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Door Spaces On Generalized Topology

[1]A.P. Dhana Balan and [2] RM.Sivagama Sundari

[1, 2]Department of Mathematics; Alagappa Govt. Arts College, Karaikudi - 630003; Tamil Nadu; India.
e.mail:danabalanap@yahoo.com; e.mail:kanish9621@hotmail.com;
Mob: 94882 90720.

Abstract

In this paper, the authors investigate the properties of door spaces on generalized topology. Also the concept of μ - Hausdorff door space, μ - semi Hausdorff spaces are introduced. We obtain some characterizations and properties of such a spaces. This paper also takes some investigations on $\alpha -$ set and its associated generalized door spaces.

Keywords: Door space, μ - open, μ - closed, μ - semiopen, μ - semiclosed, μ - dense.

Introduction

Generalized topological space is an important generalization of topological spaces and many interesting results have been obtained. In topology weak forms of open sets play an important role in the generalization of various forms of continuity. Using various forms of open sets, many authors have introduced and studied various types of continuity. In 1961, Levine [7] introduced the notion of weak continuity in topological spaces and obtained a decomposition of continuity. Generalized topology was first introduced by Csaszar [1].

1.Preliminaries:

Let X be a set. A subset μ of exp X is called a generalized topology (GT) on X and (X, μ) is called a generalized topological space [1] (GTS) if μ has the following properties:

(i) $\phi \in \mu$,.

(ii) Any union of elements of μ belongs to μ.

For a GTS (X, μ), the elements of μ are called μ - open sets and the complement of μ - open sets are called μ - closed sets. Consider X ={a, b, c} and μ = { ϕ, {a}, {b}, {a, b}}.The μ - closed sets are X, {b, c}, {a, c}and{c}. If A ={a, b} then A is not μ - closed.

For A $\subset$ X, we denote by c_μ(A) the intersection of all μ - closed sets containing A, that is the smallest μ - closed set containing A, and by i_μ(A), the union of all μ - open sets contained in A, that is the largest μ - open set contained in A.If X$\in$ μ then X is called strong generalized topology. Throughout this paper, a space (X, μ) or simply X for short, will always mean a strong generalized topological space with strong generalized topology μ unless otherwise stated explicitly. Intensive research on the field of generalized topological space (X, μ) was done in the past ten years as the theory was developed by A.Csaszar[1], A.P.Dhana Balan[5] and many more researchers. It is easy to observe that $c_\mu$and i_μ are idempotent and monotonic, where γ:exp X $\rightarrow$exp X is said to idempotent if and only if A $\subseteq$B$\subseteq$ X implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic if and only if A $\subseteq$ B $\subseteq$ X implies γ(A) $\subseteq$ γ(B). It is also well known that from [3, 4] that if μ is a GT on X and A $\subseteq$ X, $x\in$ X then $x \in c_\mu$(X) if and only if $x\in$ M$\in$ μ $\Rightarrow$M$\cap$ A $\neq$ ϕ and c_μ(X $-$ A) = X $-$ i_μ(A).

Let B $\subset$exp X and ϕ $\in$ B. Then B is called a base[3] for μ if {$\cup$ B': B' $\subset$B}= μ.We also say that μ is generated by B. A point $x\in$ X is called a μ - cluster point of B$\subset$ X if U$\cap$ (B - {x}) $\neq$ ϕ for each U $\in$ μ with $x\in$ μ. The set of all μ - cluster point of B is denoted by d(B).

Let X = Z_n={1, 2,......n}, μ = {ϕ, X}$\cup${A $\subset$ Z_n / A =Z_n - {i}, i=1,............n}, the co - singleton generalized topology defined on a finite set. The only μ - closed sets are, X and singleton subsets of Z_n. Some theorems are developed based on [1, 2]. This paper is partially based on [6].The end or the omission of a proof will be denoted by ■.

2. Generalized door spaces.

A generalized door space is a generalized topological space (X, μ) in which every subset is either μ - open or μ - closed.

Definition 2.1: A generalized topological space is a generalized door space if and only if every subset is either μ - open or μ - closed.

Example 2.2: Let X ={a, b, c} and μ ={ϕ, {a}, {b}, {a, b}}.The μ - closed sets are { X, {b, c}, {a, c}, {c}}.Therefore the generalized topological space (X, μ) is a generalized door space. The space X need not be strong. If X is strong generalized topological space, then both ϕ and X are μ - open and μ - closed. Discrete space is a generalized door space. Every door space is a generalized door space.

Definition 2.3: Let X be a strong generalized topological space. A subset A of X is μ - semiopen set of X, denoted by so_μ(X) if $A \subset c_\mu i_\mu$(A).The complement of μ - semiopen set is called μ - semiclosed.

Definition 2.4: Let X be a set. Let μ_s be the collection of all subsets U of X such that X - U is either singleton or all of X.Then μ_s is a generalized topology on X, called co - singleton generalized topology. Both ϕ and X are in μ_s, since X $-$ X = ϕ is finite and X $-$ ϕ is all of X. If $\{U_\alpha\}$ is an indexed collection of elements of μ_s, to show that $\cup U_\alpha$ is in μ_s, we compute X $- \cup U_\alpha$ = $\cap$(X $-$ U_α). The later set is finite or singleton because each X $-$ U_α is finite or singleton.

Let X = Z_2= {1, 2} μ = $\{\phi,$ X$\}\cup\{$A $\subset$ Z_2 / A =Z_2 - {i}, i=1, 2}, the co - singleton generalized topology defined on a finite set Z_2.That is μ = $\{\phi,$ X$\}\cup$$\{$$\{2\}$, $\{1\}\}$ μ - closed sets are $\{\phi,$ X$\}\cup$$\{$$\{1\}$, $\{2\}\}$. That is the only μ - closed sets are $\phi,$ X and singleton subsets of Z_2 and so X is a generalized door space. Now, let X = Z_3= {1, 2, 3} μ = $\{\phi,$ X$\}\cup\{$A $\subset$ Z_3 / A =Z_3 - {i}, i=1, 2, 3}, the co - singleton generalized topology defined on a finite set Z_3. μ = $\{\phi,$ X$\}\cup$$\{$2, 3}, {1, 3}, {1, 2}} μ - closed sets are $\{\phi,$ X$\}\cup$$\{$$\{1\}$, $\{2\}$, $\{3\}\}$.That is the μ - closed sets are $\phi,$ X and the singleton subsets of Z_3 and so X is a generalized door space.

In general, if we choose X = Z_n, n≥4, not necessarily a generalized door space; result follows easily, and so left to the reader.

Lemma 2.4: Let x ∈ (X, μ) then $\{x\}$∈ so_μ(X) if and only if $\{x\}$∈ μ.That is, if a singleton is μ - semiopen then it is μ - open.

Proof follows from the definition ■.

Remark 2.5: The set of all μ - closed set need not be a generalized topological space.

Lemma2.6: Let U, V ∈ (X, μ).If U ∈ so_μ(X) and V ∈ μ, then U $\cap$ V ∈ so_μ(X) ■.

Result 2.7: If A = ϕ then i_μ(A) = ϕ.

Lemma2.8: If A is non - empty in X, then i_μ(A) need not be empty.

Proof follows from the definition ■.

Example 2.9: Let X = {a, b, c}and let μ = { ϕ, {a}, {b}, {a, b}}.Let A ={a, c}. μ - interior of A is the largest μ - open subset of A.Consider a ∈ A, there exists a μ - open set $\{a\}$containing a and contained in A. consider c ∈ A, there exists no μ - open set containing c and contained in A.

Lemma 2.10: If A is μ - semiopen, B $\subset$ X and A $\subset$ B $\subset$ c_μ(A), then B is μ - semiopen ■.

Definition 2.11: A subset A of a generalized topological space X is nowhere dense if $i_\mu c_\mu$(A) = ϕ.

Lemma2.12: Let X be a generalized topological space. If A $\in so_{\mu}$(X) then A $- i_{\mu}$(A) is nowhere μ dense in X ■.

Definition 2.13: A generalized topological space X is called μ - semi Hausdorff if every two disjoint points in X can be separated by disjoint μ - semi open sets. Every μ - Hausdorff space is μ - semi Hausdorff but the reverse is not necessarily true.

If (X, μ) is a GTS and A $\subset$ X, the collection μ' = {U$\cap$ A /U $\in \mu$}is a generalized topology for A, called the relative topology for A. Let (X, μ) be a generalized topological space and let A $\subset$ X. Consider μ'={U$\cap$ A /U $\in \mu$} Since ϕ = A$\cap\phi$ and A = A$\cap$X, we have ϕ and A contained in μ', where ϕ and X are elements of μ.Consider the index set J, $\cup_{\alpha\in \mathrm{J}}(U_{\alpha}\cap$A) =($\cup_{\alpha\in \mathrm{J}} U_{\alpha}$) $\cap$A $\in \mu'$, since $\cup_{\alpha\in \mathrm{J}} U_{\alpha} \in \mu$. Hence μ' is a strong generalized topology.

Theorem 2.14: Every subspace S of a generalized door space X is a generalized door space.

Proof: Let A $\in$ S. Since X is a generalized door space, we have A is either μ - open or μ - closed in X and hence in S. Thus S is also a generalized door space ■.

Theorem 2.15: A μ - semi Hausdorff generalized door space X has atmost one limit point.

Proof: Let x and y be two different limit points in X.Since X is μ - semi Hausdorff, there exists U $\in so_{\mu}$(X) and V $\in so_{\mu}$(X) such that $x \in$ U, $y \in$ V and U$\cap$V= ϕ.Since X is a generalized door space, the set A=(U\\{x}) $\cup$ {y} is either μ - open or μ - closed.Suppose A is μ - open, then by lemma 2.6, A$\cap$V = {y} is μ - semiopen in X.Now A$\cap$V = {y} is μ - open, follows from lemma 2.4.Now suppose A is μ - closed, then X $-$ A is μ - open.Thus (X $-$ A)$\cap$U ={x}is so_{μ}(X) and hence μ - open in the notion Lemma 2.6 and Lemma 2.4.Thus in both cases atleast one of the two points is isolated in X and by contradiction the claim is proved ■.

Definition 2.16: A generalized topological space X is called a generalized semi - door space if and only if every subset of X is either μ - semiopen or μ - semiclosed. Clearly every generalized door space is a generalized semi - door space but not conversely.

Theorem 2.17: A μ - Hausdorff generalized semi - door space has atmost one limit point ■.

Corollary2.18: A μ - Hausdorff door space has atmost one limit point and if x is a point which is not a limit point, then {x} is μ - open ■.

Definition 2.19: Let X be a strong generalized topological space. X is called a (i) μT_0 - space if x, $y \in$ X and $x \neq y$ then there is U $\in \mu$ such that either U$\cap${x, y}={x} or U$\cap${x, y}={y} (ii) μT_1 - space if $x, y \in$ X and $x \neq y$ then there are $U_x, U_y \in \mu$ such that

$U_x \cap \{x, y\} = \{x\}$ and $U_y \cap \{x, y\} = \{y\}$ (iii) μT_2 - space or Hausdorff space if $x, y \in$ X and $x \neq y$, then there are U_x and $U_y \in \mu$ such that $U_x \cap \{x, y\} = \{x\}$ and $U_y \cap \{x, y\} = \{y\}$

Notation: Let X be a space. For $x \in$ X, μ_x={U: $x \in$U $\in \mu$}and $c_\mu(\mu_x)$ = { c_μ(U): U$\in \mu_x$}.

Lemma 2.20: For a space the following are equivalent. (i) X is a μT_1 - space. (ii) $\{x\}$= $c_\mu(\{x\})$ for each $x \in$ X.So each singleton of X is μ - closed. (iii)If x, $y \in$ X and $x \neq y$ then $c_\mu(\{x\}) \cap c_\mu(\{y\}) = \phi$.

Proof: (i) $\Leftrightarrow$ (ii) Let X be a space and let x, $y \in$ X and $x \neq y$, then U$\cap\{x, y\}$=$\{x\}$ for some U $\in \mu_x$ and so $\mu_x \not\subset \mu_y$ and hence $x \notin c_\mu(\{y\})$. (ii) $\Longrightarrow$ (iii) Let x, $y \in$ X and $x \neq y$ then $c_\mu(\{x\})$ =$\{x\}$ and $c_\mu(\{y\})$ =$\{y\}$. So $c_\mu(\{x\}) \cap c_\mu(\{y\})$=$\{x\} \cap \{y\}$ =ϕ (iii) $\Longrightarrow$ (ii) Let $x \in$ X. If $y \in$ X and $y \neq x$, then $c_\mu(\{x\}) \cap c_\mu(\{y\})$= ϕ.It follows that $y \notin c_\mu(\{x\})$, because $y \in c_\mu(\{y\})$.So $c_\mu(\{x\}) \subset \{x\}$.On the otherhand, it is known that $\{x\} \subset c_\mu(\{x\})$.Consequently, $\{x\}$= $c_\mu(\{x\})$ ■.

It is well known that a topological space X is a T_1 - space iff each finite subset of X is closed. Can "singleton" in lemma 2.20 be replaced by a "finite subset"?. The following example shows that the answer of this question is negative.

Example 2.21: Let X={a, b, c} and μ ={ϕ, X, {a, b}, {b, c}, {c, a}}.Then (X, μ) is a μT_1 - space but A={a, b}is not μ - closed.Thus there is a μT_1 - space X with a finite subset A of X such that A is not μ - closed.

3. $\alpha -$ set and associated generalized door spaces.

Let X be a generalized topological space. A subset A of X is μ - dense in X, if c_μ(A) = X**.** Clearly, X is μ - dense in X and infact X is the only μ - closed set dense in (X, μ). A $\subset$ X is nowhere μ - dense if $i_\mu c_\mu$(A) = ϕ

Definition 3.1: A space X is submaximal if every μ - dense subset of X is μ - open.

Theorem 3.2: Every generalized door space X is submaximal.

Proof: Let A $\subset$ X be μ - dense. Suppose A is not μ –open. Then A is μ - closed, since X is a generalized door space. Then A = $\overline{\text{A}}$ = X and A is μ - open (and μ - closed).Thus X is submaximal ■.

Recall that, s. c_μ(A) denote the intersection of all μ - semiclosed set containing A.

Theorem 3.3: If A ∈ so_{μ} (X) then $s.c_{\mu}$ (A) is μ - semiopen and μ - semiclosed ■.

Definition 3.4: A space X is called resolvable if there exists a pair of disjoint μ - dense subsets of (X, μ) ; otherwise (X, μ) is called irresolvable.

A subset A of the space X is an $\alpha -$ set if A ⊂ $i_{\mu} c_{\mu} i_{\mu}$ (A).We now mention some facts which will be used in the sequel.

Lemma 3.5: (i)If μ^{α} denotes the collection of all $\alpha -$ sets of (X, μ) then μ^{α} is a generalized topology on X. (ii) A subset A of (X, μ) is an $\alpha -$ set if and only if A⋂B is μ - semiopen for each μ - semiopen set B of X. (iii) A subset A of (X, μ) is μ - semiopen if and only if i_{μ} (A)⋂U≠ ϕ for each μ - open set U with A⋂U≠ ϕ. (iv) μ= μ^{α} if and only if every nowhere μ - dense subset of (X, μ) is μ - closed in (X, μ) ■.

Definition 3.6[8]: A GTS (X, μ) is said to be μ - connected if X cannot be written as the union of two disjoint μ - open sets.

Recall that a non - void space X is irresolvable if the following are equivalent:

(i) Every two non - void μ - open subset of X intersect.

(ii) X is not the union of a finite family of μ - closed proper subsets.

(iii) Every non - void μ - open subset of X is μ - dense.

(iv) Every μ - open subset of X is μ - connected.

Theorem 3.7: Every irresolvable submaximal space (X, μ) is a generalized door space.

Proof: Let A ⊂ X. Suppose A is μ - dense in the space X. By submaximality A is μ - open in X. Suppose A is not μ - dense in X, then we can find a non - void μ - open set B ⊂ X \ A.Since X is irresolvable, then B is μ - dense and hence X \ A is also μ - dense. Again by submaximality of X, we have X \ A is μ - open or equivalently A is μ - closed in X.Thus both the cases A is either μ - open or μ - closed.Hence X is a generalized door space ■.

Theorem 3.8: Let (X_i), i ∈ I be a family of generalized topological spaces. For the generalized topological sum X = $\sum_{i \in I} X_i$ the following are equivalent:

(i) X is a generalized door space.

(ii) Each X_i is a generalized door space and X_i is non discrete for atmost one index.

Proof: (i) ⇒ (ii) By theorem 2.14 each X_i is a generalized door space. Now assume that for some indices i and j, X_i and X_j are non discrete. Thus, there exists a non μ - open set A ⊂ X_i and a non μ - closed set B ⊂ X_j. If A∪B is a μ - open subset of X, then (A∪B) ⋂ X_i= A is a μ - open subset of X_i, which is a contradiction. If A∪B is μ - closed subset of X, then B must be μ - closed subset of X_j, which is again a contradiction. Thus X_i is non discrete for atmost one index. (ii) ⇒ (i) We can assume I ≠ ϕ, since otherwise X= ϕ and the claim would be trivial. By (ii) for some j ∈ I, X_j is a generalized door space and X_i is discrete for every i≠j. Let A ⊂ X. Then A =

$\bigcup_{i\in I}(X_i \cap A)$. Also, $X_i \cap A$ is both μ - open and μ - closed in X_i for each i≠j. On the otherhand $X_j \cap A$ is either μ - open or μ - closed in X_j.Thus A is either μ - open or μ - closed in X, or equivalently X is a generalized door space ■.

References:

[1] Csaszar.A., Generalized topology, generalized continuity. Acta Mathematica Hungarica, 96 (4), 351 - 357, 2002.

[2] Csaszar.A., Extremally disconnected generalized topologies. Annales Univ. Budapest, Section Math(17) 151 - 161, 2004.

[3] Csaszar.A., Generalized open sets in generalized topologies. Acta Mathematica Hungerica, 106, 1:2, 53 - 66, 2005.

[4] Csaszar.A., δ and θ modifications of generalized topologies.Acta Mathematica Hungarica, 120 (3), 275 – 279, 2008.

[5] Dhana Balan.A.P., μ - Continuous Functions on Generalized topology and certain Allied Structures. Math. Sci. Int. Research jou; 3(1), 180 - 183, 2014.

[6] Julian Dontchev, On Door spaces, Indian.J pure appl. Math. 26(9), 873 - 881, 1995.

[7] Levine, N., A decomposition of continuity in topological spaces. Amer. Maths. Monthly 68, 44 - 66, 1961.

[8] R.X.Shen, A note on generalized connectedness, Acta Mathematica Hungerica, 122(3), 231 - 235, 2009.