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Some Fixed Point Results for Generalized Contractive Mappings on Cone Metric Spaces

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Abstract

In the present paper we improve and generalize fixed point theorems for generalized contractive type mappings in cone metric spaces are established.

Keywords: Fixed point, Generalized contractive mappings, Cone metric space.

1. Introduction and Preliminaries

Very recently Huang and Zhang [3] introduce the notion of cone metric space. He replaced real number system by ordered Banach space. He also gave the condition in the setting of cone metric spaces. The results in [3] were generalized by Sh. Rezapour and R. Hamlbarani [7] omitting the assumption of normality on the cone.

In this paper, we establish some new generalized contractive type condition for mappings defined on cone metric spaces and prove some new fixed point theorems for these mappings. Our results are generalization of results in [2,3,4,5,8].

Let (E, τ) be a topological vector space and $P \in E$. Then P is called a cone whenever,

(i) P is closed, non-empty and $P \neq \{0\}$;
(ii) $ax + by \in P \ \forall \ x, y \in P$ and non-negative real number a, b;
(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, for $x, y \in P, x \ll y$ if $y - x \in Int(P)$, I$nt\ P$ denotes the interior of P.

The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above is called the normal constant of P.

Definition 1.1. Let X be a non-empty set. Suppose that the mapping $d: X \times X \to E$ satisfies

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 1.2. Let (X, d) be a cone metric space and $\{x_n\}$ be a sequence in X, then,

(i) $\{x_n\}$ converges to $x \in X$, if for every $c \in E$ with $0 \ll c$, there is an $n_0 \in N$ the set of all natural numbers such that for all $n \geq n_0, d(x_n, x) \ll c$. It is denoted by $lim_{n\to\infty} x_n = x$ or $x_n \to x(n \to \infty)$.
(ii) If for every $c \in E$, there is a number $n_0 \in N$ such that for all $m, n \geq n_0$ $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.
(iii) (X, d) is called complete cone metric space if every Cauchy sequence in X is convergent.

Definition 1.3. A function $F: P \to P$ is called $\ll -$ increasing if, for each $x, y \in P, x \ll y$ if and only if $f(x) \ll f(y)$. Let $F: P \to P$ be a function such that

(F1) $F(t) = 0$ if and only if $t = 0$.
(F2) F is $\ll -$ increasing.
(F3) F is surjective.

We denote by $\xi(P, P)$ the family of functions satisfying $(F1), (F2), (F3)$.

Lemma 1.4[1]. Let E be a topological vector space. If $c_n \in E$ and $c_n \to 0$, then for each $c \in Int(P)$ there exists N such that $c_n \ll c$ for all $n > N$.

2. Main Results

Theorem 2.1. Let (X, d) be a complete cone metric space. Suppose that a mapping $T: X \to X$ satisfies

$$F(d(Tx, Ty)) \leq \lambda_1 F\big(d(x, y)\big) + \lambda_2 [F(d(x, Tx)) + F\big(d(y, Ty)\big)] \text{ ---- (2.1)}$$

for all $x, y \in X$, where $\lambda_1, \lambda_2 \in [0,1)$ and $F \in \xi(P, P)$ such that

(1) F is sub additive.

(2) If for $\{c_n\} \subset p, lim_{n\to\infty} F(c_n) = 0$ then $lim_{n\to\infty} c_n = 0$.

Then T has a fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}$ is convergent to the fixed point.

Moreover, if $\lambda_1 + \lambda_2 < 1$ then T has a unique fixed point in X.

Proof. Let $x_o \in X$ be fixed. Let $x_1 = Tx_0$ and let $x_{n+1} = Tx_n = T^{n+1}\ x_0$ for all $n \in N$. If there exist $n \in N$ such that $x_{n+1} = x_n$, then $Tx_n = x_{n+1} = x_n$ and so T has a fixed point. Hence the proof is complete.

Hence we have that $x_{n+1} \neq x_n$ for any $n \in N \cup \{0\}$ from (2.1) with $x = x_n$ and $y = x_{n-1}$, we have

$$F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1}))$$

$$\leq \lambda_1 F(d(x_n, x_{n-1})) + \lambda_2[F(d(x_n, Tx_n)) + F(d(x_{n-1}, Tx_{n-1}))]$$

$$\leq \lambda_1 F(d(x_n, x_{n-1})) + \lambda_2[F(d(x_n, x_{n+1})) + F(d(x_{n-1}, x_n))]$$

$$(1 - \lambda_2)F(d(x_{n+1}, x_n)) \leq (\lambda_1 + \lambda_2)F(d(x_n, x_{n-1}))$$

$$\Rightarrow F(d(x_{n+1}, x_n)) \leq \left(\frac{\lambda_1+\lambda_2}{1-\lambda_2}\right) F(d(x_n, x_{n-1}))$$

$$\leq k(F(d(x_n, x_{n-1})))$$

where $k = \frac{\lambda_1+\lambda_2}{1-\lambda_2} < 1$.

Thus we obtain, $F(d(x_{n+1}, x_n)) \leq kF(d(x_n, x_{n-1}))$ for all $n \in N$.

Hence $F(d(x_{n+1}, x_n)) \leq kF(d(x_n, x_{n-1}))$

$$\leq k^2 F\big(d(x_{n-1}, x_{n-2})\big)$$

$$\leq k^n F\big(d(x_1, x_0)\big).$$

We now show that $\{x_n\}$ is a Cauchy sequence in X.

For $m > n$, we have that

$$F(d(x_n, x_m)) \leq F(d(x_n, x_{n+1})) + F(d(x_{n+1}, x_{n+2})) + F(d(x_{n+2}, x_{n+3}))$$

$$+ - - + F(d(x_{m-1}, x_m))$$

$$\leq k^n F\big(d(x_1, x_0)\big) + k^{n+1} 0 F\big(d(x_1, x_0)\big) + k^{n+2} F\big(d(x_1, x_0)\big) + - - - +$$

$$+ k^{m-1} F\big(d(x_1, x_0)\big)$$

$$\leq k^n F\big(d(x_1, x_0)\big)[1 + k + k^2 + - - - + k^{m-n-1}]$$

$$\leq k^n F\big(d(x_1, x_0)\big) \left(\frac{1-k^{m-n}}{1-k}\right)$$

$$\leq \left(\frac{k^n}{1-k}\right) F(d(x_1, x_0)) \to 0.$$

Applying Lemma (1.4), $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exist $z \in X$ such that $lim_{n\to\infty} x_n = z$.

Let $c \in Int\ (P)$ be given, we can choose $N \in \mathbb{N}$ such that $d(x_{n-1}, z) \ll F^{-1}(\frac{c}{2})$ for all $n > N$,

By (F_2) and (F_3),

$F(d(x_{n-1}, z)) \ll \frac{c}{2}$ for all $n > N$.

Thus for all $n > N$, we obtain

$$F(d(z, Tz)) \leq F\big(d(z, x_n)\big) + F(d(x_n, Tz))$$

$$\leq F\big(d(z, x_n)\big) + F(d(Tx_{n-1}, Tz))$$

$$\leq F\big(d(z, x_n)\big) + \lambda_1 F\big(d(x_{n-1}, z)\big) + \lambda_2[F\big(d(x_{n-1}, Tx_{n-1})\big)$$

$$+F\big(d(z, Tz)\big)$$

$$\leq F\big(d(z, x_n)\big) + \lambda_1 F\big(d(x_{n-1}, z)\big) + \lambda_2 F\big(d(x_{n-1}, x_n)\big)$$

$$+\lambda_2 F\big(d(z, Tz)\big)$$

$$(1-\lambda_2)F\big(d(z, Tz)\big) \leq F\big(d(x_n, z)\big) + \lambda_1 F\big(d(x_{n-1}, z)\big) + \lambda_2[F\big(d(x_{n-1}, z)\big)$$

$$+F(d(z, x_n))]$$

$$(1-\lambda_2)F\big(d(z, Tz)\big) \leq F\big(d(x_n, z)\big) + \lambda_1 F\big(d(x_{n-1}, z)\big) + \lambda_2 F\big(d(x_{n-1}, z)\big)$$

$$+\lambda_2 F\big(d(x_n, z)\big)$$

$$(1-\lambda_2)F\big(d(z, Tz)\big) \leq (1+\lambda_2)F(d(x_n, z)) + (\lambda_1+\lambda_2)F(d(x_{n-1}, z)$$

$$F\big(d(z, Tz)\big) \leq \frac{1+\lambda_2}{1-\lambda_2} F(d(x_n, z)) + \frac{\lambda_1+\lambda_2}{(1-\lambda_2)} F(d(x_{n-1}, z))$$

$$F\big(d(z, Tz)\big) \leq \frac{c}{2} + \frac{c}{2} = c.$$

Thus, $F(d(Tz, z)) \ll \frac{c}{n}$ for all $n \in N$, and so $\frac{c}{n} - F(d(Tz, z)) \in P$. Since $\frac{c}{n} \to 0$ and P is closed, $-F(d(Tz, z)) \in P$. Hence $F(d(Tz, z)) = 0$.

By $(F1)$, $d(Tz, z) = 0$ and so $z = Tz$.

Assume that u is another fixed point of T.

Then $F\big(d(z, u)\big) = F(d(Tz, Tu))$

$$\leq \lambda_1 F(d(z, u)) + \lambda_2[F(d(z, Tz)) + F(d(u, Tu))]$$

$$\leq \lambda_1 F(d(z, u)) + \lambda_2[F(d(z, z)) + F(d(u, u))]$$

$\leq \lambda_1 F(d(z,u))$

Thus $(\lambda_1 - 1)F(d(u,z)) \in P$, since $0 \leq \lambda_1 - 1 < 1$,

$(\lambda_1 - 1)F(d(u,z)) \in -P$ Hence $F(d(z,u)) = 0$.

By (F_1) $d(z,u) = 0$ and so $z = u$.

Therefore, T has a unique fixed point in X.

Theorem 2.2. Let (X,d) be a complete cone metric space. Suppose that a mapping $T: X \to X$ satisfies

$$F(d(Tx,Ty)) \leq \lambda_1 F(d(x,y)) + \lambda_2 F(d(x,Tx)) + \lambda_3 F(d(y,Ty)) + \lambda_4[F(d(y,Tx)) + F(d(x,Ty))] \quad \text{……..(2.2)}$$

for all $x, y \in X$; where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in [0,1)$ and $F \in \xi(P,P)$ such that

(1) F is sub-additive;

(2) If, for $\{c_n\} \subset P$, $\lim_{n\to\infty} F(c_n) = 0$ then $\lim_{n\to\infty} c_n = 0$.

Then T has a fixed point in X. For each $x \in X$, the iterative sequence $\{T^n x\}$ is convergent to the fixed point.

Moreover, if $\lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_4 < 1$ then T has a unique fixed in X.

Proof. Let $x_o \in X$ be fixed. Let $x_1 = Tx_0$ and let $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}$.

If there exist $n \in \mathbb{N}$ such that $x_{n+1} = x_n$ then $Tx_n = x_{n+1} = x_n$ and so T has a fixed point. Hence the proof is complete.

Hence we have that $x_{n+1} \neq x_n$ for any $n \in \mathbb{N} \cup \{0\}$.

From (2.2) with $x = x_n$ and $y = x_{n-1}$, we have

$$F(d(x_{n+1}, x_n)) = F(d(Tx_n, Tx_{n-1}))$$

$$\leq \lambda_1 F(d(x_n, x_{n-1})) + \lambda_2 F(d(x_n, Tx_n)) + \lambda_3 F(d(x_{n-1}, Tx_{n-1})) +$$

$$\lambda_4[F(d(x_{n-1}, Tx_n)) + F(d(x_n, Tx_{n-1}))]$$

$$\leq \lambda_1 F(d(x_n, x_{n-1})) + \lambda_2 F(d(x_n, x_{n+1})) + \lambda_3 F(d(x_{n-1}, x_n)) +$$

$$\lambda_4[F(d(x_{n-1}, x_{n+1}) + F(d(x_n, x_n))]$$

$$\leq (\lambda_1 + \lambda_3)F(d(x_n, x_{n-1})) + \lambda_2 F(d(x_n, x_{n+1})) + \lambda_4 F(d(x_{n-1}, x_n)) +$$

$$+\lambda_4 F(d(x_n, x_{n+1}))$$

$$\leq (\lambda_1 + \lambda_3 + \lambda_4)F\left(d(x_n, x_{n-1})\right) + (\lambda_2 + \lambda_4)F(d(x_n, x_{n+1}))$$

$$(1-\lambda_2-\lambda_4)F\left(d(x_n,x_{n+1})\right)$$
$$\leq (\lambda_1+\lambda_3+\lambda_4)F(d(x_n,x_{n-1}))\ F(d(x_n,x_{n+1}))$$
$$\leq \left(\frac{\lambda_1+\lambda_3+\lambda_4}{1-\lambda_2-\lambda_4}\right)F\left(d(x_n,x_{n-1})\right)$$

Thus $F\left(d(x_n,x_{n+1})\right) \leq hF\left(d(x_n,x_{n-1})\right) \forall\ n \in \mathbb{N}$ where $h=\frac{\lambda_1+\lambda_3+\lambda_4}{1-\lambda_2-\lambda_4} < 1$

as $\lambda_1+\lambda_2+\lambda_3+2\lambda_4 < 1 \Rightarrow \lambda_1+\lambda_3+\lambda_4 < 1-\lambda_2+\lambda_4$

$\Rightarrow \frac{\lambda_1+\lambda_3+\lambda_4}{1-\lambda_2-\lambda_4} < 1$

Thus we obtain,

$$F\left(d(x_n,x_{n+1})\right) \leq hF\left(d(x_n,x_{n-1})\right)$$
$$\leq h^2F(d(x_{n-2},x_{n-3}))$$
$$\leq h^nF(d(x_1,x_o)).$$

We now show that $\{x_n\}$ is a Cauchy sequence in X. For $m>n$, we have

$$F\left(d(x_n,x_m)\right) \leq F(d(x_n,x_{n+1})) + -- +F(d(x_{m-1},x_m))$$
$$\leq h^nF\left(d(x_1,x_0)\right) + h^{n+1}F(d(x_1,x_0)) + ---$$
$$----- +h^{m-1}F(d(x_1,x_0))$$
$$\leq h^nF(d(x_1,x_0))[1+h+h^2+ ---- +h^{m-n-1}]$$
$$\leq h^nF(d(x_1,x_0))\left[\frac{1-h^{m-n}}{1-h}\right]$$
$$\leq \frac{h^n}{1-h}F(d(x_1,x_0)) \to 0.$$

Hence $\lim_{n,m\to\infty} d(x_n,x_m)=0$.

Applying Lemma (1.4) $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Let $c \in int\ (P)$ be given. We can choose $N \in \mathbb{N}$ such that

$d(x_n,z) \ll F^{-1}\left(\frac{c}{2}\right)$ for all n>N,

$$F(d(Tz,z)) \leq F(d(Tz,Tx_{n-1})) + F(d(Tx_{n-1},z))$$
$$\leq \lambda_1F(d(z,x_{n-1})) + \lambda_2F\left(d(z,Tz)\right) + \lambda_3F\left(d(x_{n-1},Tx_{n-1})\right) +$$
$$\lambda_4[F\left(d(x_{n-1},Tz)\right) + F(d(z,Tx_{n-1}))] + F\left(d(Tx_{n-1},z)\right)$$
$$\Rightarrow (1-\lambda_2)F(d(Tz,z)) \leq \lambda_1F(d(z,x_{n-1})) + \lambda_3F(d(x_{n-1},z)) + \lambda_3F(d(z,x_n))$$

$+\lambda_4[F(d(x_{n-1},Tz)) + F(d(z,x_n))] + F(d(x_n,z))$

$\Rightarrow (1-\lambda_2)F(d(Tz,z)) \leq \lambda_1 F(d(x_{n-1},z)) + \lambda_3 F(d(x_{n-1},z)) + \lambda_3 F(d(x_n,z))$

$+\lambda_4[F\ (d(x_{n-1},Tz))] + \lambda_4[F(d(z,x_n))] + F(d(x_n,z))$

$\Rightarrow (1-\lambda_2)F(d(Tz,z)) \leq \lambda_1 F(d(x_{n-1},z)) + \lambda_3 F(d(x_{n-1},z)) + \lambda_3 F(d(x_n,z))$

$+\lambda_4[F\ d(x_n,z))] + F(d(x_n,z) + \lambda_4[F(d(x_{n-1},z)) + F(d(z,Tz))]$

$(1-\lambda_2-\lambda_4)F(d(Tz,z)) \leq (\lambda_1 + \lambda_3 + \lambda_4)F(d(x_{n-1},z))$

$+(\lambda_3 + \lambda_4 + 1)F(d(x_n,z))$

$$\Rightarrow F(d(Tz,z)) \leq \frac{(\lambda_1 + \lambda_3 + \lambda_4)}{(1-\lambda_2-\lambda_4)}F(d(x_{n-1},z)) + \frac{(1+\lambda_3+\lambda_4)}{1-\lambda_2-\lambda_4}F(d(x_n,z))$$

$$\ll \frac{c}{2} + \frac{c}{2} = c.$$

Thus $F(d(Tz,z)) \ll \frac{c}{n}$ for all $n \in N$ $\therefore \frac{c}{n} - F(d(Tz,z)) \in P$
Since $\frac{c}{n} \to 0$ and P is closed, $-F(d(Tz,z)) \in P$. Hence $F(d(Tz,z)) = 0$.
By $(F1)$, $d(Tz,z) = 0$. Hence $z = Tz$.
Assume that u is another fixed point of T then

$F(d(z,u)) = F(d(Tz,Tu))$

$\leq \lambda_1 F(d(z,u)) + \lambda_2 F(d(z,Tz)) + \lambda_3 F(d(u,Tu)) +$

$\lambda_4[F(d(u,Tz)) + F(d(z,Tu))]$

$\leq \lambda_1 F(d(z,u)) + \lambda_4[F(d(u,z)) + F(d(z,u))]$

$$\leq (\lambda_1 + 2\lambda_4)F(d(u,z))$$

Thus $(\lambda_1 + 2\lambda_4 - 1)F(d(u,z)) \in P, \because 0 \leq \lambda_1 + 2\lambda_4 < 1, (\lambda_1 + 2\lambda_4 - 1) \in -P$
Hence $F(d(z,u)) = 0$.
By (F_1), $d(z,u) = 0$ and so $z = u$.
Therefore, T has a unique fixed point in X.

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