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Generalized Fractional Integral Associated with H-function

Vijay Kumar Singhal

Department of Mathematics, Dr. Radhakrishnan Institute of Technology, Jaipur–302037, India

Abstract

The present paper aims at the study and derivation of Saigo generalized fractional integral operator involving product of H-function of one variable and general class of polynomials. On account of the most general nature of the operator, H-function and general class of polynomials occurring in the main result, a large number of known and new results involving Rieman-Liouville, Erdélyi- Kober Fractional differential operators, Bessel function, Mittag-leffler function, Wright hypergeometric function follows as special cases of our main finding.

Keywords: Saigo fractional integral operator, H-function, general class of polynomials, Appel function.

1. Introduction

H-function of one variable is defined by Srivastava, Gupta and Goyal [17]

$$H_{P,Q}^{M,N}\left[z \,\middle|\, \begin{matrix} (e_P, E_P) \\ (f_Q, F_Q) \end{matrix} \right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds \qquad \text{...(1.1)}$$

where

$$\Phi(s) = \frac{\prod\limits_{\substack{j=1 \\ j\neq g}}^{M} \Gamma(f_j - F_j s) \prod\limits_{j=1}^{N} \Gamma(1 - e_j + E_j s)}{\prod\limits_{j=M+1}^{Q} \Gamma(1 - f_j + F_j s) \prod\limits_{j=N+1}^{P} \Gamma(e_j - E_j s)}$$

with all conditions given in [17].

The H-function of several complex variables introduced by Srivastava and Panda [18, p.265]. This function is defined and represented in the following manner:

$$H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\ldots;[B^{(r)},D^{(r)}]}$$

$$\left[\begin{array}{l|l} [(a):\theta';\ldots;\theta^{(r)}]:[b':\phi'];\ldots;[b^{(r)}:\phi^{(r)}] & \\ & z_1,\ldots,z_r \\ [(c):\psi';\ldots;\psi^{(r)}]:[d':\delta'];\ldots;[d^{(r)}:\delta^{(r)}] & \end{array}\right]$$

$$=\frac{1}{(2\pi w)^r}\int_{L_1}\cdots\int_{L_r} T(s_1,\ldots,s_r)\,R_1(s_1)\cdots R_r(s_r)z_1^{s_1}\cdots z_r^{s_r}\,ds_1\cdots ds_r \quad (1.2)$$

...

where

$$w=\sqrt{(-1)}\ ,$$

$$R_i(s_i)=\frac{\prod_{j=1}^{u^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}s_i)\prod_{j=1}^{v^{(i)}}\Gamma(1-b_j^{(i)}+\phi_j^{(i)}s_i)}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}s_i)}\ ,\forall\ (i=1,2,\ldots,r),$$

$$T(s_1,\ldots,s_r)=\frac{\prod_{j=1}^{\lambda}\Gamma(1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}s_i)}{\prod_{j=\lambda+1}^{A}\Gamma(a_j-\sum_{i=1}^{r}\theta_j^{(i)}s_i)\prod_{j=1}^{C}\Gamma(1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}s_i)},$$

and an empty product is interpreted as unity.

The general class of polynomials introduced and studied by Srivastava [15] as follows:

$$S_N^M[x]=\sum_{k=0}^{[N/M]}(-N)_{Mk}\,A_{N,k}\,\frac{x^k}{k!}\ ,N=0,1,2,\ldots \quad .(1.3)$$

where m is an arbitrary positive integer and the coefficients $A_{N,k}$ $(N,k\geq 0)$ are arbitrary constants, real or complex.

The Saigo fractional integral operator ([11], [19]) is defined as

$$I_{0,x}^{p,q,\gamma}f(x)=\begin{cases}\dfrac{x^{-p-q}}{\Gamma(p)}\displaystyle\int_0^x(x-t)^{p-1}F(p+q,-\gamma;p;1-\frac{t}{x})f(t)dt & (\mathrm{Re}(p)>0)\\ \dfrac{d^r}{dx^r}I_{0,x}^{p+r,q-r,\gamma-r}f(x), & (\mathrm{Re}(p)\leq 0,\ 0<\mathrm{Re}(p)+r\leq 1, r=1,2,\ldots)\end{cases} \quad (1.4)$$

...

where F is the Gauss hypergeometric function.

Saigo fractional integral operator contains as special cases the Riemann-Liouville and Erdély- Kober operators of Fractional Integration of order α > 0 ([14], [5]):

$$I_{0,z}^{\alpha,-\alpha,-\alpha} f(z) = R^{\alpha} f(z) = \frac{z^{\alpha}}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(tz)dt$$

$$z^{-\alpha-\gamma} I_{0,z}^{\alpha,-\alpha-\gamma,-\alpha} f(z) = I_1^{\gamma,\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} t^{\gamma} f(zt\ \ )dt \qquad (\alpha > 0, \gamma \in \mathrm{R})$$

Let α, α', β, β' ∈ R and γ > 0, then Saigo generalized fractional integral operator [11] of a function $f(x)$ is defined by

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma} f(z) = \frac{z^{-\alpha}}{\Gamma(\gamma)} \int_0^z (z-t)^{\gamma-1} t^{-\alpha'} F_3(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{t}{z},1-\frac{z}{t}) f(t)dt \qquad \gamma > 0 \qquad .(1.5)$$

Where $f(z)$ is analytic in a simply connected region of z-plane. Principal value for $0 \le \arg(z-t) \le 2\pi$ is denoted by $(z-t)^{\gamma-1}$

F₃ denote the Appell hypergeometric function of third type, also known as Horn's **F₃** function,

$$F_3(\alpha,\alpha';\beta,\beta';\gamma;z,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_m(\alpha')_n(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{z^m t^n}{m!n!} \qquad |z| < 1, |t| < 1$$

Following Lemma [11, p.394]; see also [6] will be required in the sequel:

Lemma: Let $\mathrm{Re}(\gamma) > 0, k > \max[0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), \mathrm{Re}(\alpha'-\beta')] - 1$ then

$$I_{0,z}^{\alpha,\alpha',\beta,\beta',\gamma}[z^k] = \frac{\Gamma(1+k)\Gamma(1+k-\alpha'+\beta')\Gamma(1+k-\alpha-\alpha'-\beta+\gamma)}{\Gamma(1+k+\beta')\Gamma(1+k-\alpha'-\beta+\gamma)\Gamma(1+k-\alpha-\alpha'+\gamma)} z^{k-\alpha-\alpha'+\gamma} \qquad ..(1.6)$$

2. Main Result

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\mu-1}(b-at)^{-\upsilon} S_N^M\left(t^{\lambda}(b-at)^{-\delta}\right) H_{p,q}^{m,n}\left(wt^{\sigma}(b-at)^{-s} \middle| \begin{matrix} (a_j,A_j)_{1,p} \\ (b_j,B_j)_{1,q} \end{matrix} \right) \right]$$

$$= b^{-\upsilon} t^{\mu-\alpha-\alpha'+\upsilon-1} \sum_{k=0}^{[N/M]} A_{N,k} \frac{(-N)_{Mk}}{k!} b^{-\delta k} t^{\lambda k}$$

$$H_{4,3:p,q+1;0,1}^{0,4:m,n;0,0}\left[\begin{matrix} wb^{-s}t^{\sigma} \\ -\dfrac{a}{b}t \end{matrix} \middle| \begin{matrix} (1-\mu-\lambda k;\sigma,1),(1-\mu-\lambda k+\alpha'-\beta';\sigma,1) \\ (1-\mu-\lambda k-\beta';\sigma,1),(1-\mu-\lambda k+\alpha'+\beta'-\upsilon;\sigma,1) \end{matrix} \right]$$

$$\left.\begin{matrix}(1-\mu-\lambda k+\alpha+\alpha'+\beta;\sigma,1),(1-\upsilon-\delta k;s,1):(a_j,A_j)_{1,p};(0,0)\\(1-\mu-\lambda k+\alpha+\alpha'-\gamma;\sigma,1):(b_j,B_j)_{1,q},(1-\upsilon-\delta k;s);(0,1)\end{matrix}\right] \qquad ...(2.1)$$

Provided

1) $\alpha,\alpha',\beta,\beta',\gamma,\mu,\upsilon,s,\sigma,a,b\in C;\lambda,\delta>0$

2) $|\arg w|<\frac{1}{2}\Omega\pi;\Omega>0$

Where $\Omega=\sum_{j=1}^{m}B_j+\sum_{j=1}^{n}A_j-\sum_{j=m+1}^{q}B_j-\sum_{j=n+1}^{p}A_j$

3) $\gamma>0,\mathrm{Re}(\mu)+\sigma\min_{1\le j\le m}\left[\mathrm{Re}(\frac{b_j}{B_j}\right]>\max\{0,\alpha'-\beta',\alpha+\beta-\gamma\}-1$

$\mathrm{Re}(\upsilon)+s\min_{1\le j\le m}\left[\mathrm{Re}(\frac{b_j}{B_j}\right]>\max\{0,\alpha'-\beta',\alpha+\beta-\gamma\}-1$

4) $\left|\frac{a}{b}t\right|<1$

Proof: In order to prove (2.1), we first express the general class of polynomials in series form given by (1.3) , the H-function in terms of Mellin-Barnes type of contour integrals given by (1.1) and interchange the order of summations, integration and fractional derivative operator, which is permissible under the stated conditions. Now using the result (1.6)we arrive at the desired result after a little simplification.

3. Interesting Special Cases

On account of the most general character of the *H*-function and general class of polynomials occurring in the main result, many special cases of the result can be derived but, for the sake of brevity, a few interesting special cases will be given below:

(i) If $\alpha=u+v,\alpha'=\beta'=0,\beta'=-w',\gamma=u$ then

$$I_{0,z}^{u+v,0,-w',0,u}=I_{0,z}^{u,v,w'}$$

which is saigo type fractional integral operator[19]. Hence the main result (2.1) takes the form

$$I_{0,t}^{u,v,w'}\left[t^{\mu\text{-}1}(b-at)^{-\upsilon}S_{\mathrm{N}}^{\mathrm{M}}\left(t^{\lambda}(b-at)^{-\delta}\right)\mathrm{H}_{\mathrm{p,q}}^{\mathrm{m,n}}\left(wt^{\sigma}(b-at)^{-s}\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right.\right)\right]$$

$$=b^{-\upsilon}t^{\mu-u-v+\upsilon-1}\sum_{k=0}^{[N/M]}A_{N,k}\frac{(-N)_{Mk}}{k!}b^{-\delta k}t^{\lambda k}$$

$$H_{4,3:p,q+1;0,1}^{0,4:m,n;0,0}\left[\begin{matrix}\frac{w}{b}t\\-\frac{a}{b}t\end{matrix}\left|\begin{matrix}(1-\mu-\lambda k),(1-\mu-\lambda k)\\(1-\mu-\lambda k),(1-\mu-\lambda k-\upsilon)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\mu-\lambda k+u+v-w'),(1-\upsilon-\delta k):(a_j,A_j)_{1,p};(0,0)\\(1-\mu-\lambda k+v;\sigma,1):(b_j,B_j)_{1,q},(1-\upsilon-\delta k;s);(0,1)\end{matrix}\right] \qquad \ldots(3.1)$$

valid under the same conditions surrounding (2.1)

(ii) Setting $M=1,A_{N,k}=\binom{N+\lambda'}{N}\frac{(\lambda'+\mu'+N+1)_k}{(\lambda'+1)_k}$ then main result takes the form

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\mu\text{-}1}(b-at)^{-\upsilon}P_{\mathrm{N}}^{(\lambda',\mu')}\left(1-2t^{\lambda}(b-at)^{-\delta}\right)\mathrm{H}_{\mathrm{p,q}}^{\mathrm{m,n}}\left(wt^{\sigma}(b-at)^{-s}\left|\begin{matrix}(a_j,A_j)_{1,p}\\(b_j,B_j)_{1,q}\end{matrix}\right.\right)\right]$$

$$=b^{-\upsilon}t^{\mu-\alpha-\alpha'+\upsilon-1}\sum_{k=0}^{N}\binom{N+\lambda'}{N}\frac{(\lambda'+\mu'+N+1)_k}{(\lambda'+1)_k}\frac{(-N)_k}{k!}b^{-\delta k}t^{\lambda k}$$

$$H_{4,3:p,q+1;0,1}^{0,4:m,n;0,0}\left[\begin{matrix}wb^{-s}t^{\sigma}\\-\frac{a}{b}t\end{matrix}\left|\begin{matrix}(1-\mu-\lambda k;\sigma,1),(1-\mu-\lambda k+\alpha'-\beta';\sigma,1)\\(1-\mu-\lambda k-\beta';\sigma,1),(1-\mu-\lambda k+\alpha'+\beta'-\upsilon;\sigma,1)\end{matrix}\right.\right.$$

$$\left.\begin{matrix}(1-\mu-\lambda k+\alpha+\alpha'+\beta;\sigma,1),(1-\upsilon-\delta k;s,1):(a_j,A_j)_{1,p};(0,0)\\(1-\mu-\lambda k+\alpha+\alpha'-\gamma;\sigma,1):(b_j,B_j)_{1,q},(1-\upsilon-\delta k;s);(0,1)\end{matrix}\right] \qquad .(3.2)$$

..

at $a_j=0=A_j;b_j=b;B_j=B$

(iii) main result reduces to

$$I_{0,t}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\mu\text{-}1}(b-at)^{-\upsilon}w^{\frac{b}{B}}t^{\frac{\sigma b}{B}}(b-at)^{-\frac{sb}{B}}e^{-\left(w^{\frac{1}{B}}t^{\frac{\sigma}{B}}(b-at)^{-\frac{s}{B}}\right)}S_{\mathrm{N}}^{\mathrm{M}}\left(t^{\lambda}(b-at)^{-\delta}\right)\right]$$

$$= b^{-\upsilon} t^{\mu-\alpha-\alpha'+\upsilon-1} \sum_{k=0}^{[N/M]} A_{N,k} \frac{(-N)_{Mk}}{k!} b^{-\delta k} t^{\lambda k}$$

$$H^{0,4:1,0;0,0}_{4,3:0,2;0,1}\left[\begin{matrix} wb^{-s}t^{\sigma} \\ -\dfrac{a}{b}t \end{matrix}\left|\begin{matrix} (1-\mu-\lambda k;\sigma,1),(1-\mu-\lambda k+\alpha'-\beta';\sigma,1) \\ (1-\mu-\lambda k-\beta';\sigma,1),(1-\mu-\lambda k+\alpha'+\beta'-\upsilon;\sigma,1) \end{matrix}\right.\right.$$

$$\left.\begin{matrix} (1-\mu-\lambda k+\alpha+\alpha'+\beta;\sigma,1),(1-\upsilon-\delta k;s,1):--;(0,0) \\ (1-\mu-\lambda k+\alpha+\alpha'-\gamma;\sigma,1):(b,B),(1-\upsilon-\delta k;s);(0,1) \end{matrix}\right] \qquad ...(3.3)$$

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