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Another Approach for Exponential Matrix

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Abstract

The exponential matrix e^{tA} is a very important subclass of matrix functions. It is a very useful tool on solving linear systems of first order which help us in the analysis of controllability and observability of a linear system from first order of differential equations and control theory. In this paper, we introduce new approach or method to calculate the matrix exponential e^{tA} where the matrix $A \in C^{n \times n}$ has an eigenvalue $\lambda_1 = 0$. In the last part of this paper we provide some examples to show the effectiveness of this method.

Keywords: matrix theory, matrix exponential, eigenvalues, Vandermonde matrix.

1. INTRODUCTION

The exponential matrix e^{tA} is a very important a particular matrix functions. It is a very useful tool for solving linear systems, which help us in the analysis of controllability and observability of a linear system and control theory .Basically, matrix functions are widely used in science area especially in matrix analysis. It provides a formula for closed solutions, which help us in the analysis of controllability and observability of a linear system [1,6]. There are several methods for calculating the exponential matrix, some of these methods are effective and others not effective. The differential equation $x'(t) = Ax(t)$ has the solution $e^{tA}x(0),$ which plays an important role in linear system and control theory. It is well known that the exponential matrix e^{At} can be defined by a convergent power series $e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$

However, calculating the powers of matrix A which is an infinite sum , makes the researchers design methods to get the accurate solution of e^{tA}. In recent years, there exist many methods for computing e^{tA} for more details [2,3,4,5,7,8]. Among these methods, for example, the explicit formulae can overcome the truncation errors which are widely used in these papers [2, 4, 5,7,8], which are based on this paper [4]. In this paper, we introduce new method to compute the matrix exponential e^{tA} where the matrix $A \in C^{n\times n}$ has an eigenvalues $\lambda_1 = 0$. This method comput**e** the accurate solution of e^{tA} where the matrix $A \in C^{n\times n}$ satisfy this condition $A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2}$, Where ω_1 *and* ω_2 are parameters. When $\omega_1 = 0$ or $\omega_2 = 0$, then the matrix A is the same as the matrix in [4]. In the last part of this chapter, we provide examples to show the effectiveness of this method.

2. THE MAIN RESULTS

It is well known that the exponential matrix e^{At} can be defined by a convergent power series

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = I + tA + \frac{(tA)^2}{2!} + \ldots\ldots + \frac{(tA)^{n-1}}{(n-1)!} + \ldots\ldots\ldots \tag{1}$$

Some authors and researchers gave explicit formulae to compute the exponential matrix, for example, Bernstein and So in his papers [2,7] found a new method to compute the exponential matrix e^{tA} when $A^2 = A$, $A^2 = \omega I_n$ *and* $A^3 = \omega A$ and Wu, B. B. [4] found a new method to compute the exponential matrix e^{tA} when:

$$A^{n+1} = \omega A^n \ , A^{n+2} = \omega^2 A^n \ and \ A^{n+3} = \omega^3 A^n .$$

In this paper, we will generalize a method which presented in Wu, B. B. [4] to compute the exponential matrix e^{At} when $A \in C^{n\times n}$ has this form

$$A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3} + \ldots\ldots. + \omega_k A^{n-k} \ ,k < n\,.$$

Suppose $A^n = A^{n-1} + A^{n-2}$, then $A^{n+1} = 2A^{n-1} + A^{n-2}$ continue this process until we get $A^{n+k} = \alpha_1 A^{n-1} + \alpha_2 A^{n-2}$, by using this formulae, we rewrite equation (1) as follow

$$e^{At} = I + tA + \frac{t^2 A^2}{2!} + \ldots\ldots\ldots + \alpha_1(t)A^{n-2} + \alpha_2(t)A^{n-1} \tag{2}$$

To compute the parameters $\alpha_1(t), \alpha_2(t)$ as the following process:

From equations (1,2) and putting $A^n = A^{n-1} + A^{n-2}$, $A^{n+1} = 2A^{n-1} + A^{n-2}$ and so on in all terms and collect them in the same kind we get

$$I + \frac{tA}{1!} + \ldots.. + \frac{t^{n-2}A^{n-2}}{(n-2)!} + \frac{t^{n-1}A^{n-1}}{(n-1)!} + \ldots. = I + \frac{tA}{1!} + \ldots. + \alpha_1(t)A^{n-2} + \alpha_2(t)A^{n-1}$$

By comparing all factors we get

$$\alpha_1(t) = \frac{t^{n-2}}{(n-2)!} + \frac{t^n}{(n)!} + \frac{t^{n+1}}{(n+1)!} + \frac{2t^{n+2}}{(n+2)!} + \ldots.. \tag{3}$$

$$\alpha_2(t) = \frac{t^{n-1}}{(n-1)!} + \frac{t^n}{(n)!} + \frac{2t^{n+1}}{(n+1)!} + \frac{3t^{n+2}}{(n+2)!} + \ldots.. \tag{4}$$

By looking to equations (3) and (4) we see that infinite series, so it is difficult to find the exact form for $\alpha_1(t), \alpha_2(t)$, which we will discuss them in next section.

Definition (1)

A matrix

$$\begin{bmatrix} 1 & s_1 & s_1^2 & \cdots & s_1^{n-1} \\ 1 & s_2 & s_2^2 & \cdots & s_2^{n-1} \\ 1 & s_3 & s_3^2 & \cdots & s_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & s_n^2 & \cdots & s_n^{n-1} \end{bmatrix}$$

is called Vandermonde matrix, and

$$\begin{vmatrix} 1 & s_1 & s_1^2 & \cdots & s_1^{n-1} \\ 1 & s_2 & s_2^2 & \cdots & s_2^{n-1} \\ 1 & s_3 & s_3^2 & \cdots & s_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_n & s_n^2 & \cdots & s_n^{n-1} \end{vmatrix} = \prod_{1 \le j < i \le n} (s_i - s_j)$$

is the determinant of the Vandermonde matrix .

Definition (2)

The characteristic polynomial of the matrix $A \in C^{n \times n}$ is $f(\lambda)$, then

$$f(A) = A^n - c_{n-1}A^{n-1} - c_{n-2}A^{n-2} - \dots\dots\dots - c_1 I = 0$$

By applying definition (2) to equation (6.1) we have the following equation:

$$e^{At} = c_0(t)I + c_1(t)A + c_2(t)A^2 + \dots + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \qquad (5)$$

Because the matrix A satisfies the assumption $A^n = A^{n-1} + A^{n-2}$, then we find the characteristic polynomial of the matrix A and determine the eigenvalues of the matrix A , which are three distinct eigenvalues $\lambda_1 = 0$ *and* λ_2, λ_3. The first eigenvalue $\lambda_1 = 0$ of the matrix A is $(n-2)$ eigenvalues. To find the coefficients $c_0(t), c_1(t), \dots, c_{n-1}(t)$ by comparing the equations (2) and (5) we have the matrix equation as the following:

$$(c_0(t) - 1)I + (c_1(t) - \frac{t}{1!})\lambda_i + (c_2(t) - \frac{t^2}{2!})\lambda_i^2 + \dots +$$
$$+ (c_{n-2}(t) - \alpha_1(t))\lambda_i^{n-2} + (c_{n-1}(t) - \alpha_2(t))\lambda_i^{n-1} = 0 \qquad (6)$$

As we mentioned above the matrix A has three eigenvalues and by using Vandermonde matrix and replace $\lambda_i = s_i$ we construct the matrix equation as the following:

$$\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{n-1} \\ 1 & \lambda_3 & \lambda_3^2 & \ldots & \lambda_3^{n-1} \end{bmatrix} \begin{bmatrix} c_0(t)-1 \\ c_1(t)-\dfrac{t}{1!} \\ \vdots \\ c_{n-2}(t)-\alpha_1(t) \\ c_{n-1}(t)-\alpha_2(t) \end{bmatrix} = 0 \quad ,n>3 \tag{7}$$

By looking to the matrix equation (7), there are two or more than two solutions, so we can't obtain the exact form of $\alpha_1(t),\alpha_2(t)$ by using $c_0(t),c_1(t),c_2(t),.....c_{n-2},c_{n-1}(t)$. To avoid this problem and compute $c_0(t),c_1(t),c_2(t),.....c_{n-2},c_{n-1}(t),$we substitute $\lambda_1 = 0$ in (6), we get $c_0(t)=1$, and after the derivative of (6) with respect to λ_1 we get

$$\begin{aligned}&(c_1(t)-\frac{t}{1!})+2(c_2(t)-\frac{t^2}{2!})\lambda_i^{\ 2}+....+(n-2)(c_{n-2}(t)-\\ &-\alpha_1(t))\lambda_i^{\ n-3}+(n-1)(c_{n-1}(t)-\alpha_2(t))\lambda_i^{\ n-2}=0\end{aligned} \tag{8}$$

By substituting $\lambda_1 = 0$ in (8) we get $c_1(t)=\dfrac{t}{1!}$, continue this process (n-3) derivation and substitute $\lambda_1 = 0$ to get $c_2(t),.....c_{n-3}(t)$. Finally, we get

$$(c_{n-2}(t)-\alpha_1(t))\lambda_i^{\ n-2}+(c_{n-1}(t)-\alpha_2(t))\lambda_i^{\ n-1}=0 \tag{9}$$

Now, we construct the matrix equation as a linear system by taking λ_2,λ_3 in (9) we get

$$\begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ \lambda_3^{n-2} & \lambda_3^{n-1} \end{bmatrix} \begin{bmatrix} c_{n-2}(t)-\alpha_1(t) \\ c_{n-1}(t)-\alpha_2(t) \end{bmatrix} = 0 \tag{10}$$

Where determinate of (6.10) doesn't equal zero. Then, we conclude that $c_{n-2}(t)=\alpha_1(t)\ \ and\ \ c_{n-1}(t)=\alpha_2(t)$.

Lemma (1)

Let $A \in C^{n\times n}$ and satisfies this condition $A^n = A^{n-1} + A^{n-2}$, then the exponential matrix computed by

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \tag{11}$$

Where $c_{n-2}(t)$ *and* $c_{n-1}(t)$ are computed by

$$\begin{pmatrix} c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ \lambda_3^{n-2} & \lambda_3^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_3^i}{i!} \end{pmatrix}$$

$$\text{and } \lambda_2 = \frac{1+\sqrt{5}}{2}, \lambda_3 = \frac{1-\sqrt{5}}{2} \tag{12}$$

Proof

From equations (2) and (5) and we can rewrite them and get

$$e^{At} = I + tA + \frac{t^2 A^2}{2!} + \dots\dots + \alpha_1(t)A^{n-2} + \alpha_2(t)A^{n-1}$$

$$e^{At} = c_0(t)I + c_1(t)A + c_2(t)A^2 + \dots + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1}$$

and from equation (8) we have from that process

$c_0(t) = I, c_1(t) = \frac{t}{1!}, c_2 = \frac{t^2}{2!} \dots\dots\dots \text{and } c_{n-2}(t) = \alpha_1(t), c_{n-1}(t) = \alpha_2(t)$, then after comparing them and substitution by $c_{n-2}(t) = \alpha_1(t), c_{n-1}(t) = \alpha_2(t)$ in equation (2) we get,

$$e^{At} = I + tA + \frac{t^2 A^2}{2!} + \dots\dots + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1}$$

Now, replacing A by $\lambda_i, i = 2,3$, we get the exponential matrix as following:

$$e^{\lambda_i t} = I + t\lambda_1 + \frac{t^2 \lambda_i^2}{2!} + \dots\dots + c_{n-2}(t)\lambda_i^{n-2} + c_{n-1}(t)\lambda_i^{n-1} \tag{13}$$

To compute $c_{n-2}(t)$ *and* $c_{n-1}(t)$ from equation (6.13) after simplifying it we get,

$$e^{\lambda_i t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_i^i}{i!} = c_{n-2}(t)\lambda_i^{n-2} + c_{n-1}(t)\lambda_i^{n-1} \tag{14}$$

Take λ_2 in (14) and take λ_3 to construct the matrix equation to compute $c_{n-2}(t)\, and\;\; c_{n-1}(t)$ as following:

$$\begin{pmatrix} c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ \lambda_3^{n-2} & \lambda_3^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-3} \dfrac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-3} \dfrac{t^i \lambda_3^i}{i!} \end{pmatrix} \quad . \tag{15}$$

Lemma (2)

Let $A \in C^{n\times n}$ and satisfies $A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2}$, then the exponential matrix computed by,

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \ldots. + c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \tag{16}$$

Where $c_{n-2}(t)\;\; and\; c_{n-1}(t)$ are computed by,

$$\begin{pmatrix} c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ \lambda_3^{n-2} & \lambda_3^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-3} \dfrac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-3} \dfrac{t^i \lambda_3^i}{i!} \end{pmatrix} \tag{17}$$

now, if ω_1 *and* ω_2 *satisfy this equation* $\omega_1^2 + 4\omega_2 > 0$ *then, we compute* λ_2 *and* λ_3 *as* $\lambda_2 = (\omega_1 + \sqrt{(\omega_1^2 + 4\omega_2)}\;/2)$ *and* $\lambda_3 = (\omega_1 - \sqrt{(\omega_1^2 + 4\omega_2)}\;/2)$.

now, if ω_1 *and* ω_2 *satisfy this equation* $\omega_1^2 + 4\omega_2 < 0$ *then, we compute* λ_2 *and* λ_3 *as* $\lambda_2 = (\omega_1 + \sqrt{(-\omega_1^2 - 4\omega_2 i)}\;/2)$ *and* $\lambda_3 = (\omega_1 - \sqrt{(-\omega_1^2 - 4\omega_2 i)}\;/2)$.

Notation (1): If $\omega_1 = \omega_2$ this mean $\omega_1^2 + 4\omega_2 = 0,$ in this case, we need to compute the $c_{n-2}(t)\;\; and\; c_{n-1}(t)$ by derivative,

$$e^{\lambda_2 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_2^i}{i!} = c_{n-2}(t)\lambda_2^{n-2} + c_{n-1}(t)\lambda_2^{n-1}$$

With respect to λ_2 in Equation (6.8), we get,

$$te^{\lambda_2 t} - \sum_{i=1}^{n-3} \frac{it^i \lambda_2^{i-1}}{i!} = (n-2)c_{n-2}(t)\lambda_2^{n-3} + (n-1)c_{n-1}(t)\lambda_2^{n-2}.$$

Where $c_{n-2}(t)$ and $c_{n-1}(t)$ are computed by

$$\begin{pmatrix} c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-2} & \lambda_2^{n-1} \\ (n-2)\lambda_2^{n-3} & (n-1)\lambda_2^{n-2} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-3} \frac{t^i \lambda_2^i}{i!} \\ te^{\lambda_2 t} - \sum_{i=1}^{n-3} \frac{it^i \lambda_3^{i-1}}{i!} \end{pmatrix} . \tag{18}$$

Lemma (3)

Let $A \in C^{n\times n}$ and satisfies this condition $A^n = A^{n-1} + A^{n-2} + A^{n-3}$, then the exponential matrix is computed by,

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots + c_{n-3}(t)A^{n-3} c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \tag{19}$$

Where $c_{n-3}(t), c_{n-2}(t)$ and $c_{n-1}(t)$ are computed by,

$$\begin{pmatrix} c_{n-3}(t) \\ c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-3} & \lambda_2^{n-2} & \lambda_3^{n-1} \\ \lambda_3^{n-3} & \lambda_3^{n-2} & \lambda_3^{n-1} \\ \lambda_4^{n-3} & \lambda_4^{n-2} & \lambda_4^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_4 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \end{pmatrix}$$

$$\text{where } \lambda_2, \lambda_3 \text{ and } \lambda_4 \text{ are root of } \lambda^3 - \lambda^2 - \lambda - 1 = 0 \quad . \tag{20}$$

Lemma (4)

Let $A \in C^{n\times n}$ and satisfies $A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3}$, then the exponential matrix is computed by,

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \dots + c_{n-3}(t)A^{n-3} c_{n-2}(t)A^{n-2} + c_{n-1}(t)A^{n-1} \tag{21}$$

Where $c_{n-3}, c_{n-2}(t)$ and $c_{n-1}(t)$ are computed by

$$\begin{pmatrix} c_{n-3}(t) \\ c_{n-2}(t) \\ c_{n-1}(t) \end{pmatrix} = \begin{bmatrix} \lambda_2^{n-3} & \lambda_2^{n-2} & \lambda_3^{n-1} \\ \lambda_3^{n-3} & \lambda_3^{n-2} & \lambda_3^{n-1} \\ \lambda_4^{n-3} & \lambda_4^{n-2} & \lambda_4^{n-1} \end{bmatrix}^{-1} \begin{pmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_4 t} - \sum_{i=0}^{n-4} \frac{t^i \lambda_2^i}{i!} \end{pmatrix} \tag{22}$$

where λ_2, λ_3 *and* λ_4 *are the different roots of* $\lambda^3 - \omega_1 \lambda^2 - \omega_2 \lambda - \omega_3 1 = 0$

Notation (2): If $\lambda_2, \lambda_3 and\ \lambda_4$ are not the different roots, we can also compute the parameters $c_{n-3}, c_{n-2}(t)\ and\ c_{n-1}(t)$ by using the method as in notation (1).

Lemma (5)

Let $A \in C^{n \times n}$ and satisfies this condition

$$A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3} + \ldots .. + \omega_k A^{n-k}, k < n,$$

Then the exponential matrix is computed by,

$$e^{At} = I + tA + \frac{t^2}{2!} A^2 + \ldots . + \frac{t^{n-k-1}}{(n-k-1)!} A^{n-k-1} + c_{n-k}(t) A^{n-k} + \\ + c_{n-k+1}(t) A^{n-k+1} + \ldots . + c_{n-1}(t) A^{n-1} \tag{23}$$

Where $c_{n-k}(t), \ldots \ldots .., c_{n-3}(t), c_{n-2}(t)\ and\ c_{n-1}(t)$ are computed by,

$$\begin{bmatrix} c_{n-k}(t) \\ c_{n-k+1}(t) \\ \vdots \\ c_{n-1}(t) \end{bmatrix} = \begin{bmatrix} \lambda_2^{n-k} & \lambda_2^{n-k+1} & \cdots & \lambda_2^{n-1} \\ \lambda_3^{n-k} & \lambda_3^{n-k+1} & \cdots & \lambda_3^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{k+1}^{n-k+1} & \lambda_{k+1}^{n-k+1} & \cdots & \lambda_{k+1}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} e^{\lambda_2 t} - \sum_{i=0}^{n-k-1} \frac{t^i \lambda_2^i}{i!} \\ e^{\lambda_3 t} - \sum_{i=0}^{n-k-1} \frac{t^i \lambda_3^i}{i!} \\ \vdots \\ e^{\lambda_{k+1} t} - \sum_{i=0}^{n-k-1} \frac{t^i \lambda_{k+1}^i}{i!} \end{bmatrix} \tag{24}$$

$\lambda_2, \lambda_3, \ldots ., \lambda_{k+1}$ *are the different roots of*
$\lambda^k - \omega_1 \lambda^{k-1} - \ldots \ldots - \omega_{k-1} \lambda - \omega_k 1 = 0$.

3. APPLICATIONS OF METHOD

Here, we will give some example to show the procedure and effectiveness this method.

Example (1) :

Let A a matrix $(4\times 4),\ \ A=\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

After some calculations we get $A^4 = A^3 + 2A^2$, then, $\omega_1 = 1 \ and\ \omega_2 = 2$. From lemma (2), we have $\omega_1\ and\ \omega_2$ satisfy $\omega_1 + 4\omega_2^2 > 0$.then, we compute $\lambda_1\ and\ \lambda_2$ as in lemma (2) which equals to $\omega_1 = 2\ and\ \omega_2 = -1$. Then, the exponential matrix will be as the following:

$$e^{At} = I_4 + tA + c_2(t)A^2 + c_3(t)A^3$$

$$= I_4 + tA + \left(\frac{1}{12}e^{2t} + \frac{2}{3}e^{-t} + \frac{1}{2}t - \frac{3}{4}\right)A^2 + \left(\frac{1}{12}e^{2t} - \frac{1}{3}e^{-t} - \frac{1}{2}t + \frac{1}{4}\right)A^3$$

Where $c_2 = \left(\frac{1}{12}e^{2t} + \frac{2}{3}e^{-t} + \frac{1}{2}t - \frac{3}{4}\right), c_3(t) = \left(\frac{1}{12}e^{2t} - \frac{1}{3}e^{-t} - \frac{1}{2}t + \frac{1}{4}\right)$

then, the exponential matrix is

$$e^{At} = \begin{bmatrix} 1 & 2t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ -e^{-t} & 2e^{-t} + 2t - 2 & 0 & e^{-t} \end{bmatrix}$$

Example (2) :

Let A a matrix $(4\times 4), A=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

As is the above example, the matrix A satisfy $A^4 = 2A^3 + A^2 - 2A$, then, $\omega_1 = 2, \omega_2 = 1 \text{ and } \omega_3 = -2$ from lemma (4), we compute $\lambda_2, \lambda_3 \text{ and } \lambda_4$ then, the exponential matrix will be as the following:

$$
\begin{aligned}
e^{At} &= I_4 + c_1(t)A + c_2(t)A^2 + c_3(t)A^3 \\
&= I_4 + \left(\frac{-1}{6}e^{2t} + e^t - \frac{1}{3}e^{-t} - \frac{1}{2}\right)A + \left(\frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1\right)A^2 + \\
&\quad + \left(\frac{1}{6}e^{2t} - \frac{1}{2}e^t - \frac{1}{6}e^{-t} + \frac{1}{2}\right)A^3
\end{aligned}
$$

Where

$$
c_1 = \left(\frac{-1}{6}e^{2t} + e^t - \frac{1}{3}e^{-t} - \frac{1}{2}\right), c_2 = \left(\frac{1}{2}e^t + \frac{1}{2}e^{-t} - 1\right)
$$

$$
\text{and } c_3(t) = \left(\frac{1}{6}e^{2t} - \frac{1}{2}e^t - \frac{1}{6}e^{-t} + \frac{1}{2}\right)
$$

then the exponential matrix is

$$
e^{At} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 \\ 0 & e^t & e^t - 1 & e^t + e^{-t} - 2 \\ 0 & 0 & 1 & -2e^{-t} + 2 \\ 0 & 0 & 0 & e^{-t} \end{bmatrix}
$$

4. CONCLUSIONS

In this chapter we presented new method to compute the exponential matrix e^{At} as accurate solution. The basic idea of this method is based on the matrix theory; the matrices satisfy the special case

$$
A^n = \omega_1 A^{n-1} + \omega_2 A^{n-2} + \omega_3 A^{n-3} + \ldots.. + \omega_k A^{n-k}, k < n.
$$

Furthermore, this method can be extended to the more general case

$$
A^n = \omega_1 A^{k-1} + \omega_2 A^{k-2} + \omega_3 A^{k-3} + \ldots.. + \omega_k A^{k-m}, k < n, m < k
$$

REFERENCES

[1]- P. J. Antsaklis & A. N. Michel; A Linear Systems Primer. Birkhauser, Boston (2007).

[2]- D. S. Bernstein and W. So, some explicit formulas for the matrix exponential, IEEE transaction on Automatic control, vol38.No 8, 1228–1232.August 1993.

[3]- Cleve B. Moler and Charles F. Van Loan, Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later. SIAM Rev., 45(1),3-49, 2003.

[4]- Wu, Beibei. (2011). Explicit formulas for the exponentials of some special matrices. Applied Mathematics Letters, 24(5), 642–647.

[5]- Cheng, H. W., & Yau, S. S.-T. (1997). More explicit formulas for the matrix exponential. Linear Algebra and Its Applications, 262(1), 131–163.

[6]- D. S. Bernstein, "Some open problems in matrix theory arising in linear systems and control," Linear Algebra its Appl., pp. 162-164, pp. 409- 432, 1992.

[7]- Wasin So, "Exponential Formulas and Spectral Indices," Ph.D. dissertation, University of California at Santa Barbara, 1991.

[8]- R. C. Thompson, "Special cases of a matrix exponential formula, Linear Algebra Its Appl., vol. 107, pp. 283-292, 1988.