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Exact Solution of multi-pantograph Equation using Differential Transform Method

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Abstract:

In this study the differential transform method is applied to solve the multi-pantograph equation . This method involves less computational work and can, thus, be easily applied to initial value problems. Firstly, we stated the definition of the one dimensional transform method, and some related theorems. Then some illustrative examples are given, The numerical results obtained by these examples are found to be the same

Keywords: System of Pantograph equation , the differential transform method, numerical method.

I. Introduction

The differential transform method has been successfully utilized by Zhou (1986) to solve linear and nonlinear initial value problem in electric circuit analysis. Moreover, Chen and Ho developed this method for partial differential equations and Ayaz applied it to system of differential equations. The current Projected differential transform method (PDTM) is an improvement of the former differential version[1]-[2]-[4]-[6]-[14]

The purpose of this paper is to employ the differential transformation method to systems of differential equations which are often encounter in many branches of physics, chemical and engineering. A variety of methods, exact, approximate, and purely numerical are available for the solution of systems of differential equations. Most of these methods are computationally intensive because they are trial-and-error in nature, or need complicated symbolic computations **[**11 **]**

The differential transform method (DTM) is one of the approximate methods which can be easily applied to many linear and nonlinear problems and is capable of reducing the size of computational work. Differential transform method is a semi-numerical analytic technique that formalizes the Taylor series in a totally different manner. With this method, the given differential equation and related initial conditions are transformed into a recurrence equation, that finally leads to a system of algebraic equations which can easily be solved. In this method no need for linearization or perturbations, large computational work and round- off errors are avoided In resent years many researchers apply the DTM for solving differential equations[9]-[13]

This method constructs, for differential equations an analytical solution in the form of a polynomial. Not like the traditional high order Taylor series method that requires symbolic computations. Another important advantage is that this method reduces the size of computational work while the Taylor series method is computationally taking long time for large orders. This method is well addressed in[3]-[5]

II. One dimensional Differential transform method:

The basic definitions and fundamental theorems of one dimensional differential transform method are defined and proved in[7] and will be stated brief in this paper.

Differential transform of function $y\left(px\right)$ is defined as follows:

$$y\left(k\right)=\frac{1}{k!}\left[\frac{d^{k}y\left(px\right)}{dx^{k}}\right]_{x=0} \tag{1}$$

Where $y\left(px\right)$ the original is function and $y\left(k\right)$ is the transformed function, which is also called the T- function.

The inverse differential transform of $y\left(k\right)$ is defined as.

$$y\left(x\right)=\sum_{k=0}^{\infty}y\left(k\right)x^{k} \tag{2}$$

Combining equations (1) and (2) we have

$$y\left(x\right)=\sum_{k=0}^{\infty}\left[\frac{d^{k}y\left(px\right)}{dx^{k}}\right]_{x=0}\frac{x^{k}}{k!} \tag{3}$$

The fundamental theorems of the one dimensional differential transform are:

Theorems :

(1) $If\ z\left(x\right)=u\left(px\right)\pm v\left(px\right)\ Then\ z\left(k\right)=p^{k}u\left(k\right)\pm p^{k}v\left(k\right)$

(2) $If\ z\left(x\right)=cu\left(px\right)\ Then\ z\left(k\right)=cp^{k}u\left(k\right)$

(3) $If\ z\left(x\right)=\dfrac{du\left(px\right)}{dx}\ Then\ z\left(k\right)=\left(k+1\right)p^{k+1}u\left(k+1\right)$

(4) $If\ z\left(x\right)=\dfrac{d^{n}u\left(px\right)}{dx^{n}}Then\ z\left(k\right)=\dfrac{\left(k+n\right)!}{k!}p^{k+n}u\left(k+n\right)$

(5) *If* $z(x) = u(px)v(qx)$ *Then* $z(k) = \sum_{m=0}^{k} p^m q^{k-m} u(m) v(k-m)$

(6) *If* $z(x) = u_1(p_1 x) u_2(p_2 x) \ldots u_n(p_n x)$ *Then*

$$z(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} p_1^{k_1} p_2^{k_2-k_1} \ldots p_n^{k-k_{n-1}} u_1(k_1) u_2(k_2-k_1) \ldots u_{n-1}(k_{n-1}-k_{n-2}) u_n(k-k_{n-1})$$

(7) *If* $z(t) = \frac{du(pt)}{dt} \frac{dv(qt)}{dt}$ *Then* $z(k) = \sum_{m=0}^{k} p^{m+1} q^{k-m} (k+1)(m+1) u(m+1) v(k-m)$

(8) *If* $(x) = \left[\frac{d}{dt}\, g_1(p_1 x)\right] \left[\frac{d}{dt}\, g_2(p_2 x)\right] \ldots \left[\frac{d}{dt}\, g_n(p_n x)\right]$ *Then*

$$F(k) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \ldots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} p_1^{k_1} p_2^{k_2-k_1} \ldots p_n^{k-k_{n-1}} G_1(k_1)\, G_2(k_2-k_1) \ldots \times G_n(k-k_{n-1})$$

(9) *If* $z(x) = (px)^n$ *Then* $z(k) = p^n \delta(n-k)$ $, \delta(n-k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq n \end{cases}$

(10) *If* $z(t) = \sin(pt+\alpha)$ *Then* $z(k) = \frac{p^k}{k!} \sin\left(\frac{\pi}{2} k\right)$

(11) *If* $z(t) = \cos(pt+\alpha)$ *Then* $z(k) = \frac{p^k}{k!} \cos\left(\frac{\pi}{2} k\right)$

(12) *If* $z(t) = e^{pt}$ *Then* $z(k) = \frac{p^k}{k!}$

Note that c is Constant and n is a nonnegative integer.

III. Application:

The pantograph type equations have been studied extensively owing to the numerous applications in which these equations arise. The name pantograph originated from the work of Ockendon and Tayler on the collection of current by the pantograph head of an electric locomotive, this equations are appeared in modeling of various problems in engineering and sciences such as biology, economy, control and electrodynamics .Consider the flowing system of multi-pantograph equation[8]

$$u_i'(t) = \sum_{j=1}^{n} \alpha_{ij}(t) u_j(t) + \sum_{j=1}^{n} \beta_{ij}(t) u_j(p_j t) + f_i(t) \quad , i = 1, 2, \ldots, n \qquad (4)$$

With the initial condition

$u_i(0) = \gamma_i$

Where α_{ij}, β_{ij} and f_i are known functions , γ_i, p_i are constants such that $0 < p_i < 1$ u_i unknown analytical functions on the given interval to de determined .

Taking the projected differential transform method of equation (4) we have

$$(k+1)u_i(k+1)=\sum_{m=0}^{k}\sum_{j=1}^{n}\alpha_{ij}(m)u_j(k-m)+\sum_{m=0}^{k}\sum_{j=1}^{n}p^{k-m}\beta_{ij}(m)u_j(k-m)+f_i(k)$$

$$u_i(k)=\frac{1}{k}\left[\sum_{m=0}^{k-1}\sum_{j=1}^{n}\alpha_{ij}(m)u_j(k-m-1)+\sum_{m=0}^{k-1}\sum_{j=1}^{n}p^{k-m-1}\beta_{ij}(m)u_j(k-m-1)+f_i(k-1)\right]$$

$$k=1,2,.....$$

Substituting $u_i(k)$ into equation (2) we have

$$u_i(t)=\gamma_i+\sum_{k=1}^{\infty}\frac{1}{k}\left[\sum_{m=0}^{k-1}\sum_{j=1}^{n}\alpha_{ij}(m)u_j(k-m-1)+\sum_{m=0}^{k-1}\sum_{j=1}^{n}p^{k-m-1}\beta_{ij}(m)u_j(k-m-1)+f_i(k-1)\right]t^k$$

IV. Numerical examples :

In this section, we present three example with analytical solution to show the efficiency of methods described in the Section III

Example 4.1:[12] Consider tow-dimensional pantograph equation

$$\begin{cases}\dfrac{du_1(t)}{dt}=u_1(t)-u_2(t)+u_1\left(\dfrac{1}{2}t\right)+e^{-t}-e^{\frac{1}{2}t}\\[2ex]\dfrac{du_2(t)}{dt}=-u_1(t)-u_2(t)-u_2\left(\dfrac{1}{2}t\right)+e^{t}-e^{-\frac{1}{2}t}\end{cases}\qquad(5)$$

With the initial condition $u_1(0)=1$, $u_2(0)=1$

Applying the differential transform method to equation (5) we get

$$\begin{cases}(k+1)u_1(k+1)=u_1(k)-u_2(k)+\left(\dfrac{1}{2}\right)^k u_1(k)+A(k)-B(k)\\[2ex](k+1)u_2(k+1)=-u_1(k)-u_2(k)-\left(\dfrac{1}{2}\right)^k u_2(k)+C(k)+D(k)\end{cases}$$

Where $A(k),B(k),C(k)$ and $D(k)$ correspond to transformation of $e^{-t},e^{-\frac{1}{2}t},e^{t}$ and $e^{\frac{1}{2}t}$ respectively and this leads to

$$A(k)=\frac{(-1)^k}{k!}\ ,\quad B(k)=\frac{\left(\frac{1}{2}\right)^k}{k!}\ ,\quad C(k)=\frac{1}{k!},\quad D(k)=\frac{\left(-\frac{1}{2}\right)^k}{k!}$$

$$u_1(1)=1\ ,\ u_1(2)=\frac{1}{2!}\ ,\ u_1(3)=\frac{1}{3!}\ ,u_1(4)=\frac{1}{4!}$$

$$u_2(1)=-1,\ u_2(2)=\frac{1}{2!},\ u_2(3)=\frac{-1}{3!}\ ,u_2(4)=\frac{1}{4!}$$

and so on in general $u_1(k)=\dfrac{1}{k!}$, $u_2(k)=\dfrac{(-1)^k}{k!}$

Substituting $u_1(k)$ and $u_2(k)$ into equation (2) we get

$$u_1(t)=\sum_{k=0}^{\infty}u_1(k)t^k=\sum_{k=0}^{\infty}\frac{t^k}{k!}=e^t$$

$$u_2(t)=\sum_{k=0}^{\infty}u_2(k)t^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}=e^{-t}$$

Example 4. 2:[10] Consider tow-dimensional pantograph equation

$$\begin{cases}\dfrac{du_1(t)}{dt}=(\sin t)u_1\left(\dfrac{1}{2}t\right)+(\cos t)u_2\left(\dfrac{1}{2}t\right)+\cos t-\cos\dfrac{1}{2}t\\[2mm] \dfrac{du_2(t)}{dt}=(-\cos t)u_1\left(\dfrac{1}{2}t\right)+(\sin t)u_2\left(\dfrac{1}{2}t\right)-\sin t-\sin\dfrac{1}{2}t\end{cases}\qquad(6)$$

With the initial condition $u_1(0)=0$, $u_2(0)=1$ using the differential transform method in to equation (6) we have

$$\begin{cases}(k+1)u_1(k+1)=\displaystyle\sum_{m=0}^{k}\left(\frac{1}{2}\right)^{k-m}A(m)u_1(k-m)+\sum_{m=0}^{k}\left(\frac{1}{2}\right)^{k-m}B(m)u_2(k-m)-B(k)-C(k)\\ (k+1)u_2(k+1)=-\displaystyle\sum_{m=0}^{k}\left(\frac{1}{2}\right)^{k-m}B(m)u_1(k-m)+\sum_{m=0}^{k}\left(\frac{1}{2}\right)^{k-m}A(m)u_2(k-m)-A(k)-D(k)\end{cases}$$

Where $A(k), B(k), C(k)$ and $D(k)$ correspond to transformation of $\sin t, \cos t, \cos\frac{1}{2}t$ and $\sin\frac{1}{2}t$ respectively and this leads to

$$A(k)=\frac{1}{k!}\sin\left(\frac{\pi}{2}k\right),\ B(k)=\frac{1}{k!}\cos\left(\frac{\pi}{2}k\right),C(k)=\frac{\left(\frac{1}{2}\right)^k}{k!}\cos\left(\frac{\pi}{2}k\right),D(k)=\frac{\left(\frac{1}{2}\right)^k}{k!}\sin\left(\frac{\pi}{2}k\right)$$

$$u_1(1)=1,u_1(2)=0,\ u_1(3)=-\frac{1}{3!},\ u_1(4)=0,u_1(5)=\frac{1}{5!}$$

$$u_2(1)=0,\ u_2(2)=-\frac{1}{2!},\ u_2(3)=0,\ u_2(4)=\frac{1}{4!},u_2(5)=0$$

and so on in general

$$u_1(k)=\begin{cases}\dfrac{(-1)^{\frac{k-1}{2}}}{k!} & ,\ k\ \ is\ \ odd\\ 0 & ,\ k\ \ is\ \ even\end{cases}\quad ,u_2(k)=\begin{cases}\dfrac{(-1)^{\frac{k}{2}}}{k!} & ,\ k\ \ is\ \ even\\ 0 & ,\ k\ \ is\ \ odd\end{cases}$$

Substituting $u_1(k)$ and $u_2(k)$ into equation (2) we get

$$u_1(t)=\sum_{k=0}^{\infty}u_1(k)t^k=\sum_{k=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{k-1}{2}}}{k!}t^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}t^{2k+1}=\sin t$$

$$u_2(t)=\sum_{k=0}^{\infty}u_2(k)t^k=\sum_{k=0,2,4,\ldots}^{\infty}\frac{(-1)^{\frac{k}{2}}}{k!}t^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k}=\cos t$$

Example 4. 3:[12] Consider three-dimensional pantograph equation

$$\begin{cases}\dfrac{du_1(t)}{dt}=2u_2\left(\dfrac{1}{2}t\right)+u_3(t)-t\cos\dfrac{1}{2}t\\ \dfrac{du_2(t)}{dt}=1-t\sin t-2u_3^2\left(\dfrac{1}{2}t\right)\\ \dfrac{du_3(t)}{dt}=u_2(t)-u_1(t)-t\cos t\end{cases}\qquad(6)$$

With the initial condition $u_1(0)=-1 \quad u_2(0)=0 \quad u_3(0)=0$

Applying the differential transform method to equation (6) we get

$$\begin{cases}(k+1)u_1(k+1)=\left(\dfrac{1}{2}\right)^{k-1}u_2(k)+u_3(k)-\displaystyle\sum_{m=0}^{k}\delta(m-1)A(k-m)\\ (k+1)u_2(k+1)=\delta(k)-\displaystyle\sum_{m=0}^{k}\delta(m-1)B(k-m)-\sum_{m=0}^{k}\left(\dfrac{1}{2}\right)^{k-1}u_3(m)u_3(k-m)\\ (k+1)u_3(k+1)=u_2(k)-u_1(k)-\displaystyle\sum_{m=0}^{k}\delta(m-1)C(k-m)\end{cases}$$

Where $A(k), B(k)$ and $C(k)$ correspond to transformation of $\cos\frac{1}{2}t$, $\sin t$, $\cos t$ respectively and this leads to

$$A(k)=\frac{\left(\frac{1}{2}\right)^k}{k!}\cos\left(\frac{\pi}{2}k\right)\ ,\quad B(k)=\frac{1}{k!}\sin\left(\frac{\pi}{2}k\right)\ ,\ C(k)=\frac{1}{k!}\cos\left(\frac{\pi}{2}k\right)$$

$$\begin{cases} u_1(k)=\dfrac{1}{k}\left[\left(\dfrac{1}{2}\right)^{k-2}u_2(k-1)+u_3(k-1)-A(k-2)\right] \\ \\ u_2(k)=\dfrac{1}{k}\left[\delta(k-1)-B(k-2)-\displaystyle\sum_{m=0}^{k-1}\left(\dfrac{1}{2}\right)^{k-2}u_3(m)u_3(k-m-1)\right] \\ u_3(k)=\dfrac{1}{k}\left[u_2(k-1)-u_1(k-1)-C(k-2)\right] \end{cases}$$

$$u_1(1)=0\ ,u_1(2)=\frac{1}{2!}\ ,u_1(3)=0,\ u_1(4)=-\frac{1}{4!}\ ,u_1(5)=0$$

$$u_2(1)=1,\ u_2(2)=0,\ u_2(3)=-\frac{1}{2!},\ u_2(4)=0\ ,u_2(5)=\frac{1}{4!}$$

$$u_3(1)=1,\ u_3(2)=0,\ u_3(3)=-\frac{1}{3!},\ u_3(4)=0\ ,u_3(5)=\frac{1}{5!}$$

and so on in general

$$u_1(k)=\begin{cases}\dfrac{(-1)^{\frac{k}{2}+1}}{k\,!} & ,\ k\ \ is\ even \\ 0 & ,\ k\ \ is\ odd\end{cases}\quad ,\ u_2(k)=\begin{cases}\dfrac{(-1)^{\frac{k-1}{2}}}{(k-1)!} & ,\ k\ \ is\ odd \\ 0 & ,\ k\ \ is\ even\end{cases}$$

$$u_3(k)=\begin{cases}\dfrac{(-1)^{\frac{k-1}{2}}}{(k)!} & ,\ k\ \ is\ odd \\ 0 & ,\ k\ \ is\ even\end{cases}$$

Substituting $u_1(k),u_2(k)$ and $u_3(k)$ into equation (2) we get

$$u_1(t)=\sum_{k=0}^{\infty}u_1(k)t^k=\sum_{k=0,2,4,,\ldots}^{\infty}\frac{(-1)^{\frac{k}{2}+1}}{k\,!}t^k=-\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k}=-\cos t$$

$$u_2(t)=\sum_{k=0}^{\infty}u_2(k)t^k=\sum_{k=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{k-1}{2}}}{(k-1)!}t^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}t^{2k+1}=t\cos t$$

$$u_3(t)=\sum_{k=0}^{\infty}u_3(k)t^k=\sum_{k=1,3,5,\ldots}^{\infty}\frac{(-1)^{\frac{k-1}{2}}}{(k)!}t^k=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}t^{2k+1}=\sin t$$

V. Conclusion:-

In this work, we used the differential transform method for solving the multi-pantograph equation and compared our results with the exact solution in order to demonstrate the validity and applicability of the method. This method is better than numerical methods, since it is free from rounding off error, and does not require large computer power In summary, using differential transformation method to solve ODE, consists of three main steps. First, transformation ODE in to algebra equation, second, solve the equations, finally inverting the solution of algebraic equations to obtain a closed form series solution.

References:

1- Abdel – Hassan, I.H, 2004 Differential transformation technique for solving higher-order initial value problem. Applied math .Comput, 154-299-311.
2- Ayaz.F.2004 solution of the system of differential equations by differential transform method .Applied math.Comput, 147: 54767.
3- C. Hchen, S.H.Ho.Solving Partial differential by two dimensiona differential transform method, APPL. Math .Comput.106 (1999)171-179.
4- FatmaAyaz-Solution of the system of differential equations by differential transform method .Applied .math.Comput. 147(2004)547-567.
5- F. Kanglgil .F .A yaz. Solitary wave Solution for kdv and M kdv equations by differential transform method, chaos solutions and fractions do1:10.1016/j. Chaos 2008.02.009.
6- Hashim,I,M.SM.Noorani,R.Ahmed.S.A.Bakar.E.S.I.Ismailand A.M.Zakaria,2006.Accuracy of the Adomian decomposition method applied to the Lorenz system chaos 2005.08.135.
7- J. K. Zhou, Differential Transformation and its Application for Electrical eructs .Hunzhong university press, wuhan, china, 1986
8- J.R. Ockendon, A.B. Tayler, The dynamics of a current collection system for an electric locomotive, Proc. R. Soc. Lond. Ser. A. 322 (1971) 447-468
9- Montri Thong moon. Sasitornpusjuso.The numerical Solutions of differential transform method and the Laplace transform method for a system of differential equation. Nonlinear Analysis. Hybrid systems (2009) d0I:10.1016/J.nahs 2009.10.006.
10- 1Mohsen Alipour, 2Kobra Karimi, 3Davood Rostamy, Modified Variational Iteration Method for the Multi-pantograph Equation with Convergence Analysis , Australian Journal of Basic and Applied Sciences, 5(5): 886-893, 2011ISSN 1991-8178
11- R.L. Burrden, J.D. Faires, Numerical Analysis, PWS-KENT Publishing Company, Boston, 1993.
12- Sabir Widatalla1, 2 and Mohammed Abdulai Koroma1, Approximation Algorithm for a System of Pantograph Equations, Hindawi Publishing Corporation Journal of Applied Mathematics Volume 2012, Article ID 714681
13- Shih-Hsiang chang .I-Ling chang: Anew algorithm for calculating one Dimensional differential transform of nonlinear Functions. Appli.math .Compu195 (2008)799-808
14- T. Hayat. M. Sajid, on analytic solution for thin film flow of a fourth grade Fluid down a vertical cylinder, phys. Lett. A361 (2007)316-322