International Journal of Computer Science and Software Engineering
Volume 2, Number 1 (2016), pp. 1-6

http://www.irphouse.com

Arranging Letters of English Alphabet Randomly (A Combinatorial Problem)

Shashank Prakash Srivastava
National Institute of Technology, Goa, India.

Abstract

A close observation by arranging the alphabets (with weights assigned to each letter in increasing order from ‘A’ through ‘Z’) in all possible ways is made. It is shown that six of the letters have successive weights that either increase or decrease.
Note that the total number of such combinations that exist are equal to ‘n!’, where n=26 in this scenario.

=>Total no. of combinations possible = n!
= 26!
= 4.0329146e+26

Evidently, picking up all the test cases is beyond our scope but the result derived applies to all the 26! cases without any exception, as elucidated by the proof given below.

PROBLEM

Mr Parker is jobless at the moment, but has got a very demanding job of looking after his mischievous nine years old son, Dave, while Mrs Parker is away on her school-trip to Florida. Failing in all his attempts to control Dave from creating nuisance at home, Mr Parker comes up with a very smart puzzle for his naughty son. He brings home a pack of twenty-six cards, each one printed with a letter from the English Alphabet and asks Dave to write arbitrary numbers on the cards in increasing order from card ‘A’ through card ‘Z’. The only point to take care of, Mr Parker restates to Dave, is that the number on card ‘A’ should be smaller than the number on card ‘B’, the number on card ‘B’ should be smaller than the number on card ‘C’ and so on. Obviously, the card ‘Z’ bears the greatest number. Now, Mr Parker asks Dave to shuffle the cards and arrange them as he likes. The only condition is: ‘Six of the cards from first through last subsequently should not be in ascending or descending order.’ The order is decided by the numbers printed on the card. If Dave gets one such arrangement, he is given whatever he asks for. If he doesn’t, well, no more chocolate cookies or playing around or naughtiness. Can the son defeat the father?

Assumption: No two letters have the same weight.

Theorem Used: Erdős–Szekeres theorem.

Explanation: Erdős–Szekeres theorem is stated as follows:

“Let n=ab+1, where $a, b \in \mathbb{N}$

$x_1, \ldots, x_n$ - a sequence of ‘n’ distinct real numbers.

Then the aforementioned sequence will surely have a monotonically increasing (decreasing) subsequence of $a+1$ terms or a monotonically decreasing (increasing) subsequence of $b+1$ terms.”

***Definitions for reference*:**

1. *Finite sequence*: A finite sequence *a* of real numbers is a function *a*: $\{1, \ldots, n\} \rightarrow R$ for some $n \in N$. The length of the sequence *a* is equal to *n*.
2. *Finite subsequence*: A subsequence is a sequence restricted to a subset of its domain.

Special Case: We use a particular case of Erdős–Szekeres theorem here, where a=b. The case is restated as follows:

Let $x_1, \ldots, x_{n^2+1}$ be any sequence of distinct real numbers. Then there exists either an increasing or decreasing subsequence of length (n+1).

<u>Proof</u>:

1. **<u>By Induction</u>**- Proven by Erdős and Szekeres.

<u>Assumption</u>: The theorem is true for ‘n’ (We can show that the abovementioned theorem is true for the base case, that is, for *n=1*)

<u>Aim</u>: To prove it is true for *n+1*

Let $x_1, x_2, x_3, \ldots, x_{(n+1)^2+1}$ be a sequence of distinct real numbers. The length of this sequence is = $(n+1)^2 +1$.

Writing the above sequence a bit more explicitly,

X1, X2, X3, …, Xn2+1, Xn2+2, …. , Xn2+2n+2

Let us now define i1, . . . , i2n+2 ∈ $[n^2 + 2n + 2]$ and b1, . . . , b2n+2 ∈ {increase, decrease} as follows.

1. By the induction hypothesis, $x_1, \ldots, x_{n^2+1}$ has an increasing (or decreasing) subsequence of length *n + 1*. Let i_1 be such that x_i1 is the last element of any such subsequence. Let b_1 be 'increase' if that subsequence is increasing, and 'decrease' if that subsequence is decreasing.

2. Let us remove x_i1 from the original sequence. Now, the left part is $x_1, x_2, \ldots,$ x_i1-1,
   xi1+1, . . . , xn^2+1.
3. Let us have a gander at the first n^2+1 elements of the new sequence. Repeat the above process to obtain i2, xi2 , and b_2. Remove xi2 .
4. We keep on repeating this process until we have i1, . . . , i2n+2 and b1, . . . , b2n+2.

Let us see all the possible cases below:

<u>Case 1</u>: Assuming *n + 2* of the b's are labelled 'increase'. Having a look at just those *n +*
2 elements, which we call xj1 , xj2 , . . . , xjn+2 . Of course, these come out to be the endpoints of subsequences of length *n + 2*.

If there is an $a < b$ with xja < xjb then the increasing *n + 1*-long subsequence that ends with xja , together with xjb , forms an increasing *n + 2*-long subsequence. If no such a, b exists then xj1 , xj2 , . . . , xjn+2 is a decreasing *n + 2*-long subsequence.

<u>Case 2</u>: Assuming *n + 2* of the b's are labelled 'decrease'. This is same as Case 1.

<u>Case 3</u>: Assuming *n + 1* of the b's are labelled 'increase' and *n + 1* of the b's are labelled 'decrease'. Let bj1 , . . . , bjn+1 be labelled 'decrease' and let bk1, . . . , bkn+1 be labelled 'increase'. By the same reasoning as in Case 1 we have xj1 < · · · < xjn+1 and xk1
> · · · > xkn+1 . Hence, there are three subcases:

Subcase 3a: xjn+1 ≠ $x_{(n+1)^2+1}$ (This implies the last point in the original sequence.) There are two cases: xjn+1 < x^2_{n+1} or xjn+1 > x^2n+1. From the former case we obtain, xj1 < · · · < xjn+1 < $x_{(n+1)^2+1}$ is an increasing subsequence of length (*n+2*).

From the latter, we obtain that the decreasing subsequence of length (*n+1*) that terminates with xjn+1 can be extended to a decreasing subsequence of length (*n+2*) by adding $x_{(n+1)^2+1}$ to the end.

Subcase 3b: $xk_{n+1} \neq x_{(n+1)^2+1}$ The reasoning is as stated in Subcase 3a.

Subcase 3c: $xj_{n+1} = xk_{n+1} = x_{(n+1)^2+1}$ Note that this case is impossible to take place as once an element is picked to be one of the x's, it is removed.
Hence, Proved.

2. By Pigeonhole principle (The Strong form):

Principle: "Let $q_1, q_2, ..., q_n$ be positive integers.
If

$$q_1 + q_2 + \cdots + q_n - n + 1$$

Objects (pigeons) are distributed into n boxes (holes), then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects, ..., or the nth box contains at least q_n objects."

Coming back to our sequence,

Sequence: (a1, a2,…., an2+1).

Length: n^2+1

Assumption: The longest monotonic increasing subsequence of the original sequence has length at most *n*.

To show: Our sequence has a monotonically decreasing subsequence of length *n+1*.

Explanation: For each number *a*i in the sequence, let $F(a_i)$ be the length of the longest monotonically increasing subsequence that has a_i as its terminating point.

Given our assumption, $F(a_i) \leq$ n for every number a_i in our sequence.
Also, note that if $i<j$, and ai ≤ aj, then $F(a_i)<F(a_j)$. (This is just an extension of monotonically increasing function by increasing *i* to *j*.)
Therefore, if $i < j$ and $F(ai)=F(a_j)$, then ai > aj --→1.)

Applying Strong Pigeonhole Principle, let: Numbers in the sequence = the objects, Possible values of $F(a_i)$= the categories.
∴ number of pigeons= n^2+1 Number of holes= n
(*Note:* $1 \leq F(a_i) \leq$ n)

According to strong pigeonhole principle, some holes must have $\lceil (n^2+1)/(n) \rceil$ = n+1 pigeons.

So there is some *m* and at least *n+1* elements (pigeons) that have $F(a_i) = m$.

Using 1.) if we list those $n+1$ elements by their order in our sequence, we come to a conclusion that $a_{i1} > a_{i2} > \ldots > a_{in+1}$.
Thus, our sequence has a monotonically decreasing subsequence of length $n+1$.

Coming back to our problem, total no. of cards = 26

$$\Rightarrow n^2+1 = 26$$
$$\Rightarrow n = 5$$

One such possible arrangement of letters done by Dave with natural nos. used as weights for the letters is given below.

Let A=1, B=2, C=3, … , Y=25, Z=26.

Let a random sequence after the cards have been shuffled be:
(15, 24, 18, 14, 9, 4, 8, 26, 16, 25, 19, 17, 23, 22, 2, 12, 10, 1, 11, 3, 21, 5, 20, 13, 6, 7)

In our problem, $n = 5$,
$\Rightarrow n+1 = 6$

Erdős–Szekeres theorem guarantees that our sequence will have a monotone subsequence of length $n+1 = 6$.

<u>Calculations</u>:
$F(15)=1$, $F(24)=2$, $F(18)=2$, $F(14)=1$, $F(9)=1$, $F(4)=1$, $F(8)=2$, $F(26)=3$ $F(16)=3$, $F(25)=4$, $F(19)=4$, $F(17)=4$, $F(23)=5$, $F(22)=5$, $F(2)=1$, $F(12)=3$, $F(10)=3$, $F(1)=1$, $F(11)=4$, $F(3)=2$, $F(21)=5$, $F(5)=3$, $F(20)=5$, $F(13)=5$, $F(6)=4$, $F(7)=5$.

Given our previous assumption, the abovementioned sequence doesn't have a monotonically increasing subsequence of length $n+1=6$;

So, according to the theorem, we shall obtain a monotonically decreasing subsequence of length *6* or more.

The pigeonhole principle guides that one value of F is taken at least 6 times, which in this case is *5*. Accordingly, if we arrange all the elements a_i that have $F(a_i) = 5$ in sequential order, we will obtain a monotonically decreasing subsequence which in this case is: 23, 22, 21, 20, 13, 5.

Hence, Proved.

RESULT
Using Erdős–Szekeres theorem, we have shown that no matter how we arrange *26* distinct real numbers, there will exist a subsequence of *6* numbers that either increase or decrease. Hence, Mr Parker wins!

REFERENCE

[1] http://www.cs.umd.edu/~gasarch/BLOGPAPERS/subseq4proofs.pdf
[2] (Proof of Erdős–Szekeres theorem)
[3] http://www.skidmore.edu/~adean/MC2151401/Handouts/ErdosSzekeresProof.pdf
[4] (Proof of Erdős–Szekeres theorem using Strong Pigeonhole principle)
[5] https://en.wikipedia.org/wiki/Pigeonhole_principle#Strong_form (Pigeonhole Principle)
[6] Discrete and Combinatorial Mathematics by Ralph P. Grimaldi (Pigeonhole Principle) s