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Fuzzy 2-Bounded Linear Operators

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Abstract

This paper defines the concept of Fuzzy 2-bounded linear operator. Two types (strong and weak) fuzzy 2-boundedness are defined. Relation between strongly fuzzy 2-boundedness and weakly fuzzy 2-boundedness is studied.

Keywords: Fuzzy 2-bounded linear operator, strongly Fuzzy 2-bounded linear operator, weakly Fuzzy 2-bounded linear operator.

1. INTRODUCTION

The concept of fuzzy set was introduced by Zadeh [11] in 1965. Katsaras [6] in 1984, first introduced the notion of fuzzy norm on a linear space. In 1992, Felbin [3] introduced an idea of fuzzy norm on a linear space by assigning a fuzzy real number to each element of the linear space so that the corresponding metric associated this fuzzy norm is of Kaleva type [5] fuzzy metric. She also introduced an idea of fuzzy bounded linear operator, the norm of which is a fuzzy number. Recently Xiao and Zhu [10] redefined in a more general setting the Idea of Felbin's [3] definitions of fuzzy norm of a linear operator from a normed linear space to another fuzzy normed linear space.

A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gähler [4]. In 2009, Sundaram and Beaula[9] defined the concept of fuzzy 2-normed linear space and introduced fuzzy 2-linear operator.

In the present paper, we introduce the concept of fuzzy 2-bounded linear operator on a fuzzy2-normed linear space to another fuzzy 2-normed linear space and also two types (strong and weak) fuzzy 2-bounded linear operators are defined.

T. Bag, S.K. Samanta [2] have proved some results on fuzzy boundedness of fuzzy linear operator on a fuzzy normed linear space using fuzzy norm, we have generalized this concept to a fuzzy 2-normed linear space and discuss the relation between strong fuzzy 2-bounded linear operator and weak fuzzy 2-bounded linear operator.

2. PRELIMINARIES

Definition 2.1[4]: Let X be a real vector space of dimension greater than 1 and let $\|.,.\|$ be a real valued function on $X \times X$ satisfying the following conditions

(1) $\|x, y\| = 0$ if and only if x and y are linearly dependent.

(2) $\|x, y\| = \|y, x\|$

(3) $\|x, \alpha\, y\| = |\alpha|\,\|x, y\|$, where α is real.

(4) $\|x, z + y\| \le \|x, y\| + \|x, z\|$

$\|.,.\|$ is called 2-norm on X and the pair $(X, \|.,.\|)$ is called a 2-normed linear space.

Definition 2.2 [1]: Let X be a linear space over the field K (where K is the field of real or complex numbers). A fuzzy subset N of $X \times R$ (R is the set of real numbers) is called a fuzzy norm on X iff for all $x, u \in X$ and $c \in K$.

(N_1) for all $t \in R$, with $t \le 0$, $N(x, t) = 0$.

(N_2) for all $t \in R$, with $t > 0$, $N(x, t) = 1$ if and only if $x = 0$.

(N_3) for all $t \in R$, with $t > 0$, $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ if $c \ne 0$.

(N_4) for all $s, t \in R$, $x, u \in X$, $N(x + u, s + t) \ge \min\{N(x, s), N(u, t)\}$.

(N_5) $N(x, \bullet)$ is non-decreasing function of R and $\lim_{t\to\infty} N(x, t) = 1$.

The pair (X, N) will be referred to as a fuzzy normed linear space.

Definition 2.3 [8]: Let X be a linear space over a field F. A fuzzy subset N of $X \times X \times R$ (R is the set of real numbers) is called a fuzzy 2-norms on X if and only if

1. for all $t \in R$, with $t \le 0$, $N(x_1, x_2, t) = 0$.
2. for all $t \in R$, with $t > 0$, $N(x_1, x_2, t) = 1$ if and only if x_1 and x_2 are linearly dependent.
3. $N(x_1, x_2, t)$ is invarient under any permutation of x_1, x_2.
4. for all $t \in R$, with $t > 0$, $N(x_1, cx_2, t) = N\left(x_1, x_2, \frac{t}{|c|}\right)$ if $c \ne 0$, $c \in F$.
5. for all $s, t \in R$. $N(x_1, x_2 + x_2', s + t) \ge \min\{N(x_1, x_2, s), N(x_1, x_2', t)\}$

6. $N(x_1, x_2, \bullet)$ is non-decreasing function of R and $\lim_{t\to\infty} N(x_1, x_2, t) = 1$

then $N(X, N)$ is called a fuzzy 2-normed linear space.

Example 2.1: Let $(X, \|.,.\|)$ be 2-normed linear space define

$$N(x_1, x_2, t) = \frac{t}{t + \|x_1, x_2\|}, \quad \text{when } t > 0,\ t \in R,\ \ x_1, x_2 \in A \times B$$

$$= 0, \quad \text{when } t \le 0,\ t \in R,\ \ x_1, x_2 \in A \times B$$

Then (X, N) fuzzy 2-normed linear space.

Definition 2.4 [9]: A fuzzy 2-linear operator T is a function from $A \times B$ to $C \times D$ where A, B are subspaces of fuzzy 2-normed linear space (X, N_1) and C, D are subspaces of fuzzy 2-normed linear space (Y, N_2) such that

$$T(x_1 + x, x_2 + x') = T(x_1, x_2) + T(x_1, x') + T(x, x_2) + T(x, x')$$

and $T(\alpha x_1, \beta x_2) = \alpha\beta\, T(x_1, x_2)$.

3. Fuzzy 2-bounded linear operator

In this section we define the notion of weakly fuzzy 2-boundedness and strongly fuzzy 2-boundedness for fuzzy 2-bounded linear operators over fuzzy 2-normed linear spaces and relation between fuzzy 2-continuity and fuzzy 2-boundedness are studied.

Let X and Y be two linear spaces over the same field of scalars. Let N_1 and N_2 be two fuzzy 2 norms on X and Y respectively. Then (X, N_1) and (Y, N_2) are fuzzy 2-normed linear spaces.

Definition 3.1: Let $T : A \times B \to C \times D$ be a fuzzy 2-linear operator, where A, B are subspaces of (X, N_1) and C, D are subspaces of (Y, N_2) then T is said to be strongly fuzzy 2- bounded on $A \times B$ if and only if $\exists$ a positive real number M such that $\forall\ (x, x') \in A \times B$ and

$$\forall\ s \in R, \quad N_2[T(x, x'), s] \ge N_1\left[(x, x'), \frac{s}{M}\right].$$

Definition 3.2: Let $T : A \times B \to C \times B$ be a fuzzy 2-linear operator, where A, B are subspaces of (X, N_1) and C, D are subspaces of (Y, N_2), then T is said to be weakly fuzzy 2-bounded on $A \times B$ if for any $\alpha \in (0,1)\ \exists\ M_\alpha > 0$ such that $\forall\ (x, x') \in A \times B,\ \forall\ t \in R,$

$$N_1\left((x, x'), \frac{t}{M_\alpha}\right) \ge \alpha \Rightarrow N_2(T(x, x'), t) \ge \alpha$$

Theorem 3.1: Let $T : A \times B \to C \times B$ be a fuzzy 2-linear operator, where A, B are subspaces of (X, N_1) and C, D are subspaces of (Y, N_2), then T is strongly fuzzy-2 bounded then it is weakly fuzzy 2-bounded but not conversely.

Proof: First we suppose that T is strongly fuzzy 2-bounded. Thus $\exists\ M > 0$ such that $\forall\ (x, x') \in A \times B$ and $\forall\ s \in R$, we have

$$N_2[T(x, x'), s] \geq N_1\left[(x, x'), \frac{s}{M}\right]$$

Thus for any $\alpha \in (0,1)$, $\exists\ M_\alpha (= M) > 0$, Such that

$$N_1\left((x, x'), \frac{s}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(T(x, x'), s) \geq \alpha, \quad \forall\ (x, x') \in A \times B, \ \forall\ s \in R$$

this implies that T is weakly fuzzy 2-bounded.

For conversely, we consider the following example.

Example (3.1): Let $X = R^2$ be a linear space over R.
Let $x = (a, b)$, $x' = (a', b')$

Define

$$\|x, x'\| = |ab' - a'b| \quad \text{and} \quad |x| = |a, b| = b - a$$

then $(X, \|.,.\|)$ be a 2-normed linear space.

Now we define

N_1 and $N_2 : X \times X \times R \to [0,1]$ as

$$N_1(x, x', t) = \begin{cases} \dfrac{t^2 - (\| x, x'\|)^2}{t^2 + (\| x, x'\|)^2}, & t > \| x, x'\| \\ 0 \quad , & t < \| x, x'\| \end{cases}$$

and

$$N_2(x, x', t) = \begin{cases} \dfrac{t}{t + \| x, x'\|}, & t > 0 \\ 0 \quad , & t < 0 \end{cases}$$

We know that N_2 is a fuzzy 2-normed space. Now we want to show that N_1 is a fuzzy 2-normed linear space on X.

(i) for all $t \in R$ with $t \leq 0$, we have from definition

$$N_1(x, x', t) = 0$$

(ii) for $t > 0$, we have

$$N_1(x, x', t) = 1$$

$$\Leftrightarrow \frac{t^2 - \|x, x'\|^2}{t^2 + \|x, x'\|^2} = 1$$

$\Leftrightarrow t^2 - \|x,x'\|^2 = t^2 + \|x,x'\|^2$

$\Leftrightarrow \|x,x'\|^2 = 0$

$\Leftrightarrow \|x,x'\| = 0$

$\Leftrightarrow x,x'$ are linearly dependent.

(iii) for $t \in R$ with $t > 0$

$$N_1(x,x',t) = \frac{t^2 - \|x,x'\|^2}{t^2 + \|x,x'\|^2}$$

$$= \frac{t^2 - \|x,x'\|^2}{t^2 + \|x,x'\|^2} = \frac{t^2 - \|x',x\|^2}{t^2 + \|x',x\|^2}$$

$$= N_1(x'x,t)$$

(iv) for all $t \in R$ with $t > 0$, and $c \neq 0$, $c \in F$ (field)

$$N_1\left(x,x',\frac{t}{|c|}\right) = \frac{t^2 - |c|^2\|x,x'\|^2}{t^2 + |c|^2\|x,x'\|^2}$$

$$= \frac{t^2 - \|x,cx'\|^2}{t^2 + \|x,cx'\|^2}$$

$$= N_1(x,cx',t)$$

(v) we have to prove

$N_1(x,x'+x_0,s+t) \geq \min\{N(x,x',s),(N_1x,x_0,t)\}$

If $s \leq \|x,x'\|$ or $t \leq \|x,x_0\|$

then relation is obivious.

Suppose, $s > \|x,x'\|$ and $t > \|x,x_0\|$

without loss of generality assume , $N_1(x,x_0,t) \geq N_1(x,x',s)$

then

$$\Rightarrow t^2\|x,x'\|^2 - s^2\|x,x_0\| \geq 0 \tag{i}$$

Now

$\Rightarrow s+t > \|x,x'\| + \|x,x_0\|$

$\Rightarrow s+t \geq \|x,x',x_0\|$

so

$$N_1(x,x'+x_0,s+t) = \frac{(s+t)^2 - \|x,x'+x_0\|^2}{(s+t)^2 + \|x,x'+x_0\|^2}$$

$$\geq \frac{(s+t)^2 - (\|x,x'\| + \|x,x_0\|)^2}{(s+t)^2 + (\|x,x'\| + \|x,x_0\|)^2}$$

Again

$$\frac{(s+t)^2-\left(\|x,x'\|+\|x,x_0\|\right)^2}{(s+t)^2+\left(\|x,x'\|+\|x,x_0\|\right)^2}-\frac{s^2-\|x,x'\|^2}{s^2+\|x,x'\|^2}=\frac{2(s+t)^2\|x,x'\|^2-2s^2\left(\|x,x'\|+\|x,x_0\|\right)^2}{\left\{(s+t)^2+\left(\|x,x'\|+\|x,x_0\|\right)^2\right\}\left\{\|s^2+x,x'\|^2\right\}}$$

$$=\frac{2}{A}\left[(s+t)^2\|x,x'\|^2-2s^2\left(\|x,x'\|+\|x,x_0\|\right)^2\right]$$

Where $\mathrm{A}=\left[(\mathrm{s}+\mathrm{t})^2+\left(\|\mathrm{x},\mathrm{x}'\|+\|\mathrm{x},\mathrm{x}_0\|\right)^2\right]\left[\mathrm{s}^2+\|\mathrm{x},\mathrm{x}'\|^2\right]$

$$=\frac{2}{A}\left[t^2\|x,x'\|^2-s^2\|x,x_0\|^2+2s\|x,x'\|\left(t\|x,x'\|-s\|x,x_0\|\right)\right]\geq 0 \text{ [by (i)]}$$

Thus

$N_1(x,x'+x_0,s+t)\geq N_1(x,x',s)$ if $N_1(x,x_0,t)\geq N_1(x,x',s)$

similarly

$N_1(x,x'+x_0,s+t)\geq N_1(x,x_0,t)$ if $N_1(x,x',s)\geq N_1(x,x_0,t)$

Thus $N_1(x,x'+x_0,s+t)\geq \min\{N_1(x,x',s),N_1(x,x_0,t)\}$

(VI) for all $t_1,t_2\in R$, if $t_1<t_2\leq\|x,x'\|$ then by definition

$N_1(x,x',t_1)=N_1(x,x',t_2)=0$

suppose $t_2 > t_1 > \|x,x'\|$ then

$$\frac{t_2^2-\|x,x'\|^2}{t_2^2+\|x,x'\|^2}-\frac{t_1^2-\|x,x'\|^2}{t_1^2+\|x,x'\|^2}$$

$$=\frac{\left(t_2^2-\|x,x'\|^2\right)\left(t_1^2+\|x,x'\|^2\right)-\left(t_1^2-\|x,x'\|^2\right)\left(t_2^2+\|x,x'\|^2\right)}{\left(t_2^2+\|x,x'\|^2\right)\left(t_1^2+\|x,x'\|^2\right)}$$

≥ 0

for all $(x,x')\in A\times B$ implies

$$\frac{t_2^2-\|x,x'\|^2}{t_2^2+\|x,x'\|^2}\geq\frac{t_1^2-\|x,x'\|^2}{t_1^2+\|x,x'\|^2}$$

$\Rightarrow N_1(x,x',t_2)\geq N_1(x,x',t_1)$

Thus $N_1(x,x',t)$ is non-decreasing function.

Also

$$\lim_{t\to\infty}N_1(x,x',t)=\lim_{t\to\infty}\frac{t^2-\|x,x'\|^2}{t^2-\|x,x'\|^2}=\lim_{t\to\infty}\frac{t^2\left(1-\frac{\|x,x'\|^2}{t^2}\right)}{t^2\left(1+\frac{\|x,x'\|^2}{t^2}\right)}=1$$

Thus (X,N_1) is an fuzzy 2-normed linear space.

Now we define a fuzzy 2-linear operator

$T:A\times B\to C\times D$ be a fuzzy 2-linear operator, where A,B are subspaces of (X,N_1) and C,D are subspaces of (Y,N_2) as

$T(x,x') = (x,x') \quad \forall \ (x,x') \in A \times B$

We choose $M_\alpha = \dfrac{1}{1-\alpha} \quad \forall \ \alpha \in (0,1)$, Then for $t > \|x,x'\|$

$$N_1\left(x,x',\frac{t}{M_\alpha}\right) \geq \alpha$$

$$\Rightarrow \frac{\dfrac{t^2}{M_\alpha^2} - \|x,x'\|^2}{\dfrac{t^2}{M_\alpha^2} + \|x,x'\|^2} \geq \alpha$$

$$\Leftrightarrow \frac{t^2(1-\alpha)^2 - \|x,x'\|^2}{t^2(1-\alpha)^2 + \|x,x'\|^2} \geq \alpha$$

$$\Rightarrow t^2(1-\alpha)^2 - \|x,x'\|^2 \geq \alpha t^2(1-\alpha)^2 + \alpha\|x,x'\|^2$$

$$\Rightarrow t^2(1-\alpha)^2 - \alpha t^2(1-\alpha)^2 \geq \alpha\|x,x'\|^2 + \|x,x'\|^2$$

$$\Rightarrow t^2(1-\alpha)^2(1-\alpha) \geq \alpha\|x,x'\|^2 + \|x,x'\|^2$$

$$\Rightarrow t^2(1-\alpha)^2(1-\alpha) \geq \|x,x'\|^2(1+\alpha)$$

$$\Rightarrow \|x,x'\|^2 \leq \frac{t^2\|1-\alpha\|^2(1-\alpha)}{1+\alpha}$$

$$\Rightarrow \|x,x'\| \leq \frac{t(1-\alpha)\sqrt{1-\alpha}}{\sqrt{1+\alpha}} \qquad \text{(Since } \alpha \neq 1\text{)}$$

$$\Rightarrow t + \|x,x'\| \leq \frac{t(1-\alpha)\sqrt{1-\alpha}}{\sqrt{1+\alpha}} + t$$

$$= \frac{t(1-\alpha)\sqrt{1-\alpha} + t\sqrt{1+\alpha}}{\sqrt{1+\alpha}}$$

$$= \frac{t\left\{(1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}\right\}}{\sqrt{1+\alpha}}$$

$$\frac{t}{t+\|x,x'\|} \geq \frac{\sqrt{1+\alpha}}{\left((1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}\right)} \qquad \ldots\ldots \qquad \text{(ii)}$$

Now

$$\frac{\sqrt{1+\alpha}}{\left((1-\alpha)\sqrt{1-\alpha} + \sqrt{1+\alpha}\right)} \geq \alpha$$

$$\Leftrightarrow \sqrt{1+\alpha} \geq \alpha(1-\alpha)\sqrt{1-\alpha} + \alpha\sqrt{1+\alpha}$$

$$\Leftrightarrow (1-\alpha)\sqrt{1+\alpha} \geq \alpha(1-\alpha)\sqrt{1-\alpha}$$

$$\Leftrightarrow \sqrt{1+\alpha} \geq \alpha\sqrt{1-\alpha} \qquad \text{(Since } \alpha \neq 1\text{)}$$

$$\Leftrightarrow 1+\alpha \geq \alpha^2(1-\alpha)$$

$$\Leftrightarrow 1+\alpha+\alpha^3 \geq \alpha^2$$

This is true for all $\alpha \in (0,1)$. Thus form (ii)

We get $\frac{t}{t+\|x,x'\|} \geq \alpha$, $\quad \forall \alpha \in (0,1)$

$\Rightarrow N_2(T(x,x'),t) \geq \alpha \ if \ t > \|x,x'\|$

Again since for $t \leq \|x,x'\|$,

$$\frac{t^2 - \|x,x'\|^2}{t^2 + \|x,x'\|^2} = 0$$

If follows that, $N_1\left(x,x',\frac{t}{M_\alpha}\right) \geq \alpha$

$\Rightarrow N_2(T(x,x'),t) \geq \alpha \quad \forall \ \alpha \in (0,1)$

Thus is any case, we get

$$N_1\left(x,x',\frac{t}{M_\alpha}\right) \geq \alpha \Rightarrow N_2(T(x,x'),t) \geq \alpha \ \ \forall \alpha \in (0,1)$$

Hence T is weakly fuzzy 2-bounded.

Now for $t > \|x,x'\|$

$$N_2(T(x,x'),t) \geq N_1\left((x,x'),\frac{t}{M}\right)$$

$$\Leftrightarrow \frac{t}{t+\|x,x'\|} \geq \frac{\frac{t^2}{M^2} - \|x,x'\|^2}{\frac{t^2}{M^2} + \|x,x'\|^2}$$

$$\Leftrightarrow \frac{t}{t+\|x,x'\|} \geq \frac{t^2 - M^2\|x,x'\|^2}{t^2 + M^2\|x,x'\|^2}$$

$$\Rightarrow 2t\,M^2\|x,x'\|^2 \geq t^2\|x,x'\| - M^2\|x,x'\|\,\|x,x'\|^2$$

$$\Rightarrow M^2\|x,x'\|^2(2t + \|x,x'\|) \geq t^2\|x,x'\|$$

$$\Leftrightarrow M^2 \geq \frac{t^2}{(2t+\|x,x'\|)(\|x,x'\|)}$$

$$\Leftrightarrow M \geq \frac{t}{[(2t+\|x,x'\|)(\|x,x'\|)]^{1/2}} \qquad [\|x,x'\| > 0]$$

$\Leftrightarrow M = \infty$ as $t \to \infty$

Hence T is not strongly fuzzy 2-bounded.

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