamma_j^{(i)} \xi_i)}, \quad \forall i \in \{1, 2, \dots, r\} \quad (9)$$

$$\text{and } \Psi(\xi_1, \dots, \xi_r) = \frac{\prod_{j=1}^N \Gamma(1 - a_j + \sum_{i=1}^r \alpha_j^{(i)} \xi_i)}{\prod_{j=N+1}^P \Gamma(a_j - \sum_{i=1}^r \alpha_j^{(i)} \xi_i) \prod_{j=1}^Q \Gamma(1 - b_j + \sum_{i=1}^r \beta_j^{(i)} \xi_i)} \quad (10)$$

In equation (9), i in the superscript (i) stands for the number of primes, e.g., $b^{(1)} = b', b^{(2)} = b''$ and so on; and an empty product is interpreted as unity.

For convenience, we use the notation $(a_j; \alpha_j', \dots, \alpha_j^{(r)})_{1,P}$ for P member array $(a_1; \alpha_1', \dots, \alpha_1^{(r)}), \dots, (a_P; \alpha_P', \dots, \alpha_P^{(r)})$, while $(c_j^{(i)}, \gamma_j^{(i)})_{1,P_i}$ stands for $(c_1^{(i)}, \gamma_1^{(i)}), \dots, (c_{P_i}^{(i)}, \gamma_{P_i}^{(i)})$, $(i=1, 2, \dots, r)$ and so on. The nature of contours $L_1, \dots, L_r$ in (8), the various special cases and other details of the function are given in the book referred above. Also, it is assumed throughout the present work that the various multivariable H- function occurring herein always satisfy the conditions of their existence corresponding appropriately to those mentioned in the reference given above.

2. Preliminary Results:

Lemma 1. Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ be such that $\Re(\gamma) > 0$, $\Re(\sigma) > -\min.[0, \Re(\alpha + \alpha' + \beta' - \gamma), \Re(\alpha - \beta)]$

Then there holds the relation

$$\left(D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma, t^{\sigma-1}}\right)(x) = \frac{\Gamma(\sigma)\Gamma(\sigma-\gamma+\alpha+\alpha'+\beta')\Gamma(\sigma-\beta+\alpha)}{\Gamma(\sigma-\gamma+\alpha+\beta')\Gamma(\sigma-\gamma+\alpha+\alpha')\Gamma(\sigma-\beta)} x^{\sigma-\gamma+\alpha+\alpha'-1} \quad (11)$$

Lemma 2. Let $\alpha, \alpha', \beta, \beta', \gamma \in C$ be such that $\Re(\gamma) > 0$,

$$\Re(\sigma) < 1 + \min.[\Re(\beta'), \Re(-\alpha' - \beta + \gamma), \Re(-\alpha - \alpha' + \gamma - n)] \quad ; (n = [\Re(\gamma)] + 1)$$

Then there holds the relation

$$\left(D_{-}^{\alpha, \alpha', \beta, \beta', \gamma, t^{\sigma-1}}\right)(x) = \frac{\Gamma(1-\alpha'-\beta+\gamma-\sigma)\Gamma(1+\beta'-\sigma)\Gamma(1-\alpha-\alpha'+\gamma-\sigma)}{\Gamma(1-\sigma)\Gamma(1-\alpha-\alpha'-\beta+\gamma-\sigma)\Gamma(1-\alpha'+\beta'-\sigma)} x^{\sigma-\gamma+\alpha+\alpha'-1} \quad (12)$$

3. Main Results

We establish image formulae for the multivariable H-function involving Saigo and Maeda fractional differential operators (1) and (2) in terms of the multivariable H-function.

Theorem 3.1. Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma, z_i \in C, x > 0, \rho_i > 0, \forall i \in \{1, 2, \dots, r\}$ be such that

$$\Re(\gamma) > 0, \Re(\sigma) + \sum_{i=1}^r \rho_i \min_{1 \leq j \leq M_i} \Re\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -\min[0, \Re(\alpha + \alpha' + \beta' - \gamma), \Re(\alpha - \beta)]$$

and $|\arg z_i| < \frac{1}{2}\Omega_i\pi$, $\Omega_i > 0$

where

$$\Omega_i = -\sum_{j=N+1}^P \alpha_j^{(i)} - \sum_{j=1}^Q \beta_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=N_i+1}^{P_i} \gamma_j^{(i)} + \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=M_i+1}^{Q_i} \delta_j^{(i)} > 0, \forall i \in \{1, 2, \dots, r\}$$

then the following formula holds

$$\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left[t^{\sigma-1} {}_H^{0, N: M_1, N_1; \dots; M_r, N_r}_{P, Q: P_1, Q_1; \dots; P_r, Q_r} \left[\begin{matrix} z_1 t^{\rho_1} \\ \vdots \\ z_r t^{\rho_r} \end{matrix} \left| \begin{matrix} (a_j; \alpha_j', \dots, \alpha_j^{(r)})_{1, P} : (c_j', \gamma_j')_{1, P_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)})_{1, P_r} \\ (b_j; \beta_j', \dots, \beta_j^{(r)})_{1, Q} : (d_j', \delta_j')_{1, Q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1, Q_r} \end{matrix} \right. \right] \right] \right\}^{(x)}$$

$$= {}_x^{\sigma - \gamma + \alpha + \alpha' - 1} {}_H^{0, N+3: M_1, N_1; \dots; M_r, N_r}_{P+3, Q+3: P_1, Q_1; \dots; P_r, Q_r} \left[\begin{matrix} z_1^x \rho_1 \\ \vdots \\ z_r^x \rho_r \end{matrix} \left| \begin{matrix} A^* : C^* \\ B^* : D^* \end{matrix} \right. \right] \quad (13)$$

where

$$A^* = (1-\sigma, \rho_1, \dots, \rho_r), (1-\sigma+\gamma-\alpha-\alpha'-\beta'; \rho_1, \dots, \rho_r), (1-\sigma-\alpha+\beta; \rho_1, \dots, \rho_r), (a_j; \alpha_j', \dots, \alpha_j^{(r)})_{1, P}$$

$$B^* = (1-\sigma+\gamma-\alpha-\beta'; \rho_1, \dots, \rho_r), (1-\sigma+\gamma-\alpha-\alpha'; \rho_1, \dots, \rho_r), (1-\sigma+\beta; \rho_1, \dots, \rho_r), (b_j; \beta_j', \dots, \beta_j^{(r)})_{1, Q}$$

$$C^* = (c_j', \gamma_j')_{1, P_1} ; \dots ; (c_j^{(r)}, \gamma_j^{(r)})_{1, P_r}, \quad D^* = (d_j', \delta_j')_{1, Q_1} ; \dots ; (d_j^{(r)}, \delta_j^{(r)})_{1, Q_r}$$

Proof: In order to prove (13), we first express the multivariable H-function occurring in the left hand side of (13) in terms of Mellin-Barnes contour integral with the help of equation (8) and interchanging the order of integration, we obtain (say I_1)

$$I_1 = \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \Psi(\xi_1, \dots, \xi_r) \prod_{i=1}^r \left\{ \theta_i(\xi_i) z_i^{\xi_i} \right\} \left(D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\sigma + \rho_1 \xi_1 + \dots + \rho_r \xi_r - 1} \right) (x) d\xi_1 \dots d\xi_r$$

Now, applying the equation (11) with σ replaced by $\sigma + \rho_1 \xi_1 + \dots + \rho_r \xi_r$, we arrive at

$$I_1 = \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \Psi(\xi_1, \dots, \xi_r) \prod_{i=1}^r \left\{ \theta_i(\xi_i) z_i^{\xi_i} \right\} \frac{\Gamma(\sigma + \rho_1 \xi_1 + \dots + \rho_r \xi_r) \Gamma(\sigma - \gamma + \alpha + \alpha' + \beta + \rho_1 \xi_1 + \dots + \rho_r \xi_r)}{\int \Gamma(\sigma - \gamma + \alpha + \beta + \rho_1 \xi_1 + \dots + \rho_r \xi_r) \Gamma(\sigma - \beta + \rho_1 \xi_1 + \dots + \rho_r \xi_r)}$$

$$\times \frac{\Gamma(\sigma - \beta + \alpha + \rho_1 \xi_1 + \dots + \rho_r \xi_r)}{\Gamma(\sigma - \gamma + \alpha + \alpha' + \rho_1 \xi_1 + \dots + \rho_r \xi_r)} x^{\sigma - \gamma + \alpha + \alpha' + \rho_1 \xi_1 + \dots + \rho_r \xi_r - 1} d\xi_1 \dots d\xi_r$$

Finally, interpreting the Mellin-Barnes contour integral thus obtained in terms of the multivariable H-function as given in (8), we obtain the result as given in (13).

4. Special Cases of Theorem 3.1:

- i. If we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in Theorem 3.1, we get known result obtained by Gupta et al. [3, p. 48, Eq. (18)], as

$$\left\{ D_{0+}^{\alpha, \beta, \eta} \left(t^{\sigma-1} H_{P, Q: P_1, Q_1; \dots; P_r, Q_r}^{0, N: M_1, N_1; \dots; M_r, N_r} \left[\begin{matrix} \rho_1 \\ z_1 t \\ \vdots \\ \rho_r \\ z_r t \end{matrix} \right] \right) \right\} (x) \\ = x^{\sigma + \beta - 1} H_{P+2, Q+2: P_1, Q_1; \dots; P_r, Q_r}^{0, N+2: M_1, N_1; \dots; M_r, N_r} \left[\begin{matrix} \rho_1 \\ z_1 x \\ \vdots \\ \rho_r \\ z_r x \end{matrix} \middle| \begin{matrix} A_1^* : C^* \\ B_1^* : D^* \end{matrix} \right] \quad (14)$$

where

$$A_1^* = (1 - \sigma; \rho_1, \dots, \rho_r), (1 - \sigma - \alpha - \beta - \eta; \rho_1, \dots, \rho_r), \left(a_j; \alpha_j', \dots, \alpha_j^{(r)} \right)_{1, P} \\ B_1^* = (1 - \sigma - \beta; \rho_1, \dots, \rho_r), (1 - \sigma - \eta; \rho_1, \dots, \rho_r), \left(b_j; \beta_j', \dots, \beta_j^{(r)} \right)_{1, Q}$$

C^* and D^* are same as given in (13) and the conditions of existence of the above result can be easily derived with the help of Theorem 3.1.

- ii. If we take $N = P = Q = 0$ in equation (13), the multivariable H-function reduces to the product of r-different Fox H-functions [18, p. (viii), Eq. (A.14)] following as:

$$\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} \prod_{i=1}^r H_{P_i, Q_i}^{M_i, N_i} \left[\begin{matrix} \rho_i \\ z_i t \end{matrix} \right] \right) \right\} (x) \\ = x^{\sigma - \gamma + \alpha + \alpha' - 1} H_{3, 3: P_1, Q_1; \dots; P_r, Q_r}^{0, 3: M_1, N_1; \dots; M_r, N_r} \left[\begin{matrix} \rho_1 \\ z_1 x \\ \vdots \\ \rho_r \\ z_r x \end{matrix} \middle| \begin{matrix} A_2^* : C^* \\ B_2^* : D^* \end{matrix} \right] \quad (15)$$

where

$$A_2^* = (1 - \sigma; \rho_1, \dots, \rho_r), (1 - \sigma + \gamma - \alpha - \alpha' - \beta'; \rho_1, \dots, \rho_r), (1 - \sigma - \alpha + \beta; \rho_1, \dots, \rho_r) \\ B_2^* = (1 - \sigma + \gamma - \alpha - \beta'; \rho_1, \dots, \rho_r), (1 - \sigma + \gamma - \alpha - \alpha'; \rho_1, \dots, \rho_r), (1 - \sigma + \beta; \rho_1, \dots, \rho_r)$$

C^* and D^* are same as mentioned in equation (13) and the conditions of validity of the above result can be easily derived with the help of Theorem 3.1.

iii. If we reduce product of r - different H-functions occurring in the left hand side of (15), to the r - product of Bessel function of first kind [10, p.18, Eq. (2.6.5)], we get

$$\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} \prod_{j=1}^r J_{\nu_j} \left(a_j t^{\rho_j} \right) \right) \right\} (x) \\ = x^{\sigma-\gamma+\alpha+\alpha'-1} \left\{ \prod_{j=1}^r \left(a_j x^{\rho_j} / 2 \right)^{\nu_j} \right\} H_{3,3:1,0;...;1,0}^{0,3:0,2;...;0,2} \left[\begin{matrix} \left(a_1 x^{\rho_1} / 2 \right)^2 \\ \vdots \\ \left(a_r x^{\rho_r} / 2 \right)^2 \end{matrix} \middle| \begin{matrix} A_3^* :- \\ B_3^* : D_3^* \end{matrix} \right] \quad (16)$$

where

$$A_3^* = \left(1 - \sigma - \sum_{j=1}^r \rho_j \nu_j; 2\rho_1, \dots, 2\rho_r \right), \left(1 - \sigma + \gamma - \alpha - \alpha' - \beta' - \sum_{j=1}^r \rho_j \nu_j; 2\rho_1, \dots, 2\rho_r \right), \left(1 - \sigma - \alpha + \beta - \sum_{j=1}^r \rho_j \nu_j; 2\rho_1, \dots, 2\rho_r \right) \\ B_3^* = \left(1 - \sigma + \gamma - \alpha - \beta' - \sum_{j=1}^r \rho_j \nu_j; 2\rho_1, \dots, 2\rho_r \right), \left(1 - \sigma + \gamma - \alpha - \alpha' - \sum_{j=1}^r \rho_j \nu_j; 2\rho_1, \dots, 2\rho_r \right), \left(1 - \sigma + \beta - \sum_{j=1}^r \rho_j \nu_j; 2\rho_1, \dots, 2\rho_r \right) \\ D_3^* = (0, 1), (-\nu_1, 1); \dots; (0, 1), (-\nu_r, 1)$$

iv. Now, reducing the H-function of several variables occurring in the right hand side of (16) to generalized Lauricella function [19], we get

$$\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} \prod_{j=1}^r J_{\nu_j} \left(a_j t^{\rho_j} \right) \right) \right\} (x) = x^{\sigma-\gamma+\alpha+\alpha'-1} \left\{ \prod_{j=1}^r \frac{\left(\frac{a_j x^{\rho_j}}{2} \right)^{\nu_j}}{\Gamma(\nu_j + 1)} \right\} \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(p)\Gamma(q)\Gamma(s)} \\ \times F_{3:1;...;1}^{3:0;...;0} \left[\begin{matrix} (l; 2\rho_1, \dots, 2\rho_r), (m; 2\rho_1, \dots, 2\rho_r), (n; 2\rho_1, \dots, 2\rho_r) \\ (p; 2\rho_1, \dots, 2\rho_r), (q; 2\rho_1, \dots, 2\rho_r), (s; 2\rho_1, \dots, 2\rho_r) : (v_1 + 1, 1); \dots; (v_r + 1, 1) \end{matrix} \middle| - \left(\frac{a_1 x^{\rho_1}}{2} \right)^2, \dots, - \left(\frac{a_r x^{\rho_r}}{2} \right)^2 \right] \quad (17)$$

where

$$l = \sigma + \sum_{j=1}^r \rho_j \nu_j, m = \sigma - \gamma + \alpha + \alpha' + \beta' + \sum_{j=1}^r \rho_j \nu_j, n = \sigma + \alpha - \beta + \sum_{j=1}^r \rho_j \nu_j \\ p = \sigma - \gamma + \alpha + \beta' + \sum_{j=1}^r \rho_j \nu_j, q = \sigma - \gamma + \alpha + \alpha' + \sum_{j=1}^r \rho_j \nu_j, s = \sigma - \beta + \sum_{j=1}^r \rho_j \nu_j$$

v. If we take $r = 1, a_1 = 1, \rho_1 = 1, \nu_1 = \nu$ in above equation (17), we get

$$\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} J_{\nu}(t) \right) \right\} (x) = \frac{x^{\sigma+\nu-\gamma+\alpha+\alpha'-1}}{2^{\nu}}$$

$$\times {}_3\psi_4 \left[\begin{matrix} (\sigma+v, 2), (\sigma+v-\gamma+\alpha+\alpha'+\beta', 2), (\sigma+v+\alpha-\beta, 2) \\ (\sigma+v-\gamma+\alpha+\beta', 2), (\sigma+v-\gamma+\alpha+\alpha', 2), (\sigma+v-\beta, 2), (v+1, 1) \end{matrix}; -\frac{x^2}{4} \right] \quad (18)$$

Further, taking $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in above equation (18), we get known result due to Kilbas and Sebastian [6, p. 330, Eq. (32)], as

$$\left\{ D_{0+}^{\alpha, \beta, \eta} \left(t^{\sigma-1} J_v(t) \right) \right\} (x) = \frac{x^{\sigma+v+\beta-1}}{2^v} {}_2\psi_3 \left[\begin{matrix} (\sigma+v, 2), (\sigma+v+\alpha+\beta+\eta, 2) \\ (\sigma+v+\beta, 2), (\sigma+v+\eta, 2), (v+1, 1) \end{matrix}; -\frac{x^2}{4} \right]$$

vi. If we reduce the multivariable H- function into the product of two Fox H-function and then reduce one H- function to the exponential function by taking $\rho_1 = 1$ in equation (13), we get the following result after a little simplification:

$$\begin{aligned} & \left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} e^{-z_1 t} H_{P_2, Q_2}^{M_2, N_2} \left[z_2 t^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P_2} \\ (d_j, \delta_j)_{1, Q_2} \end{matrix} \right. \right] \right] \right\} (x) \\ &= {}_x^{\sigma-\gamma+\alpha+\alpha'-1} H_{3, 3: 0, 1; P_2, Q_2}^{0, 3: 1, 0; M_2, N_2} \left[\begin{matrix} z_1 x \\ z_2 x^{\rho_2} \end{matrix} \left| \begin{matrix} A_4^* : -; (c_j, \gamma_j)_{1, P_2} \\ B_4^* : (0, 1); (d_j, \delta_j)_{1, Q_2} \end{matrix} \right. \right] \end{aligned} \quad (19)$$

where

$$\begin{aligned} A_4^* &= (1-\sigma; 1, \rho_2), (1-\sigma+\gamma-\alpha-\alpha'-\beta'; 1, \rho_2), (1-\sigma-\alpha+\beta; 1, \rho_2) \\ B_4^* &= (1-\sigma+\gamma-\alpha-\beta'; 1, \rho_2), (1-\sigma+\gamma-\alpha-\alpha'; 1, \rho_2), (1-\sigma+\beta; 1, \rho_2) \end{aligned}$$

The conditions of validity of the above result can be derived with the help of Theorem 3.1.

Further, on letting $z_1 \rightarrow 0$ in the above equation (19), it becomes

$$\begin{aligned} & \left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} H_{P_2, Q_2}^{M_2, N_2} \left[z_2 t^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P_2} \\ (d_j, \delta_j)_{1, Q_2} \end{matrix} \right. \right] \right] \right\} (x) \\ &= {}_x^{\sigma-\gamma+\alpha+\alpha'-1} H_{P_2+3, Q_2+3}^{M_2, N_2+3} \left[z_2 x^{\rho_2} \left| \begin{matrix} A_5^* \\ B_5^* \end{matrix} \right. \right] \end{aligned} \quad (20)$$

where

$$\begin{aligned} A_5^* &= (c_j, \gamma_j)_{1, N_2}; (1-\sigma, \rho_2), (1-\sigma+\gamma-\alpha-\alpha'-\beta', \rho_2), (1-\sigma-\alpha+\beta, \rho_2), (c_j, \gamma_j)_{N_2+1, P_2} \\ B_5^* &= (d_j, \delta_j)_{1, Q_2}; (1-\sigma+\gamma-\alpha-\beta', \rho_2), (1-\sigma+\gamma-\alpha-\alpha', \rho_2), (1-\sigma+\beta, \rho_2) \end{aligned}$$

Now, we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in above equation (20), we get known result due to Gupta et al. [3, p. 51, Eq. (24)], as

$$\left\{ D_{0+}^{\alpha, \beta, \eta} \left[t^{\sigma-1} {}_H^{M_2, N_2}_{P_2, Q_2} \left[z_2 t^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P_2} \\ (d_j, \delta_j)_{1, Q_2} \end{matrix} \right. \right] \right] \right\} (x) \\ = {}_x^{\sigma+\beta-1} {}_H^{M_2, N_2+2}_{P_2+2, Q_2+2} \left[z_2 x^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, N_2}, (1-\sigma, \rho_2), (1-\sigma-\alpha-\beta-\eta, \rho_2), (c_j, \gamma_j)_{N_2+1, P_2} \\ (d_j, \delta_j)_{1, Q_2}, (1-\sigma-\beta, \rho_2), (1-\sigma-\eta, \rho_2) \end{matrix} \right. \right] \quad (21)$$

Again, taking $\beta = -\alpha$ in the above equation and make suitable adjustment in the parameters, we get also known result recorded in the book by Kilbas and Saigo [5, p. 55, Eq. (2.7.22)], as

$$\left\{ D_{0+}^{\alpha} \left[t^{\sigma-1} {}_H^{M_2, N_2}_{P_2, Q_2} \left[z_2 t^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P_2} \\ (d_j, \delta_j)_{1, Q_2} \end{matrix} \right. \right] \right] \right\} (x) \\ = {}_x^{\sigma-\alpha-1} {}_H^{M_2, N_2+1}_{P_2+1, Q_2+1} \left[z_2 x^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, N_2}, (1-\sigma, \rho_2), (c_j, \gamma_j)_{N_2+1, P_2} \\ (d_j, \delta_j)_{1, Q_2}, (1-\sigma+\alpha, \rho_2) \end{matrix} \right. \right] \quad (22)$$

vii. If we reduce the H- function to generalized Wright hypergeometric function [10, p. 19, Eq. (2.6.11)] in equation (20) by taking $M_2 = M, N_2 = N, P_2 = P, Q_2 = Q$, we get

$$\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left[t^{\sigma-1} {}_P\Psi_Q \left[z_2 t^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P} \\ (d_j, \delta_j)_{1, Q} \end{matrix} \right. \right] \right] \right\} (x) \\ = {}_x^{\sigma-\gamma+\alpha+\alpha'-1} {}_{P+3}\Psi_{Q+3} \left[z_2 x^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P}, (\sigma, \rho_2), (\sigma-\gamma+\alpha+\alpha'+\beta, \rho_2), (\sigma+\alpha-\beta, \rho_2) \\ (d_j, \delta_j)_{1, Q}, (\sigma-\gamma+\alpha+\beta', \rho_2), (\sigma-\gamma+\alpha+\alpha', \rho_2), (\sigma-\beta, \rho_2) \end{matrix} \right. \right] \quad (23)$$

Now, we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in equation (23), we get known result due to Gupta et al. [3, p. 52, Eq. (25)], as

$$\left\{ D_{0+}^{\alpha, \beta, \eta} \left[t^{\sigma-1} {}_P\Psi_Q \left[z_2 t^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P} \\ (d_j, \delta_j)_{1, Q} \end{matrix} \right. \right] \right] \right\} (x) = {}_x^{\sigma+\beta-1} {}_{P+2}\Psi_{Q+2} \left[z_2 x^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P}, (\sigma, \rho_2), (\sigma+\alpha+\beta+\eta, \rho_2) \\ (d_j, \delta_j)_{1, Q}, (\sigma+\beta, \rho_2), (\sigma+\eta, \rho_2) \end{matrix} \right. \right]$$

Further, taking $\beta = -\alpha$ in the above result, we get also known result obtained by Kilbas [4, p. 119, Eq. (14)], as

$$\left\{ D_{0+}^{\alpha} \left[t^{\sigma-1} {}_P\Psi_Q \left[z_2 t^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P} \\ (d_j, \delta_j)_{1, Q} \end{matrix} \right. \right] \right] \right\} (x) = {}_x^{\sigma-\alpha-1} {}_{P+1}\Psi_{Q+1} \left[z_2 x^{\rho_2} \left| \begin{matrix} (c_j, \gamma_j)_{1, P}, (\sigma, \rho_2) \\ (d_j, \delta_j)_{1, Q}, (\sigma-\alpha, \rho_2) \end{matrix} \right. \right]$$

viii. If we put $z_2 = -1, \rho_2 = 1$ in equation (20) and reducing the H-function into generalized Mittag-Leffler function [20], we get

$$\left\{ D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} E_{u,v}^{\rho} (t) \right) \right\} (x) = \frac{x^{\sigma-\gamma+\alpha+\alpha'-1}}{\Gamma(\rho)} \times H_{4,5}^{1,4} \left[-x \mid \begin{matrix} (1-\rho, 1), (1-\sigma, 1), (1-\sigma+\gamma-\alpha-\alpha'-\beta', 1), (1-\sigma-\alpha+\beta, 1) \\ (0, 1), (1-\sigma+\gamma-\alpha-\beta', 1), (1-\sigma+\gamma-\alpha-\alpha', 1), (1-\sigma+\beta, 1), (1-v, u) \end{matrix} \right] \quad (24)$$

Now, we put $\alpha = \alpha + \beta, \alpha' = \beta' = 0, \beta = -\eta, \gamma = \alpha$ in equation (24), we get known result due to Gupta et al. [3, p. 52, Eq. (27)], as

$$\left\{ D_{0+}^{\alpha, \beta, \eta} \left(t^{\sigma-1} E_{u,v}^{\rho} (t) \right) \right\} (x) = \frac{x^{\sigma+\beta-1}}{\Gamma(\rho)} H_{3,4}^{1,3} \left[-x \mid \begin{matrix} (1-\rho, 1), (1-\sigma, 1), (1-\sigma-\alpha-\beta-\eta, 1) \\ (0, 1), (1-\sigma-\beta, 1), (1-\sigma-\eta, 1), (1-v, u) \end{matrix} \right]$$

Further, taking $\beta = -\alpha$ in the above result, we get also known result due to Saxena et al. [7, p. 170, Eq. (2.6)], as

$$\left\{ D_{0+}^{\alpha} \left(t^{\sigma-1} E_{u,v}^{\rho} (t) \right) \right\} (x) = \frac{x^{\sigma-\alpha-1}}{\Gamma(\rho)} H_{2,3}^{1,2} \left[-x \mid \begin{matrix} (1-\rho, 1), (1-\sigma, 1) \\ (0, 1), (1-\sigma+\alpha, 1), (1-v, u) \end{matrix} \right]$$

ix. If we take $M_2 = N_2 = P_2 = 1, Q_2 = 2, z_2 = -a, \rho_2 = \beta, \sigma = \gamma, c_1 = 1 - \delta, \gamma_1 = 1, d_1 = 0, \delta_1 = 1, d_2 = 1 - \gamma, \delta_2 = \beta$ in equation (22), we get known result due to Saxena and Saigo [8, p. 149, Eq. (29)], as

$$\left\{ D_{0+}^{\alpha} \left(t^{\gamma-1} E_{\beta, \gamma}^{\delta} \left(at^{\beta} \right) \right) \right\} (x) = x^{\gamma-\alpha-1} E_{\beta, \gamma-\alpha}^{\delta} \left(ax^{\beta} \right)$$

Theorem 3.2. Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma, z_i \in \mathbb{C}, x > 0, \rho_i > 0, \forall i \in \{1, 2, \dots, r\}$ be such that $\Re(\gamma) > 0$,

$$\Re(\sigma) - \sum_{i=1}^r \rho_i \min_{1 \leq j \leq M_i} \Re \left(\frac{d_j^{(i)}}{\delta_j^{(i)}} \right) < 1 + \min [\Re(\beta'), \Re(-\alpha' - \beta + \gamma), \Re(-\alpha - \alpha' + \gamma - n)]$$

and

$$|\arg z_i| < \frac{1}{2} \Omega_i \pi, \Omega_i > 0$$

where

$$\Omega_i = - \sum_{j=N_i+1}^P \alpha_j^{(i)} - \sum_{j=1}^Q \beta_j^{(i)} + \sum_{j=1}^{N_i} \gamma_j^{(i)} - \sum_{j=N_i+1}^{P_i} \gamma_j^{(i)} + \sum_{j=1}^{M_i} \delta_j^{(i)} - \sum_{j=M_i+1}^{Q_i} \delta_j^{(i)} > 0, \forall i \in \{1, 2, \dots, r\}$$

then the following formula holds

$$\begin{aligned}
 & \left\{ D_{-}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\sigma-1} {}_H_{P, Q: P_1, Q_1; \dots; P_r, Q_r}^{0, N: M_1, N_1; \dots; M_r, N_r} \left[\begin{matrix} -\rho_1 \\ z_1 t \\ \vdots \\ -\rho_r \\ z_r t \end{matrix} \right] \right) \right\} (x) \\
 &= {}_x^{\sigma-\gamma+\alpha+\alpha'-1} {}_H_{P+3, Q+3: P_1, Q_1; \dots; P_r, Q_r}^{0, N+3: M_1, N_1; \dots; M_r, N_r} \left[\begin{matrix} -\rho_1 \\ z_1 x \\ \vdots \\ -\rho_r \\ z_r x \end{matrix} \middle| \begin{matrix} A^{**}: C^* \\ B^{**}: D^* \end{matrix} \right] \quad (25)
 \end{aligned}$$

where

$$\begin{aligned}
 A^{**} &= (\sigma-\gamma+\alpha'+\beta; \rho_1, \dots, \rho_r), (\sigma-\beta'; \rho_1, \dots, \rho_r), (\sigma-\gamma+\alpha+\alpha'; \rho_1, \dots, \rho_r), (a_j; \alpha'_j, \dots, \alpha_j^{(r)})_{1, P} \\
 B^{**} &= (\sigma; \rho_1, \dots, \rho_r), (\sigma-\gamma+\alpha+\alpha'+\beta; \rho_1, \dots, \rho_r), (\sigma+\alpha'-\beta'; \rho_1, \dots, \rho_r), (b_j; \beta'_j, \dots, \beta_j^{(r)})_{1, Q} \\
 C^* \text{ and } D^* &\text{ are same as given in (13).}
 \end{aligned}$$

Proof: The proof of Theorem 3.2 can be developed on the lines similar to those given with proof of Theorem 3.1 with the help of equation (12).

A number of several special cases of Theorem 3.2 can also be obtained similarly.

References:

- [1] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F.G., 1953, "Higher Transcendental Functions, " Vol. I, Mc Graw Hill Book Company, New York, Toronto and London.
- [2] Saigo, M. and Maeda, N., 1996, "More Generalization of Fractional Calculus. Transform Methods and Special Functions, " Varna, Bulgaria, pp. 386-400.
- [3] Gupta, K.C., Gupta, K. and Gupta, A., 2011, "Generalized Fractional Differentiation of the Multivariable H-Function, " J. of the Applied Mathematics, Statistics and Informatics (JAMSI), 7(2), pp. 45-54.
- [4] Kilbas, A. A., 2005, "Fractional Calculus of the Generalized Wright Function, " Fract. Calc. and Applied Anal., 8(2), pp.113-126.
- [5] Kilbas, A. A. and Saigo, M., 2004, "H-Transforms Theory and Applications, " Chapman & Hall/ CRC London, New York.
- [6] Kilbas, A. A. and Sebastian, N., 2008, "Generalized Fractional Differentiation of Bessel Function of the First Kind, " Mathematica Balkanica, 22, pp.323-346.
- [7] Saxena, R. K., Ram, J. and Suthar, D. L., 2009, "Fractional Calculus of Generalized Mittag-Leffler Functions, " J. Indian Acad. Math., 31(1), pp. 165-172.

- [8] Saxena, R. K. and Saigo, M., 2005, "Certain Properties of Fractional Calculus Operators Associated with Generalized Mittag- Leffler Function, " *Fract. Calc. and Applied Anal.*, 8(2), pp.141-154.
- [9] Kiryakova, V., 2008, "A Brief Story About the Operators of the Generalized Fractional Calculus, " *Fract. Calc. Appl. Anal.*, 11(2), pp. 203-220.
- [10] Srivastava, H. M., Gupta, K. C. and Goyal, S. P., 1982, "The H- Functions of One and Two Variables with Applications, " South Asian Publications, New Delhi, Madras.
- [11] Saxena, R. K., Ram, J. and Kumar, D., 2012, "Generalized Fractional Differentiation for Saigo Operators Involving Aleph-Function, " *J. Indian Acad. Math.*, 34(1), pp. 109-115.
- [12] Saxena, R. K., Ram, J. and Kumar, D., 2012, "Generalized Fractional Differentiation of the Aleph-Function Associated with the Appell Function F_3 , " *Acta Ciencia Indica*, 38(4), pp. 781-792.
- [13] Saigo, M., 1978, "A Remark on Integral Operators Involving the Gauss Hypergeometric Functions, " *Math. Rep. Kyushu Univ.*, 11, pp. 135-143.
- [14] Samko, S. G., Kilbas, A. A. and Marichev, O. I., 1993, "Fractional Integrals and Derivatives, Theory and Applications, " Gordon and Breach Sci. Publ., New York.
- [15] Miller, K. S., and Ross, B., 1993, "An Introduction to Fractional Calculus and Fractional Differential Equations, John Wiley and Sons Inc., New York.
- [16] Kumar, D. and Daiya, J., 2013, "Generalized Fractional Differentiation of the $\bar{H}$ -Function Involving General Class of Polynomials, " *Int. J. Pure Appl. Sci. Technol.*, 16(2), pp. 42-53.
- [17] Saxena, R. K. and Saigo, M., 2001, "Generalized Fractional Calculus of the H-Function Associated with Appell Function F_3 , " *J. Fract. Calc.*, 19, pp. 89-104.
- [18] Mukherjee, R., 2004, "A Study of General Sequence of Functions, Multidimensional Fractional Calculus Operators and the General H- Function of One or More Variables with Applications, " Ph.D. thesis, Univ. of Rajasthan, Jaipur, India.
- [19] Srivastava, H. M., Daoust, M. C., 1972, "A Note on the Convergence of Kampe De Fariet's Double Hypergeometric Series, " *Math. Nachr.*, 53, pp.151-159.
- [20] Prabhakar, T. R., 1971, "A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel, " *Yokohama Math. J.*, 19, pp.7-15.