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Number Of Real Zeros Of Random Trigonometric Polynomial

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THEOREM:-Let EN (T, Φ', Φ'') denote the average number of real zeros of the random trigonometric polynomial

$$T = T_n(\theta,\omega) = \sum_{k=1}^{n} a_k(\omega) b_k \cos k\theta$$

in the interval (Φ', Φ''). Assuming $a_k(w)$ are independent random variables identically distributed according to the normal law that $b_k = k^p\,(p \geq 0)$ are positive constants we show that

$$EN(T;0,2\pi) \sim \left\{\left(\frac{2p+1}{2p+3}\right)(1-\varepsilon n^2\right\}^{1/2} 2n + 0(\log n)\,.$$

Outside an exceptional set of measure at most (2/n) where

$$\varepsilon^2{}_n = \frac{4\beta^2(2p+1)(2p+3)}{SS'(\log n)^2}$$

β=constant   s~1   S'~1

1. INTRODUCTION:- Let N (T, Φ', Φ'') be the number of real roots of the trigonometric polynomial

$$T = T_n(\theta,\omega) = \sum_{k=1}^{n} a_k(\omega) b_k \cos k\theta \quad (1)$$

in the interval (Φ', Φ'') where the coefficients $a_k(w)$ are mutually independent random variables identically distributed according to the normal law, $b_k=k^p$ are positive constants and when multiple roots are counted only once. Let EN (T, Φ', Φ'') denote the exception of N (T, Φ', Φ''). Obviously, $T_n(\theta, w)$ can have at most 2n zeros in the interval (0, 2π). Das [1] studied the class of polynomials

$$\sum_{k=1}^{n} k^p (g_{2k-1} \cos k\theta + g_{2k-1} \sin k\theta) \quad (2)$$

where g_k are independent normal random variables for fixed p>-1/2 and proved that in the interval (0, 2π) the function (2) have

$$[(2p+1)/(2p+3)]^{1/2} 2n + O\left(n^{11/13 + 4q/13 + \varepsilon_1} \right)$$

number of real roots when n is large. Here, q= max (0, -p) and $\varepsilon_1 < (2/13)(1-2q)$. The measure of the exceptional set does not exceed $n^{-2\varepsilon_1}$.

Das [2] took the polynomial (1) where $a_k(w)$ are independent normal random variables identically distributed with mean zero and variance done. He proves that in the interval $0 \le \theta \le 2\pi$, average number of real zeros of polynomials (1) is

$$[(2p+1)/(2p+3)]^{1/2} 2n + O(n) \quad (3)$$

For $b_k = k^p \left(p > \frac{1}{2} \right)$ and of the order of $n^{p+3/2}$ if $p \le -1/2$ for large n.

In this paper we consider the polynomial (1) with conditions as in das [2] and use the Kac-Rice formula for the exception of the number of real zeros and obtain that for p≥0.

$$EN(T;0,2\pi) \sim \left\{ \left(\frac{2p+1}{2p+3} \right) (1 - \varepsilon_n{}^2 \right\}^{1/2} 2n + 0(\log n)$$

Where $$\varepsilon^2{}_n = \frac{4\beta^2 (2p+1)(2p+3)}{SS'(\log n)^2}$$

β=constant S~1 S'~1

Our asymptotic estimate implies that Das's estimate in [1] is approached from below.Also our error term is smaller.

The particular case for p=0 has been considered by Dunnage [3] and Pratihari and Bhanja [4]. Dunnage has shown that in the interval $0 \le \theta \le 2\pi$ all save a certain exceptional set of the functions $(T_n(\theta,w)$ have

$$\frac{2n}{\sqrt{3}} + O(n^{11/13} (\log n)^{3/13}) \quad (4)$$

zeros when n is large. The measure of the exceptional set does not exceed $(\log n)^{-1}$. Using the Kac-Rice formula we tried to obtain in [4] that

$$EN(T;0,2\pi) \sim \left\{\left(\frac{2n}{\sqrt{6}}\right)\right\} + 0(\log n) \qquad (5)$$

Professor Dunnage [5] comments that our result is incorrect. He is quite right when he says than an asymptotic estimate is unique and both results (4) and (5) cannot be correct. But in his calculations given in paragraph 4 and 5 he seems to have imported a factor 2 and the correct calculation would give $I' \sim 2\pi n/\sqrt{3}$. Accepting his own statement in paragraph 3 that I<I', our point is clear. However, since $I' \sim 2\pi n/\sqrt{3}$ on direct integration, our estimation of EN as found in [4], contained in the statement (5) above, must be wrong. We are sorry about our mistake. In this paper we consider our original integral I and evaluate it directly instead of placing it between two integrals as in [4], the second one being possibly suspect. This rectification eventually raises our estimate for EN but, all the same, keeps it below Dunnage's estimate stated in (4) above. The purpose of our result is that EN approaches the value $2n/\sqrt{3}$ from below. This is something meaningful. We prove the following theorem.

Theorem. The average number of real zeros in the interval $(0, 2\pi)$ of the class of random trigonometric polynomials of the form

$$\sum_{k=1}^{n} a_k(\omega) b_k \cos k\theta$$

where $a_k(w)$ are mutually independent random variables identically distributed according to the normal law with mean zero and variance one and $b_k=k^p (p\geq 0)$ are positive constants, is asymptotically equal to

$$\left\{\left(\frac{2p+1}{2p+3}\right)(1-\varepsilon_n{}^2\right\}^{1/2} 2n + 0(\log n)$$

outside an exceptional set of measure at most (2/n) where

$$\varepsilon^2{}_n = \frac{4\beta^2(2p+1)(2p+3)}{SS'(\log n)^2}$$

β=constant S~1 S'~1

2. THE APPROXIMATION FOR $EN(T;0,2\pi) =$ Let L(n) be a positive-valued function of n such that L(n) and n/L(n) both approach infinity with n. We take $\varepsilon = L(n)/n$ throughout.

Outside a small exceptional set of values of w, $(T_n(\theta w)$ has a negligible number of zeros in each of the intervals $(0,\varepsilon), (\pi-\varepsilon, \pi+\varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$. By periodicity, the number of zeros in $(0,\varepsilon)$ and $(2\pi-\varepsilon, 2\pi)$ is the same as the number in $(-\varepsilon,\varepsilon)$. We shall use the following lemma, which is due to Das[2].

***LEMMA* :-** The probability that T has more than $1+(\log n)^{-1}(\log n+\log D_n+4n\varepsilon)$ zeros in $\omega-\varepsilon\le\theta\le\omega+\varepsilon$ does not exceed 2 exp (-nε), where $D_n=\sum_{k=1}^{n} b_k$

The step in this section follow closely those in section 2 of 4. Therefore, we indicate only the modifications necessary. In this case we have

$$T=\sum_{k=1}^{n} a_k(\omega)b_k\cos k\theta \qquad T'=-\sum_{k=1}^{n} ka_k(\omega)b_k\sin k\theta$$

$$\varphi(y,z)=\exp\left\{-\frac{1}{2}\sum_{k=1}^{n}(y\cos k\theta-zk\sin k\theta)^2 b_k{}^2\right\}$$

$$p(0,\eta)=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}\exp(i\eta z)\exp\left\{-\frac{1}{2}\sum_{k=1}^{n}(y\cos k\theta-zk\sin k\theta)^2 b_k{}^2\right\}dy$$

and finally

$$\int_{-\infty}^{\infty}|\eta|p(0,\eta)dn=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{k=1}^{n}(y\cos k\theta-zk\sin k\theta)^2 b_k{}^2}{(Au+B)^2}\right\}du \qquad (6).$$

for *fixed non-zero real constants A and B to be chosen.*

3. ESTIMATION OF THE INTEGRAL OF EQUATION :-Consider the integral

$$I=\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{k=1}^{n}(u\,\mathrm{Cos}\,k\theta+k\,\mathrm{Sin}\,k\theta)^2}{(Au+B)^2}\right\}du$$

which exists in general as a principal value if , $A^2=\sum_{k=1}^{n}\cos^2 k\theta$

Let $B^2=\sum_{1}^{n}k^2\sin^2 k\theta$ and $C^2=\sum_{1}^{n}k\cos k\theta\sin k\theta$.

As in Das[2] letting $b_k=k^p (p\ge 0)$ we get

$$A^2=\frac{1}{2}\frac{n^{2p+1}}{2p+1}\left\{1+0\left(\frac{1}{L(n)}\right)\right\}=\frac{1}{2}S\frac{n^{2p+1}}{2p+1} \qquad \textit{say,}$$

$$B^2=\frac{1}{2}\frac{n^{2p+3}}{2p+3}\left\{1+0\left(\frac{1}{L(n)}\right)\right\}=\frac{1}{2}S'\frac{n^{2p+3}}{2p+3} \qquad \textit{say,}$$

$$C^2=0\left(\frac{n^{2p+2}}{L(n)}\right)=\frac{\beta n^{2p+2}}{L(n)} \qquad (\beta=\textit{constant})$$

Outside the set $\{0,\pm\pi,\pm 2\pi\}$ of the values of θ, $AB>C^2$. We have

$$I=\int_{-\infty}^{\infty}\log\left\{\frac{\sum_{k=1}^{n}(u\cos k\theta+k\sin k\theta)^2 b^2{}_k}{(Au+B)^2}\right\}du$$

$$=\int_{-\infty}^{\infty}\log\left\{\frac{u^2+2c^2u+b^2}{u^2+2bu+b^2}\right\}du$$

where c= (C/A) and b=(B/A). Now by integration by parts,

$$\int_0^x\log(u^2+2c^2u+b^2)du=[u\log(u^2+2c^2u+b^2)]_0^X-\int_{\infty}^{x}\left\{\frac{2u^2+2c^2u}{u^2+2c^2u+b^2}\right\}du$$

$$=2X\log X+2c^2-2X+2\int_0^x\frac{c^2u+b2}{u^2+2c^2u+b^2}du+0\left(\frac{1}{X}\right)$$

and

$$\int_{-x}^{0}\log(u^2+2c^2u+b^2)du=\int_0^x\log(u^2-2c^2u+b^2)du$$

$$=2X\log X-2c^2-2X+2\int_0^x\frac{-c^2u+b2}{u^2-2c^2u+b^2}du+0\left(\frac{1}{X}\right)$$

Therefore

$$\int_{-x}^{x}\log(u^2+2c^2u+b^2)du=4X\log X-4X+2\int_{-x}^{x}\frac{c^2u+b^2}{u^2+2c^2u+b^2}du+0\left(\frac{1}{X}\right)$$

Again

$$\int_{-x}^{x}\log(u^2+2bu+b^2)du=4X\log X-4X+2\int_{-x}^{x}\frac{bu+b^2}{u^2+2bu+b^2}du+0\left(\frac{1}{X}\right)$$

Hence the integral

$$\int_{-x}^{x}\log\left(\frac{u^2+2c^2u+b^2}{u^2+2bu+b^2}\right)du=2\int_{-x}^{x}\frac{c^2u+b^2}{u^2+2c^2u+b^2}du-2\int_{-x}^{x}\frac{bu+b^2}{u^2+2bu+b^2}du+0\left(\frac{1}{X}\right)$$

Obviously

$$2\int_{-x}^{x}\left(\frac{c^2u+b^2}{u^2+2c^2u+b^2}\right)du=c^2\left[\log(u^2+2c^2u+b^2)\right]_{-x}^{x}+2\sqrt{(b^2-c^4)}\left[\tan^{-1}\frac{u^2+2c^2u+b^2}{\sqrt{(b^2-c^4)}}\right]_{-X}^{X}$$

$$c^2\log\left(\frac{X^2+2c^2X+b^2}{X^2-2c^2X+b^2}\right)+2\sqrt{(b^2-c^4)}\tan^{-1}\left(\frac{2X+\sqrt{(b^2-c^4)}}{b^2-X^2}\right)$$

When $X\rightarrow\infty$, we have

$$\int_{-\infty}^{\infty}\left(\frac{c^2u+b^2}{u^2+2c^2u+b^2}\right)du = \pi\sqrt{\left(b^2-c^4\right)}$$

$$\int_{-\infty}^{\infty}\left(\frac{bu+b^2}{u^2+2bu+b^2}\right)du = 0$$

And $\quad 0(\frac{1}{X}) = 0$

Therefore

$$I = \lim_{x\to\infty}\int_{-\infty}^{\infty}........\int_{-\infty}^{\infty}\log\left(\frac{u^2+2c^2u+b^2}{u^2+2bu+b^2}\right)du = 2\pi(b^2-c^4)^{1/2} = 2\pi\left(\frac{A^2B^2-C^4}{A^4}\right) \quad (7)$$

$$= \left\{\left(\frac{S'}{S}\right)\left(\frac{2p+1}{2p+3}\right)(1-\in^2{}_n)\right\}2\pi n$$

where

$$\in^2{}_n = \frac{4\beta^2(2p+1)(2p+3)}{SS'\{L(n)\}^2} \qquad S\sim 1 \qquad S''\sim 1$$

Thus

$$I \sim \left[(2p+1)/(2p+3)^{1/2}\, 2\pi n(1-\in^2{}_n)\right]^{1/2}$$

Now the result follows from the section 4 of [4], choosing L(n)=log n. The cases where –1/2< p<0 and p=-1/2 can be similarly dealt with and results can be obtained to show that Das's estimate are approached .

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