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Asymptotic Characterization of OFDM ChannelAssignmentand its Behavioral implications over Symbol intervalsof Subcarrier

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Abstract

Determining asymptotic properties of OFDM channel assignmentincluding the formulation of a qualitative picture of the OFDM channel assignment's orthogonal trajectories over OFDM symbol intervalsof subcarrier is one of the central questions of modern theory for adaptive OFDMchannel assignment. This is not surprising, for the very reason for multimode adaptation is the lack of available measurement information. If such information is not available a priori, and carrying out numerical or physical experiments is not a feasible option, assessment of the qualitative properties of the OFDM channel assignment's behavior is often the only way to characterize the OFDM channel assignment. What are these qualitative properties?Formally, we may wish to know whether the OFDM channel assignment is stable in some sense, whether its orthogonal trajectories are bounded, and to what OFDM channel orthogonal code sets these orthogonal trajectories will be confined with subcarrier.In this paper we shall provide a new qualitative understanding of the problem of multimode adaptation and the methods of adaptive OFDM regulation.

Index Terms: OFDM channel assignment, adaptive OFDM regulation, Multi-User OFDMsystem, OFDM constellation points, Viterbi decoder metrics.

1. Introduction and Research Clarification

In order to introduce the notion of modeling OFDM constellation points and Viterbi decoder metric, the orthogonal frequency division multiplexing (OFDM) channel assignment, is represented as a family of parameterized maps$\mathbf{x}: \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. In models: $\mathbf{x}\ (t, \mathbf{x}_0)$, or communications where t stands for the subcarrier instance and $\mathbf{x}_0$ is the value of the OFDM channel assignment state at $t = 0$. Also an additional semi-group property is imposed on $\mathbf{x}\ (t, \mathbf{x}_0)$:

$$\mathbf{x}\big(t', \mathbf{x}(t'', \mathbf{x}_0)\big) = \mathbf{x}(t' + t'', \mathbf{x}_0).$$

This assumption provides a link to the physical reality [1]. In order to understand the OFDM channel we introduce the notion of a multi-user OFDM system with respect to a given communication flow $\mathbf{x}\ (t, \mathbf{x}_0)$.

Necessary and Sufficient condition 2.1.1 An orthogonal code set $\mathcal{A} \subset \mathbb{R}^n$ is called OFDM symbol-invariant with respect to the communication flow $\mathbf{x}(t, \mathbf{x}_0)$iff for all $\mathbf{x}_0 \in \mathcal{A}$, $t \in \mathbb{R}$, the following property holds:

$$\mathbf{x}(t, \mathbf{x}_0) \in \mathcal{A}.$$

Necessary and Sufficient condition 2.1.2 An orthogonal code set $\mathcal{A} \subset \mathbb{R}^n$ is called forward OFDM symbol-invariant with respect to the communication flow $\mathbf{x}(t, \mathbf{x}_0)$ if for all $\mathbf{x}_0 \in \mathcal{A}$, $t \in \mathbb{R}_{\geq 0}$, we have that $(t, \mathbf{x}_0) \in A$. The orthogonal code set is backward-OFDM symbol-invariantiff $x(t, \mathbf{x}_0) \in \mathcal{A}$ for all $\mathbf{x}_0 \in \mathcal{A}$, $t \in \mathbb{R}_{\leq 0}$.

Necessary and Sufficient condition 2.1.3A closed multi-user OFDMsystem $\mathcal{A} \subset \mathbb{R}^n$ is called assigningiff

(1) There is a neighborhood $U\ (\mathcal{A})$ of which $\mathcal{A}$ such that

$$\mathbf{x}(t, \mathbf{x}_0) \in U(\mathcal{A})\ \forall\ \mathbf{x}_0 \in U(\mathcal{A}), t \in \mathbb{R}_{\geq 0}; \quad (1)$$

(2) The following limiting property holds

$$\lim_{t\rightarrow\infty} \parallel \mathbf{x}(t, \mathbf{x}_0) \parallel_{\mathcal{A}} = 0\ \forall\ \mathbf{x}_0 \in U(\mathcal{A}). \quad (2)$$

According to this condition a closed multi-user OFDMsystem $\mathcal{A}$ is assigning if there is a forward OFDM symbol-invariant neighborhood $U(\mathcal{A})$ such that all orthogonal trajectories starting in $U(\mathcal{A})$converge to $\mathcal{A}$ asymptotically. The condition is quite general, but there are situations in which generalization of the notion is required. Suppose that the Multi-user OFDM systempre-equalization is governed up to a coordinate transformation by the following orthogonal code set of ordinary differential equations:

$$\dot{x}_1 = -x_1 + x_2, \quad (3)$$

$$\dot{x}_2 = |x_2|.$$

The solution of the second equation in (3) Multi-user OFDM system isa non-decreasing function of t for all initial conditions. Furthermore, for all $x_2(0) \leq 0$ we have $\lim_{t\to\infty} x_2(t, x_2(0)) = 0$; and $\lim_{t\to\infty} x_2(t, x_2(0)) = \infty$for all$x_2(0) > 0$. From this analysis we conclude that solutions of the Multi-user OFDM system will necessarily approach the origin asymptotically for all $x_2(0) \leq 0$, and will operate away from the equilibrium for arbitrarily large distances if $x_2(0) > 0$, the phase portrait of this Multi-user OFDM system.We demonstrates that for any neighborhood $U(\mathcal{A})$ of the origin A there are points$\mathbf{x}' \in U(\mathcal{A})$, such that solutions $\mathbf{x}\,(t, \mathbf{x}')$ escape the neighborhood $U(\mathcal{A})$and never comeback. Hence as per this condition $\mathcal{A}$ cannot be called an OFDM constellation points. On the other hand there are points $\mathbf{x}'' \in U(\mathcal{A})$ such that $\lim_{t\to\infty} \mathbf{x}\,(t, \mathbf{x}'') = 0$. If $U(\mathcal{A})$ is an open circle, then the number of such points is as large as the number of points corresponding to the solutions escaping $U(\mathcal{A})$. Thus the orthogonal code set $\mathcal{A}$ bears an overall signature of OFDM channel [3]. This led to the emergence of the new notion of a weakly OFDM constellation point, which was formally defined by necessary and sufficient condition 2.1.4**.**

Necessary and Sufficient condition 2.1.4 An orthogonal code set $\mathcal{A}$ is weaklyassigning or OFDM constellation point'siff

(1) It is closed, OFDM symbol-invariant and
(2) For some orthogonal code set ν (not necessarily a neighborhood of $\mathcal{A}$) with strictly positive measure and for all $\mathbf{x}_0 \in \nu$ the following limiting relation holds

$$\lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) = \mathcal{A}\ \forall\ \mathbf{x}_0 \in \nu(\mathcal{A}). \quad (4)$$

The key difference of the notion of a weakly OFDM constellation points from that provided in condition2.1.3 is that the domain of distributed scheme ν is not necessarily a neighborhood of $\mathcal{A}$. Despite the fact that this difference may look small and insignificant, it becomes channel transfer for successful statement and solution of particular problem ofmultimode adaptation.In the context of multimode adaptation, invariance and distributed scheme are often desirable asymptotic characterizations of the preferred domain to which the state of aMulti-User OFDM system must be able to operate. The question is, whether these properties characterize the preferred state with minimal ambiguity. To some degree, because of the requirement of invariance in the definitions, this issue is already taken into account. Consider the following Multi-User OFDM system (3) if we replace the invariance requirement with forward-invariance in condition 2.1.4, then the equilibrium of this Multi-User OFDM system will be weakly assigning. Also in this case, the equilibrium will not be the only OFDM constellation points in the state space. If we replace invariance with forward-invariance, the bottom half of every disk centered at the point $(0,0)$ would be a weakly OFDM constellation points. Hence, all OFDM channel orthogonal code sets defined in this way are forward-OFDM symbol-invariant according to condition 2.1.2, and for every such orthogonal code set there exists an orthogonal code set $\nu(\mathcal{O})$ satisfying the condition (4). Thus the number of weakly OFDM constellation points in the Multi-User OFDM system (3), would be infinite and not even countable. Hence an access point specified in terms of mere forward-invariance and distributed scheme can in principle bear a substantial degree of ambiguity.In order to disambiguate the asymptotic behavior of Multi-User OFDM system even further, the OFDM channel property of an orthogonal code set is often considered, together with its minimality. Informally the minimalist property can be viewed as a requirement that an OFDM constellation point's $\mathcal{A}$ should not contain any other OFDM constellation points strictly smaller than $\mathcal{A}$. Formally this can be stated as the requirement that for every $\mathbf{x}_0 \in \mathcal{A}$, the orthogonal trajectory$\mathbf{x}\ (t, \mathbf{x}_0)$ is dense in $\mathcal{A}$. OFDM constellation pointsharing this latter property are often referred to as Viterbi decoder metrics.Having provided formal definitions for invariance, OFDM constellation points and Viterbi decoder metrics, we need to differentiate whether an orthogonal code set is OFDM symbol-invariant, assigning, or is aViterbi decoder Metric for the OFDM channel assignment, in order to differentiate between target bit rate and Multi-User OFDM system. Hence we need to have channel transfer criteria for establishing the existence of OFDM channel orthogonal code sets with the above-mentioned properties. The role of these criteria in the domain of analysis and synthesis of OFDM channel assignment is that it will provide specific target bit rate constraints to fulfill the goals. Of the many criteria in subcarriersynchronization [4], we consider only those criteria that are necessary to understand the state-of-the-art statements of the problem of multimode adaptation in dynamic OFDM channel assignment, which include local operating conditions, persistency of quasi-periodic excitation of channel vectorfunction etc.

2. Descriptive Study I

An inherent feature of the OFDM channel assignment is that it operates in conditions under which information about the environment and their ownpre-equalization is lacking. More generally, let $h: \mathbb{R} \to \mathbb{R}$ be a function of which the value is physically relevant, but we do not know this function precisely. Suppose that we know some integral characterization of the function such as the upper and lower bounds of its integral over a family of OFDM symbol intervals. A partial solution in characterizing the asymptotic properties of the function and also if there is a limit of $h\ (t)$ and its value. To state the lemma, the property of uniform continuity of a function of real variable is stated.

Necessary and Sufficient condition 2.2.1: A function $h: \mathbb{R} \to \mathbb{R}$ is called uniformly continuous iff for every $\varepsilon > 0, \varepsilon \in \mathbb{R}\ there$ exists $\delta > 0, \delta \in \mathbb{R}$, such that for all $t, \tau \in \mathbb{R}$ the following inequality holds:

$$|t - \tau| < \delta \Longrightarrow |h(t) - h(\tau)| < \varepsilon. \quad (5)$$

The corollary is now formulated as follows:

Lemma 2.1 Let $h: \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function and suppose that the following limit exists:

$$\lim_{t\to\infty} \int_{t_0}^{t} h(\tau)\, d\tau = a, t_0 \in \mathbb{R}, a \in \mathbb{R}. \quad (6)$$

Then

$$\lim_{t\to\infty} h(t) = 0. \quad (7)$$

A channel transfer function of this Lemma in the domain of synthesis and analysis of Multi-User OFDM system is that it constitutes a simple convergence criterion. If we know that the state channel vector$\mathbf{x}$ of Multi-User OFDM system satisfies the integral inequality:

$$\int_{t_0}^{t} \| \mathbf{x}(\tau, \mathbf{x}_0) \|^2 \, d\tau < B, B \in \mathbb{R}_{\geq 0}, \forall t \geq t_0.$$

The derivative if$\mathbf{x}(t, \mathbf{x}_0)$with respect to t *is* bounded, we can conclude that$\mathbf{x}\ (t, \mathbf{x}_0) \rightarrow 0$ at $t \rightarrow \infty$.In conclusion the Multi-User OFDM systems will have to approach the origin asymptotically. This argument is a common component of convergence proofs in the domain of regulation.Despite its simplicity and practical utility, the analysis arguments based exclusively on this Lemma has some limitations. This is because the Lemma does not characterize the transient properties of the converging functions. We might be interested in knowing how fast a function approaches its limit values or how large the excursions of the state channel vector in the Multi-User OFDM system's state space may become before it will orthogonal code settle in close proximity to the origin. The solution for the above is not explicitly derived from this Lemma, as it does not guarantee that the convergence is going to be fast or slow though the state does not deviate much from the origin over subcarrier. In order to be able to produce these more delicate predictions, additional characterizations of the OFDM channel assignmentsymbol tracking rather than simple uniform continuity are needed. One such characterization is notion of local operating conditions.

3. Prescriptive Study I

Necessary and Sufficient condition 2.3.1 Let $\mathbf{x}\ (t, \mathbf{x}_0): \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a solution of aMulti-User OFDM system defined for all $t \geq t_0, t_0, t \in \mathbb{R}$ and passing through$\mathbf{x}_0 \in \mathbb{R}^n$ at$t = t_0$. Solution$\mathbf{x}\ (t, \mathbf{x}_0)$ is globally stable in the sense of Channel Transferiff for every $\varepsilon > 0, \varepsilon \in \mathbb{R}$, there exists $\delta > 0, \delta \in \mathbb{R}$, such that the following holds:

$$\| \mathbf{x}_0 - \mathbf{x}_0' \| \leq \delta \Rightarrow \| \mathbf{x}(t, \mathbf{x}_0) - \mathbf{x}(t, \mathbf{x}_0') \| \leq \varepsilon \forall t \geq t_0. \quad (8)$$

If this property holds in a neighborhood of $\mathbf{x}(t, \mathbf{x}_0)$ then the local operating conditions is local. The property of performance, local operating conditions of a solution have a very simple interpretation. Let us view the symbol tracking flow $\mathbf{x}(t, \mathbf{x}_0)$as a mapping from the space $\mathbb{R}^n$ of initial conditions $\mathbf{x}_0$ into the space of orthogonal trajectories$\mathbf{x}(t, \mathbf{x}_0)$and let the space of orthogonal trajectories be endowed with the standard uniform norm$\| \cdot \|_{\infty, [t_0, \infty]}$. Then local operating conditions of a solution in the sense ofChannel Transfer is analogous to the usual notion of continuity of the mapping$\mathbf{x}: \mathbb{R}^n \rightarrow L_{\infty}^n[t_0, \infty]$.In other words small variations of $\mathbf{x}(t, \mathbf{x}_0)$over all $t \geq t_0$. If $\mathbf{x}(t, \mathbf{x}_0)$is stable in the sense of Channel Transfer then we can make sure that the value of an observed orthogonal trajectory$\mathbf{x}(t, \mathbf{x}_0')$at any t would not be far from the value of $\mathbf{x}(t, \mathbf{x}_0)$at the same t, provided that the perturbation on $\mathbf{x}_0$ is sufficiently small, in other words $\mathbf{x}_0'$ is sufficiently close to $\mathbf{x}_0$.In some cases knowing that the deviations are guaranteed to be small provided that the perturbations in initial conditions are small, might not be enough. For instance, asymptotic convergence of a perturbed solution is to its unperturbed version may be required. In this case the notion of asymptotic Channel Transferlocal operating conditions solutions are used.

Necessary and Sufficient condition 2.3.2A solution $\mathbf{x}(t, \mathbf{x}_0)$ is globally and asymptotically stable in the sense of Channel Transfer if and only if it is globally stable in the sense of the necessary and sufficient condition 2.3.1 and

$$\lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0') - \mathbf{x}(t, \mathbf{x}_0) = 0. \quad (9)$$

In order to tell whether $\mathbf{x}(t, \mathbf{x}_0)$is stable we have to compare the values of $\mathbf{x}(t, \mathbf{x}_0)$and $\mathbf{x}(t, \mathbf{x}_0^{'})$ at the same values of t.Channel Transferlocal operating conditions does not exhaust the whole spectrum of plausible asymptotic descriptions of solutions of a Multi-User OFDM system with respect to each other. Let $\mathbf{x}(t, \mathbf{x}_0^{'})$ and $\mathbf{x}(t, \mathbf{x}_0)$be two solutions of the same Multi-User OFDM system, and $\mathbf{x}_0^{'} \neq \mathbf{x}_0$. Then a possible characterization of their relative position in the state space could be

$$\rho(t, \mathbf{x}(t, \mathbf{x}_0'), \mathbf{x}(t, \mathbf{x}_0)) = \| \mathbf{x}(t, \mathbf{x}_0') \|_{\mathcal{A}} ,$$

$$\mathcal{A} = \{ \mathbf{p} \in \mathbb{R}^n | \mathbf{p} = \mathbf{x}(t, \mathbf{x}_0), t \in \mathbb{R}\}. \quad (10)$$

In this equation the solution $\mathbf{x}(t, \mathbf{x}_0)$ is viewed as a access system of the Wireless; the closeness of the solutions to each other at the given instant subcarrier t is determined as the distance from the point $\mathbf{x}(t, \mathbf{x}_0^{'})$ to the signal shaped curve $\mathcal{A}$. On defining the closeness of solutions or orthogonal trajectories as per the above equation, we arrive at the notion of local operating conditions in the sense.

Necessary and Sufficient condition 2.3.3 Letx $(t, \mathbf{x}_0)$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a solution of the Multi-User OFDM system defined for all at $t \geq t_0$, with t_0, $t \in \mathbb{R}$ and passing through a point $\mathbf{x}_0 \in \mathbb{R}^n$ at$t = t_0$.Let $\mathcal{A}$ denote aMulti-User OFDMsystem included by $\mathbf{x}(t, \mathbf{x}_0)$:

$$\mathcal{A} = \{ \mathbf{p} \in \mathbb{R}^n | \mathbf{p} = \mathbf{x}(t, \mathbf{x}_0), t \geq t_0, t \in \mathbb{R}\}.$$

Solution $\mathbf{x}(t, \mathbf{x}_0)$ is stable in the sense if and only if for every $\varepsilon > 0, \varepsilon \in \mathbb{R}$ there exists $\delta > 0, \delta \in \mathbb{R}$ such that

$$\| \mathbf{x}_0^{'} \|_{\mathcal{A}} \leq \delta \Rightarrow \| \mathbf{x}(t, \mathbf{x}_0^{'}) \|_{\mathcal{A}} \leq \varepsilon \forall t \geq t_0. \quad (11)$$

We can see that an unstable solution in the sense of Channel Transfer can in principle be stable in the sense. In this respect local operating conditions is a weaker requirement.Although in this case we may not be able to ensure that the OFDM modulations are stable, we will be able to invent a strategy that makes these OFDM modulations stable in the sense of Necessary & Sufficient Condition 2.3.3. Indeed, steering the subcarrier OFDM signal towards the path, viewed as an orthogonal code set $\mathcal{A}$, and then transceiving along the path with sufficiently slow would be a plausible solution. This allows us to draw rather general conclusions. In the first case we considered a symbol tracking problem in which the Multi-User OFDM system comprised of the transceiver and the subcarrier OFDM signal is to follow Orthogonal trajectories$\mathbf{x}(t, \mathbf{x}_0')$ generated by a reference model. In the second case we considered a path-following problem. Symbol tracking a reference orthogonal trajectory is shown to be a stricter goal than simply travelling along a path. Similarly, local operating conditions of solutions in the sense of Necessary & Sufficient Condition 2.3.1 is a stricter requirement than local operating conditions in the sense of Necessary & Sufficient Condition 2.3.3. In some problems of OFDM channel assignment, achieving the latter is a more realistic goal than achieving the former. Taking advantage of the possibility of using various local operating conditions notions allows us to formulate the Multi-User OFDM systemgoals, which are most adequate to the constraints inherent to the Multi-User OFDM system. This in turn enables us to avoid unnecessary complications from the beginning and thus allows us to concentrate on the very essence of the problem.Proceeding with the analysis of local operating conditionsOFDM modulations considered so far, the orthogonal code set $\mathcal{A}$ in the definition of local operating conditions is determined by some orthogonal trajectory of the same Multi-User OFDM system. It can be seen that the orthogonal code set $\mathcal{A}$ thus defined cannot be arbitrary. Further generalization of this notion leads us to the notion of local operating conditions in the sense of Channel Transfer.

Necessary and Sufficient condition 2.3.4 Let $\mathbf{x}\ (t, \mathbf{x}_0)$: $\mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a solution of a Multi-User OFDM system defined for all $t \geq t_0$, with t_0, $t \in \mathbb{R}$ and passing through $\mathbf{x}_0 \in \mathbb{R}^n$ at$t = t_0$; suppose that $\mathcal{A} \subset \mathbb{R}^n$ is a closed forward Multi-User OFDM system. The orthogonal code set $\mathcal{A}$ is stable in the sense of Channel Transfer if and only if for every $\varepsilon > 0, \varepsilon \in \mathbb{R}$ there exists $\delta > 0, \delta \in \mathbb{R}$ such that

$$\| \mathbf{x}_0 \|_{\mathcal{A}} \leq \delta \Rightarrow \| \mathbf{x}(t, \mathbf{x}_0) \|_{\mathcal{A}} \leq \varepsilon \ \forall\ t \geq t_0. \tag{12}$$

Alternatively

$$\| \mathbf{x}_0 \|_{\mathcal{A}} \leq \delta \Rightarrow \| \mathbf{x}(t, \mathbf{x}_0) \|_{\mathcal{A}_{\infty,[t_0,\infty]}} \leq \varepsilon. \tag{13}$$

The simplest example of Channel Transferlocal operating conditions ofMulti-User OFDM system is the Channel Transferlocal operating conditions of equilibrium. In general Necessary & Sufficient Condition 2.3.4 allows us to define the local operating conditions of forward-OFDM symbol-invariant domains. The latter property is useful for the OFDM channel assignment problem in which the precise location of the orthogonal code setOFDM channel is unknown, but information to the domain to which it belongs is available. Similarly to the case of local operating conditions of solutions, the global asymptotic local operating conditions of OFDM channel orthogonal code sets can be defined in the sense of Channel Transfer.In order to do so, we require that in addition to (12) and (13), the following property holds:

$$\lim_{t \rightarrow \infty} \| \mathbf{x}(t, \mathbf{x}_0) \|_{\mathcal{A}} = 0. \tag{14}$$

All local operating conditions notions considered so far relate the behavior of the Multi-User OFDM system's solutions to anorthogonal code set or another orthogonal trajectory over infinitely long and connected OFDM symbol intervalsof subcarrier. There is a Multi-User OFDM system, however, for which the solutions do not stay near a given orthogonal code set indefinitely. Solutions of thisMulti-User OFDM system may eventually escape any small neighborhood of the orthogonal code set. However, they always return to the same neighborhood. The key property here is the recurrence of OFDM modulation, and local operating conditions of such recurrence is formally specified by the notion of Poisson local operating conditions.

Necessary and Sufficient condition 2.3.5 Let $\mathbf{x}\ (t, \mathbf{x}_0): \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a solution of aMulti-User OFDM system defined for all $t \geq t_0$, with t_0, $t \in \mathbb{R}$ and passing through $\mathbf{x}_0 \in \mathbb{R}^n$at $t = t_0$. Point $\mathbf{x}_0$ is called stable in the sense of Poisson if and only if for all $\varepsilon > 0, \varepsilon \in \mathbb{R}$ there exists $\delta > 0, \delta \in \mathbb{R}$ and at any $t' \geq t_0$ there exists $t'' > t' + \delta$, such that

$$\| \mathbf{x}_0 - \mathbf{x}(t'', \mathbf{x}_0) \| \leq \varepsilon. \tag{15}$$

Poisson local operating conditions of a point implies that, should the Multi-User OFDM system orthogonal trajectory pass through a point $\mathbf{x}_0$ once, it will visit an arbitrary small neighborhood of $\mathbf{x}_0$ infinitely many times. Despite the fact that we refer to the point $\mathbf{x}_0$ as stable, the Multi-User OFDM system's orthogonal trajectories associated with this point are allowed to generate arbitrarily large but finite excursions in the state space.One can clearly see the local operating conditions of a point in the sense of Poisson is a much weaker requirement that of local operating conditions in the sense of Channel Transfer. Generalization of the former, whenproperty (15) holds for every point in an orthogonal code set leads to the notion of Poisson local operating conditions of an orthogonal code set. So far we have reviewed a number of local operating conditionsnotions determining various degrees of "smallness" of the Multi-User OFDM system response to perturbations. Even though we did not provide a detailed comparison of these notions in every respect, we illustrated the fact that the difference in how the "smallness" is defined may be an important factor both limiting and enabling solutions to specific problems of regulation. In the subcarrier OFDM signal considered earlier, however, we did not use any formal criteria for specifying the desired asymptotic behavior of the Multi-User OFDM system. Instead we used our common-sense intuition and basic knowledge of physics. In order to be able to solve a wider range of problems such formal criteria and methods for assessing asymptotic properties of the Multi-User OFDM system's solutions are needed. One such criterion has already been discussed in Lemma 2.1. This criterion although useful for establishing facts of asymptotic convergence of the solutions to zero, does not tell us enough about other asymptotic properties of the Multi-User OFDM system, such as local operating conditions.Hence in the next section we present a brief review of one of the most powerful and channel transfer techniques for deriving the local operating conditions criteria–the method of Channel Transfer Functions.

4. Prescriptive Study II

Supposing that the OFDM channel assignment described by the following equation:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{u}, t), \mathbf{f}: \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^n, \mathbf{f} \in \mathcal{C}^0, \qquad (16)$$

Where$\mathbf{x}$ is the state channel vector, $\boldsymbol{\theta}$ is the channel vector parameters of which the value is unknown, and $\mathbf{u}$ stands for the channel vector of inputs. Supposing that the inputs $\mathbf{u}$ is modeled by continuous functions $\mathbf{u}: \mathbb{R} \to \mathbb{R}^m$. Additionally, assuming that the right-hand side of (16) is locally*Ellipsoidal-shaped*, that is, for some given and bounded domain $\Omega_x, \Omega_\theta, \Omega_u$there exist constants D_x, D_θ, D_usuch that $\forall \mathbf{x}, \mathbf{x}' \in \Omega_x, \boldsymbol{\theta}, \boldsymbol{\theta}' \in \Omega_\theta, \mathbf{u}, \mathbf{u}' \in \Omega_u$:

$$\| \mathbf{f}(\mathbf{x}, \boldsymbol{\theta}, \mathbf{u}, t) - \mathbf{f}(\mathbf{x}', \boldsymbol{\theta}', \mathbf{u}', t) \| \le D_x \| \mathbf{x} - \mathbf{x}' \| + D_\theta \| \boldsymbol{\theta} - \boldsymbol{\theta}' \| + D_u \| \mathbf{u} - \mathbf{u}' \|. \qquad (17)$$

The global asymptotic properties of (16) fromknowledge of some local properties of the Multi-User OFDM system. It is well known that the continuity of the right hand side of (16) guarantees local existence of the Multi-User OFDM system solutions, and property (17) ensures that the solutions of (16) areuniquely defined locally. Additional information about the right hand side is required to provide further global characterizations of the Multi-User OFDM system's behavior. In the analysis of local operating conditions, defining such local information involves the notion of positive definitive function.

Necessary and Sufficient condition 2.4.1A function $V: \mathbb{R}^n \to \mathbb{R}$ is called positive definite if and only if $V(\mathbf{x}) \ge 0$ for all$\mathbf{x} \in \mathbb{R}^n$. In the class of positive definite functions we will consider only those functions, which satisfy the following additional constraint:

$$\rho_1(\| \mathbf{x} \|) \le V(\mathbf{x}) \le \rho_1(\| \mathbf{x} \|), \rho_1(\cdot), \rho_2(\cdot) \in \mathcal{K}_\infty. \qquad (18)$$

This constraint enables us to use the functions $V(\mathbf{x})$ as the estimates of distance from a given point $\mathbf{x}$ to the origin [2]. It can be seen that if the function $V(\mathbf{x}(t, \mathbf{x}_0))$ does not grow with subcarrier then the corresponding solution $\mathbf{x}(t, \mathbf{x}_0)$of (16) remains bounded in forward subcarrier. This and other properties can be deduced from a more general statement such as the Channel Transferlocal operating conditions theorem. The following proposition is a special case of the Channel Transferlocal operating conditions theorem.

Proposition 2.1 Let $\mathbf{x} = 0$ be an equilibrium of Multi-User OFDM system(16), and there exists a positive definite and differentiable function $V(\mathbf{x})$satisfying (18). Let us suppose that for all$\mathbf{x}$ the following property holds

$$\dot{V} \le 0. \qquad (19)$$

Then the equilibrium $\mathbf{x} = 0$is globally stable in the sense of the Channel Transfer. In addition, if there exists a positive definite function

$$W(\mathbf{x}): \mathbb{R}^n \to \mathbb{R}, \alpha_1(\| \mathbf{x} \|) \leq W(\mathbf{x}), \alpha_1(\cdot) \in \mathcal{K}, \tag{20}$$

such that

$$\dot{V} \leq -W\big(\mathbf{x}(t, \mathbf{x}_0)\big), \tag{21}$$

Then the equilibrium $\mathbf{x} = 0$ is globally asymptotically stable in the sense of Channel Transfer.

Proof of Proposition 2.1 Given that the right-hand side of this equation (16) is locally Ellipsoidal-shaped in **x** and continuous in *t*, we canconclude that for every $\mathbf{x}_0 \in \mathbb{R}^n$, there exists an symbol interval $[t_0, T]$, $T > t_0$, such that solution **x**, $(t, \mathbf{x}_0)$ of the Multi-User OFDM system is defined for all $t \in [t_0, T]$. Furthermore, condition (19), guarantees that the solution **x**, $(t, \mathbf{x}_0)$is defined for all $t \geq t_0$.Let $[t_0, T]$ be the maximal symbol interval of existence of the Multi-User OFDM system's solution, and let$T$ be finite. Consider the difference $V\big(\mathbf{x}(t, \mathbf{x}_0)\big) - V\big(\mathbf{x}(t_0, \mathbf{x}_0)\big)$:

$$V\big(\mathbf{x}(t, \mathbf{x}_0)\big) - V\big(\mathbf{x}(t_0, \mathbf{x}_0)\big) = \int_{\mathbf{t_0}}^{\boldsymbol{t}} \frac{\partial V}{\partial \mathbf{X}} \mathbf{f}\big(\mathbf{x}(\tau, \mathbf{x}_0)\big), \boldsymbol{\theta}, \mathbf{u}(\tau), \tau) d\tau.$$

On taking (19), into account we obtain

$$V\big(\mathbf{x}(t, \mathbf{x}_0)\big) - V\big(\mathbf{x}(t_0, \mathbf{x}_0)\big) = \int_{t_0}^{t} \dot{V}(\tau) d\tau \leq 0.\text{Hence}$$
$$V\big(\mathbf{x}(t, \mathbf{x}_0)\big) \leq V(\mathbf{x}_0)\ \forall\, t \in [t_0, T].$$

Moreover in accordance with (18) the following holds:

$$\rho_1(\| \mathbf{x}(t, \mathbf{x}_0) \|) \leq \rho_2(\| \mathbf{x}_0 \|)\ \forall\, t \in [t_0, T] \Rightarrow$$

$$\| \mathbf{x}(t, \mathbf{x}_0) \| \leq \rho_1^{-1}(\rho_2(\| \mathbf{x}_0 \|))\ \forall\, t \in [t_0, T],$$

where$\rho_1^{-1}\big(\rho_2(s)\big) = s\ \forall\, s \geq 0$.Consider the domain $\mathcal{D} = \{(t, \mathbf{x}), t \in [t_0, T],\ x \in \mathbb{R}^n | \|\mathbf{x}\| \leq 2\rho_1^{-1}(\rho_2(\| \mathbf{x}_0 \|))\}$; $\mathcal{D}$ is compact, and hence $\mathbf{x}(t, \mathbf{x}_0)$, $t \in [t_0, T]$ can be continued until the boundary of $\mathcal{D}$, the right hand side is Ellipsoidal-shaped in**x** and **u**, and continuous in *t*. Because $\mathbf{x}(t, \mathbf{x}_0)$ cannot reach the boundary$\mathbf{x} = 2\rho_1^{-1}(\rho_2(\| \mathbf{x}_0 \|))$, it must necessarily cross theboundary $t = T$. Given that the right hand side of(16) can be increased by a finite increment Δ. This however is in contradiction with the fact that T is finite. Hence we can conclude that $\mathbf{x}(t, \mathbf{x}_0)$ is defined for all $t \geq t_0$, and that it is bounded.It can be noticed that the composite $\rho_1^{-1}\big(\rho_2(s)\big)$ is a non-decreasing function of s, and $\rho_1^{-1}\big(\rho_2(s)\big) \in \mathcal{K}_\infty$. Thus, denoting $\varepsilon(\delta) = \rho_1^{-1}\big(\rho_2(\delta)\big)$, we arrive at

$$\| \mathbf{x}_0 \| \leq \delta \Rightarrow \| \mathbf{x}(t, \mathbf{x}_0) \| \leq \rho_1^{-1}\big(\rho_2(\delta)\big) = \varepsilon(\delta).$$

The function $\varepsilon(\delta) \in \mathcal{K}_\infty$, hence it range coincides with $\mathbb{R}_{\geq 0}$. Therefore we can conclude now that for every $\tilde{\varepsilon} > 0$there exists $\delta = \varepsilon^{-1}(\tilde{\varepsilon}) > 0$ such that

$$\| \mathbf{x}_0 \| \leq \delta \Rightarrow \| \mathbf{x}(t, \mathbf{x}_0) \| \leq \varepsilon(\delta) = \varepsilon(\varepsilon^{-1}(\tilde{\varepsilon})) = \tilde{\varepsilon}.$$

In other words, according to the Necessary andSufficient Condition 2.3.4, the origin is globally stable in the sense of the Channel Transfer. To prove the second part of the proposition we will follow the argument presented in [3]. It is noticed that the inequality (21)automatically implies

$$\dot{V} \leq -\alpha_1(\| \mathbf{x} \|). \quad (22)$$

Therefore

$$V(\mathbf{x}(t, \mathbf{x}_0)) - V(\mathbf{x}_0) \leq \int_{t_0}^{t} -\alpha_1(\| \mathbf{x}(\tau, \mathbf{x}_0) \|)d\tau \ \forall \ t \geq t_0, \quad (23)$$

and hence

$$\lim_{t \to \infty} \int_{t_0}^{t} \alpha_1(\| \mathbf{x}(\tau, \mathbf{x}_0) \|)d\tau \leq V(\mathbf{x}_0) < \infty. \quad (24)$$

Moreover, $V(\mathbf{x}(t, \mathbf{x}_0))$ is a monotone function of t, and it is bounded from below because it positivedefinite. Thus there exists $a \in \mathbb{R}_{\geq 0}$, such that $\lim_{t \to \infty} V(\mathbf{x}(t, \mathbf{x}_0)) = a$. We shall now show that $a = 0$. Suppose that $a>0$. Inequality (18) implies that$\| \mathbf{x} \| \geq \rho_2^{-1}(V(\mathbf{x}))$, thus

$$\dot{V} \leq -\alpha_1(\| \mathbf{x} \|) \leq -\alpha_1(\rho_2(V(\mathbf{x}(t, \mathbf{x}_0)))) \leq -\alpha_1(\rho_2^{-1}(a)).$$

This leads to the conclusion that the function $V(\mathbf{x}(t, \mathbf{x}_0)) \leq V(\mathbf{x}(t_0, \mathbf{x}_0)) - (t - t_0)\alpha_1(\rho_2^{-1}(a))$becomesnegative in finite subcarrier. The latter, is not possible because $V(\mathbf{x})$ is assumed to be positive definite.If the function$\mathbf{u}: \mathbb{R} \to \mathbb{R}^m$on the right hand sideof (16),is bounded the function $\mathbf{f}(\cdot)$ is bounded with respect to t, and the function $\alpha_1(\cdot)$, is differentiable, then the proof of the second part of the theorem can be easily completed by using Lemma 2.1. In this case differentiability of $\alpha_1(\cdot)$, boundedness of $\mathbf{x}$, $\mathbf{u}$ and $\boldsymbol{\theta}$, and boundedness of the right-hand side of (16) with respect to $t$imply that the function $\alpha_1(\| \mathbf{x}(\tau, \mathbf{x}_0) \|)$, is uniformly continuous in t. According to Lemma 2.1, inequality (24) ensures that

$$\lim_{t \to \infty} \alpha_1(\| \mathbf{x}(\tau, \mathbf{x}_0) \|) = 0,$$

andstrict monotonicity of the function $\alpha_1(\cdot)$, implies that $\| \mathbf{x}(\tau, \mathbf{x}_0) \| \to 0$ as$t \to \infty$.The main benefit of Proposition 2.1, is that it allows us to reduce the analysis of asymptotic properties of theMulti-User OFDM system's solutions to an easier problem of checking the algebraic inequalities (19) and (21). These inequalities can serve as target bit rate constraints determining the desired behavior of aMulti-User OFDM system. It is to be noticed that these constraints do not require precise knowledge of the unknown parameters $\boldsymbol{\theta}$. In the proof of the second part of the theorem, regarding asymptotic local operating conditions, we considered a specific case illustrating how Lemma 2.1

can be used to show that $\mathbf{x}$ approaches the origin asymptotically. The main reason for using this particular technique is that the use in tandem of a local operating conditions proof ensuring bounded of the Multi-User OFDM system's solutions followed by the analysis of estimates (22), (23) and (24) lies at the core of many local operating conditions proofs in literature on subcarriersynchronization and regulation. Despite it simplicity and generality, the method of Channel Transfer Functions has an obvious advantage. In order to use the method on needs to find a function $V(\mathbf{x})$ satisfying properties (19) and (21).Finding such a function is a non-trivial operation. Yet, there are a large classes of Multi-User OFDM system for which the corresponding Channel Transfer Functionsare already known. One of the classes of Multi-User OFDM system is the linear skew-symmetric Multi-User OFDM system with subcarrier-varying coefficients.

5. Descriptive Study II

Let the Multi-User OFDM system's pre-equalization be given by the following Multi-User OFDM system of ordinary differential equations:

$$\dot{\mathbf{x}}_1 = A\mathbf{x}_1 + B\phi^{\mathrm{T}}(t)\mathbf{x}_2, \qquad (25)$$

$$\dot{\mathbf{x}}_2 = -\phi(t)C\mathbf{x}_1,$$

Where$\mathbf{x}_1 \in \mathbb{R}^q, \mathbf{x}_2 \in \mathbb{R}^p, \phi(t): \mathbb{R} \to \mathbb{R}^{p\times m}$, is a continuous function of t, and A, B, and C are $q \times q$, $q \times m$, and $m \times q$ matrices, respectively. The OFDM channel assignment problem in the domain of regulation can be reduced to the analysis of (25). Therefore, understanding the basic asymptotic properties of Multi-User OFDM system (25) is desirable.Investigating the local operating conditions of the zero equilibrium of Multi-User OFDM system (25), suppose that there exists apositive definite and symmetric matrix $P = P^{\mathrm{T}}$:

$$\mathbf{x}^{\mathrm{T}}P\mathbf{x} > 0\ \forall \mathbf{x} \neq 0 \qquad (26)$$

Such that

$$PA^{\mathrm{T}} + AP = -Q, Q = Q^{\mathrm{T}}, \mathbf{x}^{\mathrm{T}}Q\mathbf{x} > 0\ \forall \mathbf{x} \neq 0, \qquad (27)$$
$$PB = C^{\mathrm{T}}.$$

Since $P = P^{\mathrm{T}} and Q = Q^{\mathrm{T}}$ are positive definite, the Eigen values of P and Q are real positive, and moreover thefollowing property holds:

$$\lambda_{\min}(P) \parallel \mathbf{x} \parallel^2 \leq \mathbf{x}^{\mathrm{T}}P\mathbf{x} \leq \lambda_{\max}(P) \parallel \mathbf{x} \parallel^2, \qquad (28)$$

$$\lambda_{\min}(Q) \parallel \mathbf{x} \parallel^2 \leq \mathbf{x}^{\mathrm{T}}Q\mathbf{x} \leq \lambda_{\max}(Q) \parallel \mathbf{x} \parallel^2.$$

Let λ be an eigenvalue of λ, which may be possibly complex, $\mathbf{x}_\lambda$ be its corresponding eigenchannel vector and $\lambda^*$and $\mathbf{x}_\lambda^*$be the complex conjugates of$\lambda$and $\mathbf{x}_\lambda$respectively. Then according to (26), the following holds:

$$0 < \mathbf{x}_\lambda^* P\mathbf{x}_\lambda = \lambda \parallel \mathbf{x}_\lambda \parallel^2 = \mathbf{x}_\lambda^{\mathrm{T}} P\mathbf{x}_\lambda^{*\mathrm{T}} = \lambda^* \parallel \mathbf{x}_\lambda \parallel^2.$$

Therefore, $\lambda \in \mathbb{R} and \lambda > 0$. In order to see that (28) holds too, it is noticed that $P = P^{\mathrm{T}}$is Hermitian. Hence there is a non-singular orthonormal $q \times q\ matrix T$,$T^{\mathrm{T}}T = I$, considering the eigenchannel vectors of P such that $T^{\mathrm{T}}PT$ is a diagnol matrix with the Eigen values of P placed on its main diagonal. Finally, let $\mathbf{x} = T\xi$, and it can be seen that$\| \mathbf{x} \|^2 = \xi^{\mathrm{T}}T^{\mathrm{T}}T\xi = \| \xi \|^2$. Thus

$$\mathbf{x}^{\mathrm{T}}P\mathbf{x} \leq \lambda_{\max}(P) \| \xi \|^2 = \lambda_{\max}(P) \| \mathbf{x} \|^2, \tag{29}$$

$$\mathbf{x}^{\mathrm{T}}P\mathbf{x} \leq \lambda_{\min}(P) \| \xi \|^2 = \lambda_{\min}(P) \| \mathbf{x} \|^2.$$

Proceeding with the local operating conditions analysis of the zero equilibrium of (25), and using the method of Channel Transfer Functions, local operating conditions of the equilibrium is guaranteed if we find a function $V(\cdot)$ satisfying conditions (19) and (21). We don't know this function yet. Hence a plausible option is to select a candidate function that satisfies the constraint of positive definiteness. This we may call as the Channel Transfer candidate function. After completing this step we can continue with checking whether the second condition (21), holds too. Picking the following Channel Transfer candidate function for the Multi-User OFDM system of (25):

$$V(\mathbf{x}) = \mathbf{x}_1^{\mathrm{T}}P\mathbf{x}_1 + \mathbf{x}_2^{\mathrm{T}}\mathbf{x}_2. \tag{30}$$

Function $V(\mathbf{x})$ defined as in (30) satisfies condition (18). Hence Channel Transferlocal operating conditions of the origin will follow if we show that

$$\dot{V} \leq 0.$$

For this purpose we consider

$$\begin{aligned}\dot{V} &= \mathbf{x}_1^{\mathrm{T}}P(A\mathbf{x}_1 + B\phi(t)^{\mathrm{T}}\mathbf{x}_2) + (A\mathbf{x}_1 + B\phi(t)^{\mathrm{T}}\mathbf{x}_2)^{\mathrm{T}}\mathbf{x}_1 - 2\mathbf{x}_2^{\mathrm{T}}\phi(t)C\mathbf{x}_1 \\ &= \mathbf{x}_1^{\mathrm{T}}(PA + A^TP)\mathbf{x}_1 + 2\mathbf{x}_1^{\mathrm{T}}PB\phi(t)^{\mathrm{T}}\mathbf{x}_2 - 2\mathbf{x}_2^{\mathrm{T}}\phi(t)C\mathbf{x}_1.\end{aligned} \tag{31}$$

Taking (27) into account we obtain that

$$\dot{V} \leq -\mathbf{x}_1Q\mathbf{x}_1 + 2\mathbf{x}_1^{\mathrm{T}}PB\phi(t)^{\mathrm{T}}\mathbf{x}_2 - 2\mathbf{x}_2^{\mathrm{T}}\phi(t)C\mathbf{x}_1 \leq -\mathbf{x}_1^{\mathrm{T}}Q\mathbf{x}_1 \leq 0 \tag{32}$$

The latter inequality, as follows from Theorem 2.1 guarantees that the zero equilibrium of (25) is stable. Formally this statement is summarized in the following Lemma.

Lemma 2.2 ConsiderMulti-User OFDM system (25) and suppose that there exists apositive definite and symmetric matrix P = P^{T} satisfying the condition (27). Then the zero equilibrium of (25) is globally stable in the sense of the Channel Transfer. If in addition, the function $\phi(t)$ on the right-hand side of (25) is bounded uniformly in t,

$$\exists M \in \mathbb{R}: ||\phi(t)|| \le M \forall t \in \mathbb{R}\,, \tag{33}$$

Then,

$$\lim_{t\to\infty} \mathbf{x}_1(t) = 0. \tag{34}$$

Proof of Lemma 2.2 The first part of the lemma is already proven. In order to see that the second part holds too, the following estimate is employed

$$\lambda_{\min}(Q) \parallel \mathbf{x}_1 \parallel^2 \le \mathbf{x}_1^{\mathrm{T}} Q \mathbf{x}_1 \le \lambda_{\max}(Q) \parallel \mathbf{x}_1 \parallel^2$$

and apply Lemma 2.1 to

$$V(\mathbf{x}(t)) - V(\mathbf{x}(t_0)) \le -\int_{t_0}^{t} \mathbf{x}_1(\tau)^{\mathrm{T}} Q \mathbf{x}(\tau) d\tau \le \int_{t_0}^{t} \lambda_{\min}(Q) \parallel \mathbf{x}_1 \parallel^2 d\tau \Rightarrow \int_{t_0}^{t} \lambda_{\min}(Q) \parallel \mathbf{x}_1(\tau)^{\mathrm{T}} \parallel^2 d\tau \le V(\mathbf{x}(t_0)) \forall\, t > t_0. \tag{35}$$

Lemma 2.2 and its proof constitute a simple illustration of how the method of Channel Transfer Functions can be used in the analysis of local operating conditions of equilibrium in Multi-User OFDM system (25). The proof offers a useful and channel transfer interpretation in the context of OFDM channel assignmentsymbol tracking and regulation. Supposing for example that $\phi\,(t)$ is disturbance acting on theMulti-User OFDM system pre-equalization Property (34) can be viewed as the desired behavior of the Multi-User OFDM system, and $\mathbf{x}_2$ are the Multi-User OFDM system internal variables, of which the function is to minimize the influence of the disturbance on the desired behavior. The first line of condition (27) serves as an existence hypothesis stipulating the possibility that the desired behavior (34) is realizable in the absence of perturbations. The argument in the proof of Lemma 2.2 can be straightforwardly generalized to the case of non-linear Multi-User OFDM system, like OFDM channel assignment. The equations in (25) constitute a good prototype for the systematic study of such Multi-User OFDM system. We have seen how the method of Channel Transfer Functions together with Lemma2.1 can be used to derive conditions that a part of the Multi-User OFDM system's state channel vector, $\mathbf{x}_1(t)$ converges to zero asymptotically. About the rest of the Multi-User OFDM system's variables like $\mathbf{x}_2(t)$,by applying Lemma 2.1 to (33) and (34), we can conclude that,

$$\lim_{t\to\infty} \dot{\mathbf{x}}_1(t) = 0. \tag{36}$$

Hence

$$\lim_{t\to\infty} A\mathbf{x}_1(t) + B\phi(t)^{\mathrm{T}}\mathbf{x}_2(t) = \lim_{t\to\infty} B\phi(t)^{\mathrm{T}}\mathbf{x}_2(t) = 0. \tag{37}$$

If $\dot{\mathbf{x}}_2 = 0$ then the property (37) implies that the channel vector $\mathbf{x}_2(t)$ is orthogonal to all of the rows of the matrix$\alpha(t) = B\phi(t)^{\mathrm{T}}$. A channel vector that is orthogonal to all of the basis channel vectors of a row space must necessarily be zero. Hence if

$\alpha(t)$spans $\mathbb{R}^{p\times q}$, then (37) implies that $\mathbf{x}_2 = 0$. In our case $\dot{\mathbf{x}}_2 \neq 0$, but it is nevertheless asymptotically vanishing. Therefore, it is intuitively clear that if $\alpha(t)$has non-zero projections onto all matrices (channel vectors) in $\mathbb{R}^{p\times q}$, and $\alpha(t)$operates sufficiently fast, so that its; is non-vanishing, then the condition (37) could imply that $\mathbf{x}_2(t) \to 0$ as $t \to \infty$. These properties of $\alpha(t)$ having non-zero projections to every channel vector in $\mathbb{R}^{q\times p}$ together with the requirement that its is non-vanishing are captured by the notion of persistency of quasi-periodic excitation.

Necessary and Sufficient condition 2.5.1A function $\alpha: \mathbb{R} \to \mathbb{R}^{p\times q}$ is called persistently exciting if and only if there exist $T, \delta, \Delta \in \mathbb{R}_{>0}$, such that for all$t \in \mathbb{R}$ and every $\boldsymbol{\theta} \in \mathbb{R}^p$ the following inequality holds:

$$\delta \parallel \boldsymbol{\theta} \parallel^2 \leq \boldsymbol{\theta}^{\mathrm{T}} \left(\int_t^{t+T} \alpha(\tau)\alpha(\tau)^{\mathrm{T}}\, d\tau \right) \boldsymbol{\theta} \leq \Delta \parallel \boldsymbol{\theta} \parallel^2 . \tag{38}$$

Persistency of quasi-periodic excitation of a function admits a simple geometric interpretation. On applying the mean-value theorem to (38) we obtain that

$$\boldsymbol{\theta}^{\mathrm{T}} \left(\int_t^{t+T} \alpha(\tau)\alpha(\tau)^{\mathrm{T}}\, d\tau \right) \boldsymbol{\theta} = T(\boldsymbol{\theta}^{\mathrm{T}}\alpha(\tau))\,(\alpha(\tau)^{\mathrm{T}}\boldsymbol{\theta}), \tau \in [t, t+T].$$

Therefore, noticing that $\boldsymbol{\theta}^{\mathrm{T}}\alpha(\tau)^{\mathrm{T}}\alpha(\tau)\,\boldsymbol{\theta} = \parallel \alpha(\tau)\boldsymbol{\theta} \parallel^2$ and taking (38) into account we can conclude that there exist $T, \delta, \Delta \in \mathbb{R}_{>0}$:

$$\forall t \in \mathbb{R} \exists \tau \in [t, t+T]: \delta/T \parallel \boldsymbol{\theta} \parallel^2 \leq \parallel \alpha(\tau)\boldsymbol{\theta} \parallel^2 \leq \Delta/T \parallel \boldsymbol{\theta} \parallel^2 \tag{39}$$

If $\alpha(t)$ is a channel vector-function, i.e. $q = 1$, then

$$\alpha(\tau)\boldsymbol{\theta} = \parallel \alpha(\tau) \parallel \parallel \boldsymbol{\theta} \parallel \cos(\beta(\tau)) ,$$

where $\beta(\tau)$ is the angle between the channel vectors $\alpha(\tau)$and$\boldsymbol{\theta}, \tau \in [t, t+T]$. This means that for all t, there exists a subcarrier instant $\tau \in [t, t+T]$ such that the angle $\beta(\tau)$ deviates from $\pm\pi/2 + 2\pi k, k \in \mathbb{Z}$. If the length of $\parallel \alpha(\tau) \parallel$is bounded from above by $M_\alpha$then,

$$|\cos(\beta(\tau))| \geq 1/M\alpha\sqrt{(\delta/T)} > 0,$$ and

hence the channel vector $\alpha(\tau)$ must have a non-zeroprojection on $\boldsymbol{\theta}$.Let us show that persistency of quasi-periodic excitation of the function $B\phi(t)^{\mathrm{T}}$ in (25) together with condition (34) ensures that

$$\lim_{t\to\infty} \mathbf{x}_2(t) = 0.$$

This result is formulated in Lemma 2.3.

Lemma 2.3 Consider the Multi-User OFDM system

$$\dot{\mathbf{x}}_2 = -\phi(t)C\mathbf{x}_1(t),$$ where

$\mathbf{x}_1(t)$ is a bounded and continuous function satisfying conditions (34) and (36). Suppose that

$$\lim_{t\to\infty} B\phi(t)^{\mathrm{T}}\mathbf{x}_2(t) = 0,$$

where $\phi(t)$ is a bounded continuous function, and $B\phi(t)^{\mathrm{T}}$is quasi-periodic persistently exciting. Then

$$\lim_{t\to\infty} \mathbf{x}_2(t) = 0.$$

Proof of Lemma 2.3 Consider the symbol interval $[t_0,\infty]$ as aunion of the OFDM symbol intervals $[t_i, t_{i+1}]$, $t_{i+1} = t_i + T$, i =0,1,... Given that $\mathbf{x}_1(t) \to 0$ as$t \to \infty$, and that the function $\phi(t)$is bounded, there should exist a function $\delta_2(t)\colon \mathbb{R} \to \mathbb{R}^p$such that

$$\mathbf{x}_2(\tau) = \mathbf{x}_2(t_i) + \delta_2(\tau), \forall\tau \in [t_i, t_{i+1}], \lim_{t\to\infty} \delta_2(t) = 0. \quad (40)$$

Let us denote $B\phi(t) = \alpha(t)$. The function $\alpha(t)$ is quasi-periodic persistently exciting, hence there exits $\tau_i \in [t_i, t_i + T]$such that

$$\| \alpha(\tau_i)\mathbf{x}_2(t_i) \| \geq \delta/T \| \mathbf{x}_2(t_i) \|. \quad (41)$$

Substituting (40) into (41) we obtain:

$$\delta/T \| \mathbf{x}_2(t_i) \| \leq \| \alpha(\tau_i)\big(\mathbf{x}_2(\tau_i) - \delta_2(\tau_i)\big) \| \leq \| \alpha(\tau_i)\delta_2(\tau_i) \| + \| \leq \| \alpha(\tau_i)\mathbf{x}_2(\tau_i) \|,$$

and finally

$$\lim_{i\to\infty} \delta/T \| \mathbf{x}_2(t_i) \| \leq \lim_{i\to\infty}(\| \alpha(\tau_i)\delta_2(\tau_i) \| + \| \alpha(\tau_i)\mathbf{x}_2(\tau_i) \|) = 0.$$

Therefore, $\mathbf{x}_2(t_i) \to 0$ as$i \to \infty$. This together with (40) implies that $\mathbf{x}_2(t) \to 0\ ast \to \infty$.A remarkable property of Multi-User OFDM system (25) subjected to the conditions (27) is that persistency of quasi-periodic excitation of $\phi(t)$ not only assures that both $\mathbf{x}_1(t)$and $\mathbf{x}_2(t)$ converge to zero asymptotically but also guarantees that the rate of convergence is exponential. Numerous versions of this result can be found in the literature on subcarriersynchronization [5, 6]. We enumerate it here in addition the rate of convergence expressed in terms of A, B, C and $\phi(t)$.

Proposition 2.2 Consider Multi-User OFDM system (25), and suppose that conditions (27) hold. Moreover, let the function $B\phi(t)^{\mathrm{T}}$in (25) be quasi-periodic persistently exciting:

$$\exists\delta, T\colon \int_t^{t+T} \phi(\tau)\, B^{\mathrm{T}}B\phi(\tau)d\tau \geq \delta\forall t, \quad (42)$$

and,

$$\max\{\| \phi(t) \|, \| \dot{\phi}(t) \|\} \leq B_1.$$

Let $\Phi(t, t_0), \Phi(t_0, t_0) = I$ be the fundamentalMulti-User OFDM system of solutions of (25) and let $\mathbf{p}$ be a channel vector from $\mathbb{R}^{p+q}$. Then

$$\| \Phi(t_2, t_1)\mathbf{p} \| \leq e^{-\rho(t_2-t_1)} \| \mathbf{p} \| D_2, \forall t_2 \geq t_1 \geq t_0,$$

Where the parameters ρ and D_2 do not depend on t_0 and $\mathbf{p}$ and can be expressed explicitly as functions of B_1, the constants δ and T in (42) and matrices A, B, C, P and Q.**Proof of Theorem 2.2**Starting with the following Lemma,**Lemma 2.4**Let $\mathbf{x}(t)$: $\mathbb{R} \rightarrow \mathbb{R}^n$ be a function satisfying

$$\max\{\| \mathbf{x} \|_{2,[t,\infty]}, \| \mathbf{x} \|_{\infty,[t,\infty]}\} \leq \mathrm{c} \| \mathbf{x}(t) \|, \forall t \geq t_0. \quad (43)$$

Then,

$$\| \mathbf{x}(t) \| \leq ce^{1/2}e^{-(t-t_1)/2c^2} \| \mathbf{x}(t_1) \|, \forall t \geq t_1 \geq t_0. (44)$$

Proof of Lemma 2.4 It is noticed that (43) implies

$$\int_t^\infty \| \mathbf{x}(\tau) \|^2 d\tau \leq c^2 \| \mathbf{x}(t) \|^2, \| \mathbf{x} \|^2_{\infty,[t,\infty]} \leq c^2 \| \mathbf{x}(t) \|^2.$$

Let $v(t) = \int_t^\infty \| \mathbf{x}(\tau) \|^2 d\tau$, then

Denoting $\tau = t_2 + T$, we get

$$\| \mathbf{x}(\tau) \| \leq ce^{1/2}e^{-(\tau-t_1)/2c^2} \| \mathbf{x}(t_1) \| \ \forall \tau \geq t_1 + T.$$

The Lemma will be proven if we show that the estimate holds for $0 \leq t_1 \leq \tau \leq t_1 + T$. Condition (43) implies that $\| \mathbf{x}(\tau) \| \leq c \| \mathbf{x}(t_1) \| \ \forall \tau \geq t_1$. On the other hand,

$$1 = e^{1/2}e^{-T/2c^2} \leq e^{1/2}e^{-(\tau-t_1)/2c^2} \leq e^{1/2} \forall t_1 \leq \tau \leq t_1 + T.$$

Hence $\| \mathbf{x}(\tau) \| \leq c \| \mathbf{x}(t_1) \| \leq ce^{1/2}e^{-(\tau-t_1)/2c^2} \| \mathbf{x}(t_1) \|$ for $t_1 \leq \tau \geq t_1 + T$ as well.The proof of the theorem is given as follows. Consider $\mathbf{x} = \mathrm{col}\ (\mathbf{x}_1, \mathbf{x}_2)$.

If we show that there exists $c \in \mathbb{R}_{>0}$ such that (43) holds then according to Lemma 2.4, we can conclude that,

$$\| \mathbf{x}(t_2) \| \leq ce^{1/2}e^{-(t_1-t_2)/2c^2} \| \mathbf{x}(t_1) \|, \forall t_2 \geq t_1 \geq t_0.$$

The result would then follow immediately if we let $\mathbf{x}(t_1) = \mathbf{p}$ and substitute $\mathbf{x}(t_2) = \boldsymbol{\Phi}$ $(t_2,t_1)\mathbf{p}$ into the inequality abovelet us now find a constant c such that (43) holds for Multi-User OFDM system (25).Consider the positive definite function $V(\mathbf{x}) = \mathbf{x}_1^{\mathrm{T}}P\mathbf{x}_1 + \| \mathbf{x}_2 \|^2$, where P is a symmetric positive definite matrix satisfying (27). According to (30), (31) and (32), we can conclude that,

$$\min\{\lambda_{\min}(P),1\} \| \mathbf{x} \|^2 \leq V(\mathbf{x}) \leq \max\{\lambda_{\max}(P),1\} \| \mathbf{x} \|^2,$$

and that $\dot{V} \leq -\mathbf{x}_1^{\mathrm{T}}Q\mathbf{x}_1 \leq -\lambda_{\min}(Q) \| \mathbf{x}_1 \|^2$.
Thus,

$$\| \mathbf{x} \|_{2,[t,\infty]} \leq \max\{\lambda_{\max}(P),1\}^{1/2}/\lambda_{\min}(Q)^{1/2} \| \mathbf{x}(t) \| = c_1 \| \mathbf{x}(t) \|, \quad (45)$$

$$\| \mathbf{x} \|_{\infty,[t,\infty]} \leq \max\{\lambda_{\max}(P),1\}^{1/2}/\min\{\lambda_{\min}(P),1\}^{1/2} \| \mathbf{x}(t) \| = c_2 \| \mathbf{x}(t) \|, \forall t \geq t_0.$$

Let us now estimate $\| \mathbf{x} \|_{2,[t,\infty]}$. In order to do so we introduce a new variable $\mathbf{z} = \mathbf{x}_2 - \phi(t)B^{\mathrm{T}}\mathbf{x}_1$ and consider its derivative w.r.t. t:

$$\dot{\mathbf{z}} = -\phi(t)B^{\mathrm{T}}B\phi(t)^{\mathrm{T}}\mathbf{z} - \left[\phi(t)B^{\mathrm{T}}A + (\phi(t)B^{\mathrm{T}}B\phi(t)^{\mathrm{T}})\phi(t)B^{\mathrm{T}} + \phi(t)C + \dot{\phi}(t)B^{\mathrm{T}}\right]\mathbf{x}_1. \quad (46)$$

To proceed further we will need the following lemma.

Lemma 2.5 Consider the Multi-User OFDM system

$$\dot{\mathbf{z}} = -\Gamma\alpha(t)\alpha(t)^{\mathrm{T}}\mathbf{z}, \mathbf{z} \in \mathbb{R}^n, \alpha: \mathbb{R} \to \mathbb{R}^{n\times m}, \quad (47)$$

Where α is quasi-periodic persistently exciting (i.e. property (38) holds), $\| \alpha(t) \| \leq M$, and $\Gamma = \Gamma^{\mathrm{T}}$is a positive definite matrix. Let $\mathbf{z}$ (t,t_0) be a solution of (47) passing through $\mathbf{z}_0$ at $t = t_0$. Then there existλ, $D \in \mathbb{R}_{>0}$such that

$$\| \mathbf{z}(t,t_0) \| \leq De^{-\lambda(t-t_0)} \| \mathbf{z}_0 \|, t \geq t_0,$$

Where $D = (\lambda_{\max}(\Gamma)/\lambda_{\min}(\Gamma))^{1/2}e^{\lambda T}$ and

$$\lambda = \delta\lambda_{\min}(\Gamma)/T(1 + \lambda_{\max}(\Gamma)M^2T)^2$$

Independently of $\mathbf{z}_0$, t, t_0.

Proof of Lemma 2.5 Consider the following positive definite function:

$$V(\mathbf{z}) = \| \mathbf{z} \|^2_{\Gamma-1}, \lambda_{\min}(\Gamma^{-1}) \| \mathbf{z} \|^2 \leq V(\mathbf{z}) \leq \lambda_{\max}(\Gamma^{-1}) \| \mathbf{z} \|^2$$

Its derivative is $\dot{V} = -2 \| \mathbf{z}^{\mathrm{T}}\alpha(t) \|^2$. Thus taking into account that $\alpha(t)$ is quasi-periodic persistently exciting we can derive the following estimate:$V(\mathbf{z}(t_0 + T)) - V(\mathbf{z}(t_0)) \leq$
$-((\beta)/(1+\beta))((2\delta)/\lambda_{\max}(\Gamma^{-1}))V(\mathbf{z}(t_0)) + \beta\lambda_{\max}(\Gamma)^2M^4T^2\left(V(\mathbf{z}(t_0)) - V(\mathbf{z}(t_0 + T))\right) \Rightarrow$

$$V(\mathbf{z}(t_0 + T)) \leq \rho V(\mathbf{z}(t_0)),$$

Where

$$\rho = \left(1 - \left((2\delta)/\lambda_{\max}(\Gamma^{-1})\right)(\beta/(1+\beta)(1 + \beta\lambda^2_{\max}(\Gamma)M^4T^2))\right).$$

The value of ρ is minimized at $\beta = 1/(\lambda_{\max}(\Gamma)M^2T)$ in which case

$$V(t_0 + T) \leq \rho V(t_0), \rho = (1 - ((2\delta)/\lambda_{\max}(\Gamma^{-1})(1/(1 + \lambda_{\max}(\Gamma)M^2T)^2)$$

It is to be noticed that

$$-(\ln(1 - \sigma)/T) \geq \sigma/T, \sigma \in (0,1).$$

Therefore

$$V(\mathbf{z}(t_0 + T)) \leq e^{((\ln(\rho)/T)T)}V(\mathbf{z}(t_0)) \leq e^{-2\lambda T}V(\mathbf{z}(t_0)),$$

$$\lambda = (\delta/\lambda_{\max}(\Gamma^{-1}))(1/T(1 + \lambda_{\max}(\Gamma)M^2T)^2).$$

Since any $\Delta t \geq 0$ can be expressed as $\Delta t = nT + t', t' \in [0, T)$, we have that

$$V(\mathbf{z}(t_0 + \Delta t)) \leq (e^{-2\lambda(nT+t')}/e^{-2\lambda t'})V(\mathbf{z}(t_0)) \leq (e^{-2\lambda\Delta t}/e^{-2\lambda T})\, V(\mathbf{z}(t_0)).$$

Finally

$$\| \mathbf{z}(t_0 + \Delta t) \| \leq De^{-\lambda\Delta t} \| \mathbf{z}(t_0) \|,$$

$$D = (\lambda_{\max}(\Gamma^{-1})/\lambda_{\min}(\Gamma^{-1}))^{1/2} e^{\lambda T}.$$

The desired inequality now follows from obvious identities:

$$\lambda_{\min}(\Gamma^{-1}) = \frac{1}{\lambda_{\max}(\Gamma)} \text{and} \lambda_{\max}(\Gamma^{-1}) = 1/\lambda_{\min}(\Gamma).$$

Let $\Phi_1(t,\ t_0)$, $\Phi_1\ (t_0,\ t_0) = I$ be the fundamentalMulti-User OFDM system of solutions of

$$\dot{\mathbf{z}} = -\phi(t)B^{\mathrm{T}}B\phi(t)^{\mathrm{T}}\mathbf{z}.$$

According to Lemma 2.5, we have that

$$\| \Phi_1\ (t_2, t_1)\ \mathbf{z}(t_1) \| \leq c_3 \mathrm{e}^{-\tau(t_2 - t_1)} \| \mathbf{z}(t_1) \|,$$

Where

$$c_3 = e^{\tau T}, \tau = \delta/T(1 + T^2\lambda_{\max}(B^{\mathrm{T}}B)B_1^2)^2.$$

Given that any solution of (46) can be expressed as

$$\mathbf{z}(t) = \Phi_1(t,t_1)\ \mathbf{z}(t_1) + \int_{t_1}^{t} \Phi_1(t, s)\chi(s)\mathbf{x}_1(s)ds, \quad t \geq t_1 \geq t_0,$$

Where $\chi(s) = -(\phi(s)B^{\mathrm{T}}A + (\phi(s)B^{\mathrm{T}}B\phi(s)^{\mathrm{T}})\phi(s)B^{\mathrm{T}} + \phi(s)C + \dot{\phi}(s)B^{\mathrm{T}})$, and that

$$\| \chi(s) \| \leq B_1(\| A \| \| B \| + \| C \| + \| B \|) + B_1^3 \| B \|^3 = c_4,$$

Given that $z = \mathbf{x}_2 - \phi(t)B^{\mathrm{T}}\mathbf{x}_1$, the following estimate holds:

$$\| \mathbf{z} \|_{2,[t_1,t]} \geq \| \mathbf{x}_2 \|_{2,[t_1,t]} - \| \phi(\cdot)B^{\mathrm{T}}\mathbf{x}_1 \|_{2,[t_1,t]}.$$ Thus

$$\| \mathbf{x}_2 \|_{2,[t_1,t]} \leq \| \mathbf{z} \|_{2,[t_1,t]} + B_1 \| B \| \| \mathbf{x}_1 \|_{2,[t_1,t]},$$ and hence

$$\| \mathbf{x}_2 \|_{2,[t_1,t]} \leq ((c_3/\sqrt{2\tau}) \| \mathbf{z}(t_1) \| + (c_4 c_3/\tau) \| \mathbf{x}_1 \|_{2,[t_1,t]}) + B_1 \| B \| \| \mathbf{x}_1 \|_{2,[t_1,t]}.$$

Given that

$$\| \mathbf{z}(t_1) \| \leq (B_1 \| B \| \| \mathbf{x}_1(t_1) \| + \| \mathbf{x}_2(t_1) \|) \leq (1 + B_1 \| B \|) \| \mathbf{x}_1(t_1) \|,$$

We obtain

$$\| \mathbf{x}_2 \|_{2,[t_1,t]} \leq (c_3/\sqrt{2\tau})(1 + B_1 \| B \|) \| \mathbf{x}_1(t_1) \| + ((c_4 c_3/\tau) + B_1 \| B \|) \| \mathbf{x}_1 \|_{2,[t_1,t]}$$
$$= c_5 \| \mathbf{x}(t_1) \| + c_6 \| \mathbf{x}_1 \|_{2,[t_1,t]} .$$

Thus invoking (45) we can derive that

$$\| \mathbf{x} \|_{2,[t_1,t]} \leq (c_5 + c_1 c_6) \| \mathbf{x}(t_1) \|.$$

Hence

$$\max\{\| \mathbf{x} \|_{2,[t_1,t]}, \| \mathbf{x} \|_{\infty,[t_1,t]}\} \leq c_7 \| \mathbf{x}(t_1) \|,$$
$c_7 = \max\{(c_5 + c_1 c_6), c_2\}$ and
$$\| \Phi(t_0, t_1)\mathbf{p} \| \leq D_2 e^{-\rho(t-t_0)} \| \mathbf{p} \|.$$

6. Summary and Results

In this paper we articulated the new notions of asymptotic characterization of OFDM channel assignment with respect to multi-user OFDM system, OFDM constellation points, and Viterbi decoder metrics for distributed scheme. We articulated various definitions for local operating conditions and discussed tools for local operating conditions-analysis such as the method of Channel Transfer Functions and Lemma 2.1. These tools were illustrated with the local operating conditions analysis problem for OFDM channel assignment.

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