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Analysis of Occurrence of a Non-Zero Digit in All Base *b* Natural Numbers Less Than b^n

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Abstract

Adopting base b number system for any positive integer $b > 1$, all successive natural numbers, i.e., positive integers, less than b^n for any positive integer n are under consideration. A rigorous analysis of occurrence of the very first natural number and non-zero digit 1 in all numbers less than b^n is done here. The formula for the number of occurrences of 1's is developed. The first instance of 1 is easy to predict, so about the last occurrence also; whose formulations are provided. All the analysis is extended to multiple number of occurrences of 1's. These results are generalized for all non-zero digits. For specific instance purpose, base $b = 16$ is exemplified.

Keywords: All occurrences, natural numbers, non-zero digits.

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1. Introduction

Most often used natural numbers form infinite sequence 1, 2, 3, $\cdots$ which is the simplest arithmetical progression with first term 1 and common difference also 1. They are the foundation of number system [8].

In our routine course, we use decimal system, i.e., number system with base 10, which has 10 digits, viz., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Theoretically number system with any base $b > 1$ is employable. It will have b number of digits. If $b \leq 10$, we denote these digits by 0, 1, 2, $\cdots$, $b - 1$. In case $b > 10$, we denote first 10 digits by 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and remaining successive digits by A, B, C, $\cdots$, X where X is not necessarily alphabet X but merely a symbol to designate the alphabet at number equal to 9 more than its own number in the list of alphabets.

Here we use the term number with implied meaning of natural number. We adopt the modern convention that 0 is not in the set of Natural numbers N and, for $n \in N$, consider the ranges $1 - b^n$, leaving b^n, the numbers considered being m, with $1 \leq m < b^n$. The last number b^n is left as it contains more number $(n + 1)$ of digits in base b.

2. Occurrence of Digit 1

We start analysis with digit 1; it being first natural number [1] and being one that is present in number systems with all bases $b > 1$. It's all types occurrences in usual decimal system have been analyzed in [2], [3], [4] as also of 0 in [5], [6], [7].

For exemplifying computational purpose, we choose electronic computer system favoured base $b = 16$.

Counts of occurrence of single 1 and also double 1's in numbers less than 16^{15}, which is a little more than a quintillion, determined by using computer language Java program are as given below. While the base of numbers under analysis is 16, we prefer writing their count by using usual base 10!

Table 1: Number of Hexadecimal Numbers with Single and Double 1's in their Digits

Sr. No.	Numbers Range Less Than	Number of Numbers in Base 16 with single 1	Number of Numbers in Base 16 with two 1's
1.	16^1	1	0
2.	16^2	30	1
3.	16^3	675	45
4.	16^4	13,500	1,350
5.	16^5	253,125	33,750
6.	16^6	4,556,250	759,375
7.	16^7	79,734,375	15,946,875
8.	16^8	1,366,875,000	318,937,500
9.	16^9	23,066,015,625	6,150,937,500
10.	16^{10}	384,433,593,750	115,330,078,125
11.	16^{11}	6,343,154,296,875	2,114,384,765,625
12.	16^{12}	103,797,070,312,500	38,058,925,781,250
13.	16^{13}	1,686,702,392,578,125	674,680,957,031,250
14.	16^{14}	27,246,730,957,031,250	11,806,916,748,046,875
15.	16^{15}	437,893,890,380,859,360	204,350,482,177,734,375

During the first range $1 \le m < 16^1 = 16$, single 1 occurs merely once as itself a number. Its count can be seen as ${}^1C_1 15^{1-1} = 1$.

In the second range $1 \le m < 16^2 = 256$, single 1 comes 30 times. Its 15 instances are at unit's places in numbers

1, 21, 31, 41, 51, 61, 71, 81, 91, A1, B1, C1, D1, E1 and F1

and 15 instances are at ten's places in numbers

10, 12, 13, 14, 15, 16, 17, 18, 19, 1A, 1B, 1C, 1D, 1E and 1F.

So, second block has it ${}^2C_1 15^{2-1} = 2 \times 15 = 30$ times.

Also, in this range, double 1 occurs only once in number 11. This is be seen as ${}^2C_2 15^{2-2} = 1 \times 1 = 1$ time.

In the third range, $1 \le m < 16^3 = 4{,}096$, single 1 is seen total 675 times in three blocks of numbers: in

1, 21, 31,⋯, F1, 201, 221,⋯, 2F1, 301, 321,⋯, 3F1,⋯, F01, F21,⋯, FF1,

at unit's places, in

12, 13, 14,⋯, 1F, 210, 212,⋯, 21F, 310, 312,⋯, 31F,⋯, F10, F12,⋯, F1F,

at ten's places and in

100, 102, 103,⋯, 10F, 120, 122, 123,⋯, 130, 132, 133,⋯, 13F,⋯, 1F0, 1F2, 1F3,⋯, 1FF

at hundred's places.

This third block occurrence is thus ${}^3C_1 15^{3-1} = 3 \times 15^2 = 3 \times 225 = 675$ times.

Now here, double 1's occur in

211, 311,⋯, F11,

at unit's and ten's places and in

101, 121, 131,⋯, 1F1

at unit's and hundred's places and then in

110, 112, 113,⋯, 11F

at ten's and hundred's places. This count is ${}^3C_2 15^{3-2} = 3 \times 15 = 45$ times.

The counts in above table can thus be explained.

There are analogous occurrences of multiple 1's in these ranges. The pattern in these figures is prone to be captured in formula.

Notation : We introduce the generalized notation ${}^{A}_{1}\underset{b}{O}{}^{n}_{r}$ for number of numbers in base b less than b^n with r number of 1's.

Theorem 1 : If r, n and b are positive integers with $r \le n$ and $b > 1$, then the number of numbers in base b containing exactly r number of digit 1's in the range $1 \le m < b^n$ is

$$ {}_{\substack{1\\ b}}^{A}O_r^n = {}^nC_r (b-1)^{n-r} . $$

Proof. Let n, r and b be positive integers with $r \le n$ and $b > 1$. There are in total b digits available to occupy n places in all numbers in base b in range $1 \le m < b^n$. We want r places to be occupied by digit 1. The various choices for these r places for digit 1 will be ${}^nC_r$ in number. Now for each such choice, remaining $n - r$ places are to be occupied by any of the remaining $(b - 1)$ digits except 1 and there are $(b - 1)^{n-r}$ choices for each of that. This totals to ${}^nC_r(b - 1)^{n-r}$ and hence ${}_{\substack{1\\ b}}^{A}O_r^n = {}^nC_r (b-1)^{n-r}$.

This completes the proof of the theorem.

The table given above for $b = 16$ can be extended to higher occurrences of 1's by using this formula. The counts given by us are in usual decimal base!

Table 2: Number of Hexadecimal Numbers with Multiple 1's in their Digits

Sr. No.	Number Range <	Number of Numbers in Base 16 with 3 1's	Number of Numbers in Base 16 with 4 1's	Number of Numbers in Base 16 with 5 1's
1.	16^3	1	0	0
2.	16^4	60	1	0
3.	16^5	2,250	75	1
4.	16^6	67,500	3,375	90
5.	16^7	1,771,875	118,125	4,725
6.	16^8	42,525,000	3,543,750	189,000
7.	16^9	956,812,500	95,681,250	6,378,750
8.	16^{10}	20,503,125,000	2,392,031,250	191,362,500
9.	16^{11}	422,876,953,125	56,383,593,750	5,262,468,750
10.	16^{12}	8,457,539,062,500	1,268,630,859,375	135,320,625,000
11.	16^{13}	164,922,011,718,750	27,487,001,953,125	3,298,440,234,375
12.	16^{14}	3,148,511,132,812,500	577,227,041,015,625	76,963,605,468,750
13.	16^{15}	59,034,583,740,234,375	11,806,916,748,046,875	1,731,681,123,046,875

Table 2 : Continued …

Sr. No.	Number Range <	Number of Numbers in Base 16 with 6 1's	Number of Numbers in Base 16 with 7 1's	Number of Numbers in Base 16 with 8 1's
1.	16^6	1	0	0
2.	16^7	105	1	0
3.	16^8	6,300	120	1
4.	16^9	283,500	8,100	135
5.	16^{10}	10,631,250	405,000	10,125
6.	16^{11}	350,831,250	16,706,250	556,875
7.	16^{12}	10,524,937,500	601,425,000	25,059,375
8.	16^{13}	293,194,687,500	19,546,312,500	977,315,625
9.	16^{14}	7,696,360,546,875	586,389,375,000	34,206,046,875
10.	16^{15}	192,409,013,671,875	16,492,201,171,875	1,099,480,078,125

Table 2 : Continued …

Sr. No.	Number Range <	Number of Numbers in Base 16 with 9 1's	Number of Numbers in Base 16 with 10 1's	Number of Numbers in Base 16 with 11 1's
1.	16^9	1	0	0
2.	16^{10}	150	1	0
3.	16^{11}	12,375	165	1
4.	16^{12}	742,500	14,850	180
5.	16^{13}	36,196,875	965,250	17,550
6.	16^{14}	1,520,268,750	50,675,625	1,228,500
7.	16^{15}	57,010,078,125	2,280,403,125	69,103,125

Table 2 : Continued …

Sr. No.	Number Range <	Number of Numbers in Base 16 with 12 1's	Number of Numbers in Base 16 with 13 1's	Number of Numbers in Base 16 with 14 1's	Number of Numbers in Base 16 with 15 1's
1.	16^{12}	1	0	0	0
2.	16^{13}	195	1	0	0
3.	16^{14}	20,475	210	1	0
4.	16^{15}	1,535,625	23,625	225	1

3. First Occurrence of Digit 1

Regardless of base of the number system, the first number containing 1 is 1 itself. Containing 2 1's, the first number is 11, for 3 it is 111 and so on. Of course, these are in corresponding base *b* and hence it must be remembered that their values in decimal come out to be different.

Formula 1 : If *n*, *r* and *b* are natural numbers with *b* > 1, then the first occurrence of *r* number of 1's in numbers in base *b* in range $1 \le m < b^n$ is

$$f = \begin{cases} - & , \text{ if } r > n \\ \sum_{j=0}^{r-1}\left(1 \times b^j\right), & \text{ if } r \le n \end{cases}.$$

4. Last Occurrence of Digit 1

Taking same base *b* = 16, it is interesting to determine the last occurrences of 1. The last number in ranges shows an interesting pattern.

Table 3: Last Hexadecimal Numbers with Multiple 1's in their Digits in Various Ranges

Sr. No.	Number Range < → Last Number with ↓	16^1	16^2	16^3	16^4	16^5	16^6	16^7	16^8	16^9
1.	1 1	1	F1	FF1	F,FF1	FF,FF1	FFF,FF1	F,FFF,FF1	FF,FFF,FF1	FFF,FFF,FF1
2.	2 1's	-	11	F11	F,F11	FF,F11	FFF,F11	F,FFF,F11	FF,FFF,F11	FFF,FFF,F11
3.	3 1's	-	-	111	F,111	FF,111	FFF,111	F,FFF,111	FF,FFF,111	FFF,FFF,111
4.	4 1's	-	-	-	1,111	F1,111	FF1,111	F,FF1,111	FF,FF1,111	FFF,FF1,111
5.	5 1's	-	-	-	-	11,111	F11,111	F,F11,111	FF,F11,111	FFF,F11,111
6.	6 1's	-	-	-	-	-	111,111	F,111,111	FF,111,111	FFF,111,111
7.	7 1's	-	-	-	-	-	-	1,111,111	F1,111,111	FF1,111,111
8.	8 1's	-	-	-	-	-	-	-	11,111,111	F11,111,111
9.	9 1's	-	-	-	-	-	-	-	-	111,111,111

For every viable base *b*, their formulation is provided ahead.

Formula 2 : If *n*, *r* and *b* are natural numbers with *b* > 1, then the last occurrence of *r* number of 1's in numbers in base *b* in range $1 \le m < b^n$ is

$$l = \begin{cases} - & , \text{ if } r > n \\ \sum_{j=0}^{r-1}\left(1 \times b^j\right) + \begin{cases} 0 & , \text{ if } r = n \\ \sum_{j=r}^{n-1}\left((b-1) \times b^j\right), & \text{if } r < n \end{cases} \end{cases}.$$

Each integer sequence for every base gives a new example for further study.

5. Extension to Other Non-zero Digits

We conclude by noting an important thing that every discussion that has been done for occurrences of digit 1 is parallely applicable for all other non-zero digits. Denoting non-zero digit of interest by d, where $1 \le d < b$, the range under consideration is as usual $1 \le m < b^n$ with $1 \le r \le n$.

Notation : We further generalize the notation ${}^{A}_{d}\underset{b}{O}{}^{n}_{r}$ for number of numbers in base b less than b^n with r number of digits d's.

Theorem 2 : If r, n, d, and b are positive integers with $r \le n$ and $1 \le d < b$, then the number of numbers in base b containing exactly r number of digit d's in the range $1 \le m < b^n$ is

$$ {}^{A}_{d}\underset{b}{O}{}^{n}_{r} = {}^{n}C_{r}\,(b-1)^{n-r} \,. $$

Proof. Because the presence of each digit d, $1 \le d < b$, is quantitatively equal in base b numbers in the total range $1 \le m < b^n$, the proof is same as that for Theorem 1.

Formula 3 : If n, r, d and b are natural numbers with $1 \le d < b$, then the first occurrence of r number of digit d's in numbers in base b in range $1 \le m < b^n$ is

$$ f = \begin{cases} - & \text{, if } r > n \\ \sum_{j=0}^{r-1}\left(d \times b^{j}\right), & \text{if } r \le n \end{cases} . $$

Formula 4 : If n, r, d and b are natural numbers with $1 \le d < b$, then the last occurrence of r number of digit d's in numbers in base b in range $1 \le m < b^n$ is

$$ l = \begin{cases} - & \text{, if } r > n \\ \sum_{j=0}^{r-1}\left(d \times b^{j}\right) + \begin{cases} 0 & \text{, if } r = n \\ \sum_{j=r}^{n-1}\left((b-1) \times b^{j}\right), & \text{if } r < n \end{cases} \end{cases} . $$

Remark : All the work here has generalized the earlier results in [2].

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References

[1] Neeraj Anant Pande, "Numeral Systems of Great Ancient Human Civilizations", Journal of Science and Arts, Year 10, No. 2 (13), pp. 209-222, (2010).

[2] Neeraj Anant Pande, "Analysis of Occurrence of Digit 1 in Natural Numbers Less Than 10^n", Advances in Theoretical and Applied Mathematics, 11(2), pp. 99-104, (2016).

[3] Neeraj Anant Pande, "Analysis of Successive Occurrence of Digit 1 in Natural Numbers Less Than 10^n", American International Journal of Research in Science, Technology, Engineering and Mathematics, 16(1), pp. 37-41, (2016).

[4] Neeraj Anant Pande, "Analysis of Non-successive Occurrence of Digit 1 in Natural Numbers Less Than 10^n", International Journal of Advances in Mathematics and Statistics, Accepted, (2016).

[5] Neeraj Anant Pande, "Analysis of Occurrence of Digit 0 in Natural Numbers Less Than 10^n", American International Journal of Research in Formal, Applied and Natural Sciences, Communicated, (2016).

[6] Neeraj Anant Pande, "Analysis of Successive Occurrence of Digit 0 in Natural Numbers Less Than 10^n", IOSR-Journal of Mathematics, Accepted, (2016).

[7] Neeraj Anant Pande, "Analysis of Non-successive Occurrence of Digit 0 in Natural Numbers Less Than 10^n", International Journal of Emerging Technologies in Computational and Applied Sciences, Communicated, (2016).

[8] Nishit K. Sinha, "Demystifying Number System", Pearson Education, New Delhi, 2010.