

Optimization of CASP-CUSUM Schemes Based on Truncated Rayleigh Distribution using Lobatto Integration Method

Narayana Murthy B. R¹, Akhtar P. Md² and Venkata Ramudu B³

¹Lecturer in Statistics, S.V. Degree and P.G. College, Anantapur-515 001, (A.P.)

²Professor, Department of Statistics, S.K. University, Anantapur-515055, (A.P.)

³Asst. Professor of Statistics, S.S.B.N. Degree College, Anantapur-515 001, (A.P.)

Abstract

Among several methods to control the quality and reliability, Acceptance Sampling Plans are introduced to accept or reject the lots of finished products. Such types of methods are involved in the inspection quality of product which is destructive where 100% inspection is not possible such as bullets, batteries, bulbs and so on. This paper study the CASP-CUSUM Schemes based on the assumption that the continuous variable under consideration follows a truncated Rayleigh Distribution. It is used the Lobatto Integration Method to solve the truncated intergral equations. The Rayleigh Distribution plays a vital role in Statistical Quality Control, particularly in estimating reliability by considering its distribution. Optimum CASP-CUSUM Schemes are suggested based on numerical results obtained.

Keywords: CASP-CUSUM Schemes, type-C OC Curve, ARL, Truncated Rayleigh Distribution.

1. Introduction

Acceptance sampling is concerned with inspection and decision making regarding lots of product by the techniques of quality of assurance in 1930's and 1940's Acceptance Sampling Plans were one of the important statistical tools on the field of Statistical Quality Control. These techniques require specification of a probability model describing the life of the products. In this process the Log- logistic distribution has been studied by Shah and Dave [5].

Acceptance Sampling Plans are widely used in business, industry, producing engineering products etc. For example a company receives a segment of product from a vender, the product is after a component or raw material used in the manufacturing process. A sample is taken from a lot and examined the quality with respect to certain characteristics of interest. On the basis of information supplied by the sample, an optimal decision is to be made. Often a decision is taken to accept the lot or reject the lot. After the decision, accepted lots are put it into production while rejected lots may be returned to the vendor. In such case, to take optimal decisions with regard to "Acceptance Sampling Plans" are widely used from past two decades. Thus most acceptance sampling plans are working as an alarm in decision making therefore we will develop a procedure for calculating the values relevant characters P (A), ARL's of CASP-CUSUM to make the required decision.

In these lines Sobel and Tischendorf [6] studied acceptance sampling plans based on an exponential distributions. Gupta and Groll [4] developed sampling plans based on Gamma life test sample data. Goode and Kao [3] proposed samples plans based on the Weibull distribution and determined appropriate measures of CUSUM schemes. In this study, we proposed acceptance sampling method based on the truncated Log-logistic distribution under the assumption that the variable under study distributed according to truncated Log-logistic distribution under this assumption we determined an appropriate measures for CUSUM schemes by using Lobatto Method of Integration.

Akthar P.M d and Sarma K.L.A.P [1] proposed an optimization of CASP-CUSUM Schemes based on truncated two parametric Gamma distribution and evaluated L(O) L'(O) and Probability of Acceptance and also optimized CASP-CUSUM Schemes based on numerical results.

In the present paper it is proposed CASP-CUSUM Charts when the variable under study follows truncated Rayleigh Distribution. Thus it is more worthwhile to study some interesting characteristics of this distribution.

2. Rayleigh Distribution

Definition: A continuous random variable X assuming non-negative values is said to have Rayleigh Distribution with parameter $\sigma > 0$ its probability density function is given by:

$$f(x, \sigma) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}, \quad X \geq 0. \quad (2.1)$$

3. Truncated Rayleigh Distribution

Definition: The random variable X is said to follow a truncated Rayleigh Distribution as:

$$f_B(x, \sigma) = \frac{\frac{x}{\sigma^2} e^{-x^2/2\sigma^2}}{1 - \left[e^{-\frac{1}{B^2/2\sigma^2}} \right]}$$

truncated Rayleigh Distribution as:

$$, X \geq 0, \sigma > 0. \quad (3.1)$$

Where 'B' is the truncation point of the Rayleigh Distribution.

The Rayleigh distribution is very flexible for modeling of life time of any product, and used for the life modeling in reliability analysis, life testing problems and accepting sampling plans. This distribution has been experienced in many applications such as communication, engineering and life testing. Rayleigh distribution is very flexible for modeling failure rate by choosing an appropriate value of the parameter.

Rayleigh distribution is a member of continuous probability distributions and it was introduced by Lord Rayleigh. It is often used in wind speed, energy, physics, communication and lifetime analysis and quality inspection. For example, it is used to model scattered signals that reach a receiver by multiple paths. Rayleigh derived it from the amplitude of sound resulting from many independent sources.

4. Description of the Plan and Type-C OC Curve

Beattie [2] have suggested the method for constructing the continuous acceptance sampling plan. The procedure, suggested by him consists of a chosen decision interval namely, "Return interval" with the length h' , above the decision line is taken. We plot on the chart the sum $S_m = \sum (X_i - k)X_i$'s ($i = 1, 2, 3, \dots$) are distributed independently and k is the reference value. If the sum lies in the area of normal chart, the product is accepted and if it lies in the of the return chart, the product is rejected, subject to the following assumptions.

- 1) When the recently plotted point on the chart touches the decision line, then the next point to be plotted at the maximum, i.e., $h+h'$.
- 2) When the decision line is reached or crossed from above, the next point on the chart is to be plotted from the baseline.

When the CUSUM falls in the return chart, network or a change of specification may be employed rather than outright rejection.

The procedure in brief is given below.

1. Start plotting the CUSUM at 0.
2. The product is accepted when $S_m = \sum (X_i - k) < h$; when $S_m < 0$, return cumulative to 0.
3. When $h < S_m < h + h'$ the product is rejected: when S_m crosses h , i.e., when $S_m > h + h'$ and continue rejecting product until $S_m > h + h'$ return cumulative to $h + h'$

The Type-C, OC function, which is defined as the probability of acceptance of an item as a function of incoming quality, when sampling rate is same in acceptance and rejection regions. Then the probability of acceptance $P(A)$ is given by

$$P(A) = \frac{L(0)}{L(0) + L'(0)} \quad (4.1)$$

Where $L(0)$ =Average Run Length in acceptance zone and
 $L'(0)$ =Average Run Length in rejection zone.

Page, E.S [8] has introduced the formulae for $L(0)$ and $L'(0)$ as

$$L(0) = \frac{N(0)}{1 - P(0)} \quad (4.2)$$

$$L'(0) = \frac{N'(0)}{1 - P'(0)} \quad (4.3)$$

Where $P(0)$ =Probability for the test starting from zero on the normal chart,

$N(0)$ =ASN for the test starting from zero on the normal chart,

$P'(0)$ = Probability for the test on the return chart and

$N'(0)$ = ASN for the test on the return chart.

He further obtained integral equations for the quantities

$P(0)$, $N(0)$, $P'(0)$, $N'(0)$ as follows :

$$P(z) = F(k - z) + \int_0^h p(y) f(y + k - z) dy, \quad (4.4)$$

$$N(z) = 1 + \int_0^h N(y) f(y + k - z) dy, \quad (4.5)$$

$$P'(z) = \int_{k+z}^B f(y) dy + \int_0^h P'(y) f(-y + k + z) dy \quad (4.6)$$

$$N'(z) = 1 + \int_0^h N'(y) f(-y + k + z) dy, \quad (4.7)$$

$$F(k - z) = 1 + \int_A^{k-z} f(y) dy, \quad F(x) = 1 + \int_A^h f(x) dx:$$

and z is the distance of the starting of the test in the normal chart from zero.

5. Method of Solution

We first express the integral equation (4.4) in the form

$$F(x) = Q(x) + \int_c^d R(x,t) F(t) dt \quad (5.1)$$

Where $F(x) = P(z)$

$$Q(x) = F(k - z)$$

$$R(x,t) = f(y + k - z)$$

Let the integral $I = \int_c^d f(x) dx$ be transformed to

$$I = \frac{d-c}{2} \int_c^d f(y) dy = \frac{d-c}{2} \sum a_i f(t_i) \quad (5.2)$$

Where $y = \frac{2x-(c-d)}{d-c}$ where a_i 's and t_i 's respectively the weight factor and abscissa for the Lobatto polynomial.

The Lobatto Method of Integration is used to evaluate the integral in equation in (5.2) In this method the nodes and weights factors for various values of n are taken from the Nodes and weights for Lobatto Method of Integration.

Using (5.1) and (5.2), (4.4) can be written as

$$F(x) = Q(x) + \frac{d-c}{2} \sum a_i R(x, t_i) F(t_i) \quad (5.3)$$

Since equation (5.3) should be valid for all values of x in the interval (c,d), it must be true for $x=t_i$. $i=0(1)n$ then obtain

$$F(t_i) = Q(t_i) + \frac{d-c}{2} \sum_{j=0}^n a_j R(t_j, t_i) F(t_j) \quad (5.4)$$

Substituting

$F(t_i) = F_i$, $Q(t_i) = Q_i$, $i=0(1)n$, in (2.4.4), we get

$$\begin{aligned} F_0 &= Q_0 + \frac{d-c}{2} [a_0 R(t_0, t_0) F_0 + a_1 R(t_0, t_1) F_1 + \dots a_n R(t_0, t_n) F_n] \\ F_1 &= Q_1 + \frac{d-c}{2} [a_0 R(t_1, t_0) F_0 + a_1 R(t_1, t_1) F_1 + \dots a_n R(t_1, t_n) F_n] \\ F_n &= Q_n + \frac{d-c}{2} [a_0 R(t_n, t_0) F_0 + a_1 R(t_n, t_1) F_1 + \dots a_n R(t_n, t_n) F_n] \end{aligned} \quad (5.5)$$

In the system of equations except F_i , $i=0, 1 \dots n$ are known and hence can be solved for F_i . We solve the system of equations by the method of Iteration. For this we write the system (5.5) as

$$[1 - T a_0 R(t_0, t_0)] F_0 = Q_0 + T [a_0 R(t_0, t_0) F_0 + a_1 R(t_0, t_1) F_1 + \dots a_n R(t_0, t_n) F_n]$$

$$[1 - Ta_1 R(t_1, t_1)]F_1 = Q_1 + T[a_0 R(t_1, t_0)F_0 + a_1 R(t_1, t_1)F_1 + \dots a_n R(t_1, t_n)F_n]$$

$$[1 - Ta_n R(t_n, t_n)]F_n = Q_n + T[a_0 R(t_n, t_0)F_0 + a_1 R(t_n, t_1)F_1 + \dots a_n R(t_n, t_n)F_n] \quad (5.6)$$

Where $T = \frac{d - c}{2}$.

To start the Iteration process, let us put $F_1 = F_2 = \dots F_n = 0$ in the first equation of (5.6), we then obtain a rough value of F_0 . Putting this value of F_0 and $F_2 = F_3 \dots F_n = 0$ in the second equation, we get a rough value F_1 and so on. This gives the first set of values $F_i, i=0,1,2,\dots,n$ which are just the refined values of $F_i, i=0,1,2,\dots,n$. The process is continued until two consecutive sets of values are obtained up to a certain degree of accuracy. In the similar way solutions $P'(0), N(0), N'(0)$ can be obtained.

6. Computation of ARL's & P (A)

We developed computer programs to solve equations (5.1 to 5.6) and get the following tables.

Table 6.1: VALUES OF ARLs AND TYPE -C OC CURVES when $k=2, h=0.25, h'=0.25, \sigma=2$

B	L(0)	L'(0)	P(A)
2.9	2.45915	1.0189053	0.7070470
2.8	2.62206	1.0196999	0.7199982
2.7	2.83396	1.0205958	0.7352234
2.6	3.11937	1.0216091	0.7532927
2.5	3.52245	1.0227588	0.7749810
2.4	4.13155	1.0240691	0.8013685
2.3	5.15281	1.0255685	0.8340068
2.2	7.20460	1.0272933	0.8752057
2.1	13.37830	1.0292884	0.9285592
2.0	48626.37891	1.0316113	0.9999788

Table 6.2: VALUES OF ARLs AND TYPE -C OC CURVES when $k=2, h=0.5, h'=0.5, \sigma=2$

B	L(0)	L'(0)	P(A)
2.9	2.38913	1.0383999	0.6970412
2.8	2.54433	1.0400420	0.7098397
2.7	2.74619	1.0418967	0.7249547
2.6	3.01810	1.0439980	0.7429904

2.5	3.40212	1.0463878	0.7647778
2.4	3.98241	1.049117	0.7914914
2.3	4.95532	1.052244	0.8248460
2.2	6.90970	1.055841	0.8674465
2.1	12.78768	1.0600615	0.9234488
2.0	11903.35547	1.0649704	0.9999105

Table-6.3: VALUES OF ARLs AND TYPE –C OC CURVES when $k=2$ $h= 0.75$ $h'= 0.75$ $\sigma = 2$.

B	L(0)	L'(0)	P(A)
2.9	2.32063	1.0585122	0.6867509
2.8	2.46833	1.0610589	0.6993645
2.7	2.66045	1.0639398	0.7143318
2.6	2.91924	1.0672102	0.7322906
2.5	3.28474	1.0709375	0.7541285
2.4	3.83704	1.0752050	0.7811174
2.3	4.76293	1.0801170	0.8151450
2.2	6.62244	1.0858041	0.8591372
2.1	12.21075	1.0924342	0.9178817
2.0	5039.22314	1.1002243	0.9997817

Table 6.4: VALUES OF ARLs AND TYPE –C OC CURVES when $k=2$ $h= 1$ $h'= 1$ $\sigma = 2$

B	L(0)	L'(0)	P(A)
2.9	2.25349	1.0792723	0.6761630
2.8	2.39389	1.0827850	0.6885568
2.7	2.57651	1.0867652	0.7033350
2.6	2.82251	1.0912923	0.7211686
2.5	3.16996	1.0964634	0.7430019
2.4	3.69496	1.1023995	0.7702068
2.3	4.57495	1.1092520	0.8048536
2.2	6.34165	1.1172140	0.8502166
2.1	11.64518	1.1265339	0.9117947
2.0	2681.60718	1.1375383	0.9995760

Table: 6.5: VALUES OF ARLs AND TYPE –C OC CURVES when
 $k=4$ $h= 0.25$ $h'= 0.25$ $\sigma = 2$

B	L(0)	L'(0)	P(A)
4.9	10.88384	1.0128758	0.9148609
4.8	11.68377	1.0129642	0.9202185
4.7	12.73223	1.0130632	0.9262974
4.6	14.15342	1.0131741	0.9331970
4.5	16.17069	1.0132978	0.9410324
4.4	19.23069	1.0134360	0.9499393
4.3	24.37546	1.0135901	0.9600776
4.2	34.73089	1.0137619	0.9716388
4.1	65.92414	1.0139530	0.9848524
4.0	513085.53125	1.0141658	0.9999980

Table 6.6: VALUES OF ARLs AND TYPE –C OC CURVES when
 $k=4$ $h= 0.5$ $h'= 0.5$ $\sigma = 2$

B	L(0)	L'(0)	P(A)
4.9	10.67102	1.0260171	0.9122840
4.8	11.45370	1.0261977	0.9177720
4.7	12.47958	1.0264000	0.9240040
4.6	13.87011	1.0266263	0.9310838
4.5	15.84384	1.0268793	0.9391325
4.4	18.83773	1.0271618	0.9482926
4.2	34.00151	1.0278282	0.9706581
4.1	64.51011	1.0282193	0.9843112
4.0	131695.48438	1.0286549	0.9999922

7. Numerical Results and Conclusions

At the hypothetical values of the parameters σ , k , h and h' given at the top of each table, we determine optimum truncated point B at which $P(A)$ the probability of accepting an item is maximum and also obtained ARL's values which represents the acceptance zone $L(0)$ and rejection zone $L'(0)$ values. The values of truncated point B of random variable X , $L(0)$, $L'(0)$ and the values for Type–C OC Curve, i.e. $P(A)$ are given in columns I, II, III and IV respectively.

From Tables 6.1 to 6.6 we made the following conclusions

- By go through the above tables the values of h , h' increases then related values of $P(A)$ decreases. So the sizes of accepted and rejected zones and $P(A)$ are inversely related.

- By examining the above tables we found that the value of parameter of Rayleigh distribution increases then P (A) is also increases. So, the parameter of Rayleigh distribution and P (A) are positively related.
- By observing the above tables the values of h, h' increases then related values of L (0) decreases. So the sizes of accepted and rejected zones and L (0) are inversely related.
- By observing the above tables the reference value of k increase then P (A) will also increases. So, reference value and P (A) are positively related.
- By in sighting the above tables the reference value of k increase then L (0) increases. So, reference value and L (0) are positively related.
- From Table 6.1 to 6.6 it is examined that the size of the acceptance zone is increases as the reference value k increases
- From Table 6.1 to 6.6 it is examined that at the maximum level of probability P(A), the optimal truncated point B changes from 2.0 to 4.0 as the reference value changes from 2 to 4.

Table 7.1: Optimum Values of CASP-CUSUM Schemes

B	σ	k	h	h'	L (0)	L'(0)	P (A)
2.0	2	2	0.25	0.25	48626.3789	1.0316113	0.9999788
2.0	2	2	0.5	0.5	11903.3554	1.0649704	0.9999105
2.0	2	2	0.75	0.75	5039.2231	1.1002243	0.9997817
2.0	2	2	1	1	2681.6071	1.1375383	0.9995760
4.0	2	4	0.25	0.25	513085.5312	1.0141658	0.9999980
4.0	2	4	0.5	0.5	131695.4843	1.0286549	0.9999922

By examining the above table values, the value P (A) reaches maximum i.e. **0.9999980 at B=4**. We obtain optimum CASP-CUSUM Scheme as:

$$\left[\begin{array}{l} B = 4.0 \\ \sigma = 2 \\ k = 2 \\ h = 0.25 \\ h' = 0.25 \end{array} \right]$$

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