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Solving Optimal Control Problem for Linear Time-invariant Systems via Chebyshev Wavelet

Hatem Elaydi and Atya A. Abu Haya

Faculty of Engineering, Islamic University, Gaza, Palestine

Abstract

Over the last four decades, optimal control problem are solved using direct and indirect methods. Direct methods are casted in parameterization and discretization forms. Parameterizations are based on using polynomials to represent the optimal problem. The proposed direct method is based on transforming the optimal control problem into a mathematical programming problem. A wavelet-based method is used to parameterize the linear quadratic optimal control problem. The Chebyshev wavelets functions are used as the basis functions. Numerical examples are presented to show the effectiveness of the proposed method, and several optimal control problems were solved. The simulation results show that the proposed method gives good and comparable results.

Keywords: Chebyshev Wavelet, Optimal Control Problem, Time-invariant Systems

Introduction

The goal of an optimal controller is determining a control signal such that a specified performance index is optimized while satisfying the system equations and other constraints. Many different methods have been introduced to solve optimal control problem for a system with given state equations. Examples of optimal control applications include environment, engineering, economics etc. Optimal control problems can be solved by direct and indirect methods. Indirect methods solve the optimal control problems using the Riccati equation, Euler-Lagrange, Caley-Hamilton methods; however, these methods result in a set of usually complicated differential equations [1]. Direct methods use parameterization and discretization of the control and the states approximation. Over the last few decades, orthogonal functions have been extensively used in obtaining an approximate solution of problems described by differential equations [2] based on converting the differential equations into an integral equation through integration. The state and/or control involved in the equation are approximated by finite terms of orthogonal series and using an operational matrix of integration to eliminate the integral operations. The form of the operational matrix of integration depends on the choice of the orthogonal functions like Walsh functions, block pulse functions, Laguerre series, Jacobi series, Fourier series, Bessel series, Taylor series, shifted Legendry, Chebyshev polynomials, Hermit polynomials and Wavelet functions [3].

This paper proposes a solution to solve the general optimal control problem using the parameterization direct method. The Chebyshev wavelets are used as new orthogonal polynomials to parameterize the states and control of the time-varying linear problem. Then, the cost function can be casted using the parameterized states and control.

This paper is organized as follow: section 2 talks about the wavelets and scaling functions, section3 discusses using Chebyshev wavelets to approximate functions, section 4 presents the formulation of problems, section 5 gives numerical examples, and section 6 conclude this study.

Scaling Functions and Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously [4], the following family of continuous wavelets is constructed such as

$$\Psi_{a,b}(t) = |a|^{-\frac{1}{2}}\Psi\left(\frac{t-b}{a}\right), a, b \in \mathbb{R}, a \neq 0 \tag{1}$$

Chebyshev wavelets $\psi_{nm}(t) = \psi(k, m, n, t)$ have four arguments;

$k = 1,2,3, \dots,$ $n = 1,2,3, \dots, 2^k$, m is the order of Chebyshev polynomials and t is the normalized time. They are defined on the interval [0,1) by:

$$\Psi_{nm}(t) = \begin{Bmatrix} \frac{\alpha_m 2^{\frac{k}{2}}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & \frac{n-1}{2^k} \leq t \leq \frac{n}{2^k} \\ 0 & elsewhere \end{Bmatrix} \tag{2}$$

$where$

$$\alpha_m = \begin{cases} \sqrt{2} & m = 0 \\ 2\ , & m = 1,2, \dots \end{cases}$$

Here, $T_m(t)$ are the well−known Chebyshev polynomials of order m, which are orthogonal with respect to the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ and satisfy the following recursive formula [5]:

$T_0(t) = 1$

$T_1(t) = t,$

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \quad m = 1,2,3, \dots. \tag{3}$$

The set of Chebyshev wavelets are an orthogonal set with respect to the weight function

$$\omega_n(t) = \omega(2^{k+1}\, t - 2n + 1) \tag{4}$$

Function Approximation

A function f (t) is defined over [0,1) may be expanded as:

$$f(t) = \sum_{n=1} \sum_{m=0} f_{nm}\psi_{nm}(t) \tag{5}$$

where

$f_{nm} = (f(t), \psi_{nm}(t))$

If the infinite series in Eq. (5) is truncated, then it can be rewritten as

$$f(t) \cong f_{2^k, M-1} = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} f_{nm}\psi_{nm}(t) = F^T\Psi(t) \tag{6}$$

where F *and* $\psi(t)$*are* $2^k M \; x \; 1$ *matrices given by*

$$F = [f_{10}, f_{11}, \dots, f_{1,M-1}, f_{20}, \dots, f_{2,M-1}, \dots f_{2^k,0}, \dots, f_{2^k,M-1}]^T \tag{7}$$

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \dots, \psi_{1,M-1}(t), \psi_{20}(t), \dots, \psi_{2,M-1}(t), \dots \psi_{2^k,0}(t), \dots, \psi_{2^k,M-1}]^T \tag{8}$$

Chebyshev Wavelets Operational Matrix of Integration

For Chebyshev wavelet the integration of the vector $\Psi(t)$ defined in Eq. (8) can be obtained as

$$\int_0^t \Psi(s)ds \cong P\Psi(t) \tag{9}$$

where P is the $(2^k M) \; x \; (2^k M)$ operational matrix for integration and is given in [5] as

$$P = \begin{bmatrix} C & S & S & \cdots & S \\ 0 & C & S & \cdots & S \\ 0 & 0 & C & \cdots & S \\ \vdots & \vdots & \vdots & \ddots & S \\ 0 & 0 & 0 & \cdots & C \end{bmatrix} \tag{10}$$

$Where\ C\ and\ S\ are\ M\ x\ M\ matrices\ given\ by:$

$$C = \frac{1}{2^k}\begin{bmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ \frac{-1}{3\sqrt{2}} & \frac{-1}{4} & 0 & \frac{1}{12} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \frac{-1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \cdots & \frac{-1}{4(M-3)} & 0 & \frac{-1}{4(M-1)} \\ \frac{-1}{2\sqrt{2}M(M-2)} & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{4(M-2)} & 0 \end{bmatrix} \tag{11}$$

And

$$S = \frac{\sqrt{2}}{2^k}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{3} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \frac{-1}{15} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{M(M-2)} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{12}$$

Lemma 1

The integration of the product of two Chebyshev wavelet function vectors is obtained as

$for\quad k = 1,2,\ldots$ and $\mathrm{M} = 3$

$$\int_0^1 \Psi(t)\Psi^{\mathrm{T}}(t)dt = RR \tag{13}$$

$where$

$$RR = \begin{bmatrix} G & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & G \end{bmatrix}$$

and

$$G = \begin{bmatrix} \frac{2}{\pi} & 0 & -\frac{2\sqrt{2}}{3\pi} \\ 0 & \frac{4}{3\pi} & 0 \\ -\frac{2\sqrt{2}}{3\pi} & 0 & 0.594 \end{bmatrix}$$

Chebyshev Scaling Functions

$From\ Eq. (\ 2\ )\ we\ can\ obtained$

$(\ when\ M = 3, k = 2)$

$$\left.\begin{array}{l} \psi_{10}(t) = \sqrt{\frac{8}{\pi}} \\ \psi_{10}(t) = \frac{4}{\sqrt{\pi}}(8t-1) \\ \psi_{11}(t) = \frac{4}{\sqrt{\pi}}(2(8t-1)^2-1) \end{array}\right\} 0 \le t \le \frac{1}{4} \quad (14)$$

$$\left.\begin{array}{l} \psi_{20}(t) = \sqrt{\frac{8}{\pi}} \\ \psi_{21}(t) = \frac{4}{\sqrt{\pi}}(8t-3) \\ \psi_{22}(t) = \frac{4}{\sqrt{\pi}}(2(8t-3)^2-1) \end{array}\right\} \frac{1}{4} \le t \le \frac{1}{2} \quad (15)$$

$$\left.\begin{array}{l} \psi_{30}(t) = \sqrt{\frac{8}{\pi}} \\ \psi_{31}(t) = \frac{4}{\sqrt{\pi}}(8t-5) \\ \psi_{32}(t) = \frac{4}{\sqrt{\pi}}(2(8t-5)^2-1) \end{array}\right\} \frac{1}{2} \le t \le \frac{3}{4} \quad (16)$$

$$\left.\begin{array}{l} \psi_{40}(t) = \sqrt{\frac{8}{\pi}} \\ \psi_{41}(t) = \frac{4}{\sqrt{\pi}}(8t-7) \\ \psi_{42}(t) = \frac{4}{\sqrt{\pi}}(2(8t-7)^2-1) \end{array}\right\} \frac{3}{4} \le t \le 1 \quad (17)$$

Optimal Control Problem Reformulation

The linear quadratic optimal control problem can be stated as follows: Find an optimal controller $u^*(t)$ that minimizes the following quadratic performance index

$$J = \int_0^{t_f} (x^T Q x + u^T R u) dt \quad (18)$$

$subject\ to$

$$\dot{x} = Ax + Bu \tag{19}$$

$$x(0) = x_0 \tag{20}$$

Because Chebyshev wavelets are defined on the time interval $\tau \epsilon [0,1]$ and since the problem is defined on the interval $t \epsilon [0, t_f]$, it is necessary before using Chebyshev wavelets to transform the time interval of the optimal control problem into the interval $\tau \epsilon [0,1]$.

This can be done using

$$\tau = \frac{t}{t_f} \tag{21}$$

So,

$$dt = t_f d\tau \tag{22}$$

Thus, the optimal control problem becomes such as

$$J = t_f \int_0^1 (x^T Q x + u^T R u) d\tau \tag{23}$$

$$\frac{dx}{d\tau} = t_f (Ax + Bu) \tag{24}$$

Control State Parameterization

The basic idea is to approximate the state and control variables by a finite series of Chebyshev wavelets as follow [5]

$$x_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} a^i{}_{nm} \phi_{nm}(t) \qquad i = 1,2, \dots, s \tag{25}$$

$$u_i(t) = \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} b^i{}_{nm} \phi_{nm}(t) \qquad i = 1,2, \dots, r \tag{26}$$

The two equations can be written in compact form such as:

$$x(t) = \left(I_s \otimes \Phi^T(t)\right) a \tag{27}$$

$$u(t) = \left(I_r \otimes \Phi^T(t)\right) b \tag{28}$$

$where\ I_s, I_r$ are s x s and r x r identity matrices and $\Phi(t)$ is $N\ x\ 1$,

$$N = 2^k (M), vector\ of\ Chebychev\ scaling \tag{29}$$

$function\ given\ by:$

$$\Phi(t) = [\Phi_{1m-1}(t), \Phi_{2m-1}(t), \Phi_{3m-1}(t), \ldots, \phi_{2^k m-1}(t)]^T \tag{30}$$

$$\Phi_{im-1}(t) = [\phi_{i0}(t), \phi_{i1}(t), \ldots, \phi_{iM-1}(t)] \tag{31}$$

and

$$a = [\alpha^1\ \alpha^2 \ldots . \alpha^s]^T$$
$$\alpha^i = \left[a^i_{10}\ a^i_{11} \ldots . a^i_{1M-1}\ a^i_{20} \ldots . a^i_{2M-1} \ldots a^i_{2^k 0} \ldots .\ a^i_{2^k M-1}\right]$$
$$i = 1,2,\ldots,s \tag{32}$$
$$b = [\beta^1\ \beta^2 \ldots . \beta^r]^T$$
$$\beta^i = \left[b^i_{10}\ b^i_{11} \ldots . b^i_{1M-1}\ b^i_{1M-1} \ldots . b^i_{2M-1} \ldots b^i_{2^k 0} \ldots .\ b^i_{2^k M-1}\right]$$
$$i = 1,2,\ldots,r$$
(33)

$where\ a\ ,b\ are\ vectors\ of\ unknown\ parameters\ of\ dimensions\ sN\ x\ 1\ and\ rN\ x\ 1.$

To approximate the state equation via Chebyshev scaling functions equation (24) can be integrated as

$$x(t) - x_o = \int_0^t Ax(\tau)d\tau + \int_0^t Bu(\tau)d\tau \tag{34}$$

Initial Condition

The initial condition vector x_o can be expressed via Chebyshev scaling function as

$$x_o = \frac{\sqrt{\pi/2}}{2^{k/2}}(I_s \otimes \Phi^T(t))[\alpha_0^1\ \alpha_0^2 \ldots . \alpha_0^s]$$
$$= \frac{\sqrt{\pi/2}}{2^{k/2}}\left(I_s \otimes \Phi^T(t)\right)g_o \qquad (3.37) \tag{35}$$

$where\ g_o = [\alpha_0^1\ \alpha_0^2 \ldots . \alpha_0^s]\ and\ \alpha_0^i$
$= [x_i(0)\ 0\ \ 0 \ldots 0\ \ \ x_i(0)\ 0\ \ 0 \ldots 0 \ldots\ x_i(0)\ 0\ \ 0 \ldots 0]$

$We\ multiply\ Eq.(\ 35\ )by\ factor,$

$$\delta = \frac{\sqrt{\frac{\pi}{2}}}{2^{\frac{k}{2}}}$$

$because\ from\ Eq.(\ 2\ )we\ can\ obtained$

$$\phi_{n0} = \frac{2^{k/2}}{\sqrt{\pi/2}}$$

By substituting Eq. (28), (29) *and* (35) *into* (34) *and using the operational matrix, we get*

$$\left(I_s \otimes \phi^T(t)\right)a - \left(I_s \otimes \phi^T(t)\right)g_0\delta = A(I_s \otimes \phi^T(t)P^T)a + B(I_r \otimes \phi^T(t)P^T)b \tag{36}$$

Using Kronecker product properties [6] we have

$$\left(I_s \otimes \phi^T(t)\right)a = \left(I_s \otimes \phi^T(t)\right)\left(A \otimes P^T(t)\right)a + \left(I_s \otimes \phi^T(t)\right)\left(B \otimes P^T(t)\right)b + \left(I_s \otimes \phi^T(t)\right)g_0\delta \quad (3.39) \tag{37}$$

By equating the coefficients of $\left(I_s \otimes \phi^T(t)\right)$, we get

$$\left((A \otimes P^T) - I_{Ns}\right)a + \left(B \otimes P^T(t)\right)b + g_0\delta = 0 \tag{38}$$

or

$$\left[(A \otimes P^T) - I_{Ns} \quad \left(B \otimes P^T(t)\right)\right]\begin{bmatrix} a \\ b \end{bmatrix} = -g_0\delta \tag{39}$$

where I_{Ns} *is Ns x Ns identity matrix.*

Performance Index Approximation

Then, substituting (28) and (29) into (19) to get

$$J = \int_0^1 (a^T(\left(I_s \otimes \Phi(t)\right)Q\left(I_s \otimes \phi^T(t)\right)a + b^T\left(I_r \otimes \Phi(t)\right)R\left(I_r \otimes \phi^T(t)\right)b)dt \tag{40}$$

Then, to simplify it as

$$J = \int_0^1 (a^T\,(Q \otimes \Phi(t)\phi^T)a + b^T(R \otimes \Phi(t)\phi^T)b)dt \tag{41}$$

The orthogonality of Chebyshev scaling functions is shown as in Lemma1:

$$\int_0^1 \Phi(t)\Phi^T(t)dt = RR$$

Then

$$J = a^T(Q \otimes RR)a + b^T(R \otimes RR)b \tag{42}$$

Finally J can be written as

$$J = [\, a^T \quad b^T \,] \begin{bmatrix} Q \otimes RR & 0_{N_s\, x\, N_r} \\ 0_{N_r\, x\, N_s} & R \otimes RR \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \tag{43}$$

Continuity of the State Variables

To insure the continuity of the state variables between the different sections, constraints are added. There are $2^k - 1$ points at which the continuity of the state variables has to be ensured [7].

These points are:

$$t_i = \frac{i}{2^k} \qquad\qquad i = 1,2, \ldots., 2^k - 1 \tag{44}$$

So, there are $(2^k - 1)s \;\; equality\ constraints\ given\ by$:

$$\left(I_s \otimes \Phi'(t)\right)a = 0_{(2^k-1)s\, x\, 1} \tag{45}$$

Where

$$\Phi' = \begin{bmatrix} \phi_{1m-1}(t_1) & -\phi_{2m-1}(t_1) & 0 & 0 & 0 & \cdots & 0 \\ 0 & \phi_{2m-1}(t_2) & -\phi_{3m-1}(t_2) & 0 & 0 & \cdots & 0 \\ 0 & 0 & \phi_{3m}(t_3) & -\phi_{4m}(t_3) & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \phi_{(2^k-1)\mathrm{m}}\left(t_{2^k-1}\right) & -\phi_{(2^k-1)\mathrm{m}}\left(t_{2^k-1}\right) \end{bmatrix} \tag{46}$$

$\Phi' is\ (2^k - 1)\ x\ (2^k\mathrm{M})$ matrix.

Quadratic Optimal Control Transformation

By combining the equality constraints (39) with those in (45) we have

$$\begin{bmatrix} (A \otimes P^T) - I_{Ns} & (B \otimes P^T) \\ \left(I_s \otimes \Phi'\right) & 0_{(2^k-1)s\, x\, Nr} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -g_0\delta \\ 0_{(2^k-1)s\, x\, 1} \end{bmatrix} \tag{47}$$

From (43) and (47), the optimal control problem is transformed into the following quadratic programming problem

$$\min_{\mathrm{z}} z^T H z \tag{48}$$

Subject to equality constraints

$$Fz = h \tag{49}$$

$where$

$$z^T = [\, a^T \quad b^T] \tag{50}$$

$$H = \begin{bmatrix} Q \otimes RR & 0_{N_s\, x\, N_r} \\ 0_{N_r\, x\, N_s} & R \otimes RR \end{bmatrix} \tag{51}$$

$$\mathrm{F} = \begin{bmatrix} (A \otimes P^T) - I_{Ns} & (B \otimes P^T) \\ \left(I_s \otimes \Phi'\right) & 0_{(2^k-1)s\, x\, Nr} \end{bmatrix} \tag{52}$$

$$\mathrm{h} = \begin{bmatrix} -g_0\delta \\ 0_{(2^k-1)s\, x\, 1} \end{bmatrix} \tag{53}$$

Numerical Example 1

Problem Treated by Feldbaum

Find the optimal control $u^*(t)$ which minimizes

$$J = \frac{1}{2}\int_0^1 (x^2 + u^2)dt$$

subject to

$$\dot{x} = -x + u \quad , \quad x(0) = 1$$

We solved this problem when $k = 1\, , and\ M = 3\, , so\ N = 6$

Then we approximate the state and control variables as

$$x(t) = \textstyle\sum_{n=1}^{2} \sum_{m=0}^{2} a_{nm}\phi_{nm}(t) \tag{54}$$

$$\mathrm{u}(t) = \textstyle\sum_{n=1}^{2} \sum_{m=0}^{2} \mathrm{b}_{nm}\phi_{nm}(t) \tag{55}$$

For this problem

Chebyshev scaling functions for this problem are for k=1, M=3

$$\left.\begin{aligned} \psi_{10}(t) &= \frac{2}{\sqrt{\pi}} \\ \psi_{11}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}(4t-1) \\ \psi_{12}(t) &= \frac{2\sqrt{2}}{\sqrt{\pi}}(2(4t-1)^2-1) \end{aligned}\right\} \tag{56}$$

$$\left.\begin{aligned} \psi_{20}(t) &= \tfrac{2}{\sqrt{\pi}} \\ \psi_{21}(t) &= \tfrac{2\sqrt{2}}{\sqrt{\pi}}(4t-3) \\ \psi_{22}(t) &= \tfrac{2\sqrt{2}}{\sqrt{\pi}}(2(4t-3)^2-1 \end{aligned}\right\} \quad (57)$$

$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \psi_{12}(t), \psi_{20}(t), \psi_{21}(t), \psi_{22}(t)]^T$

$a = [a_{10}(t), a_{11}(t), a_{12}(t), a_{20}(t), a_{21}(t), a_{22}(t)]$

$b = [b_{10}(t), b_{11}(t), b_{12}(t), b_{20}(t), b_{21}(t), b_{22}(t)]$ (58)

$\delta = \frac{\sqrt{\pi/2}}{2^{k/2}} = \frac{\sqrt{\pi/2}}{2}$ (59)

$g_0 = [1\ 0\ 0\ 1\ 0\ 0]$

There are $2^k - 1 = 1$ point.

This point is:

$t_1 = \frac{1}{2^k} = 0.5$ $\qquad i = 1$

So there are $(2^k - 1)s = 1$ $equality\ constraint\ given\ by:$

$(I_s \otimes \Phi'(t))a = 0_{(2^k-1)s\ x\ 1}$

$\Phi' is$ $(2^k - 1)$ x $(2^k$(M)), then $[\Phi']_{1x6}$ matrix

$\Phi' = [\psi_{10}(0.5), \psi_{11}(0.5), \psi_{12}(0.5), -\psi_{20}(0.5), -\psi_{21}(0.5), -\psi_{22}(0.5)]$

Φ = [1.1284 1.5958 1.5958 -1.1284 +1.5958 -1.5958]

By solving the corresponding quadratic programming problem, the optimal value of performance index is obtained. J = 0.193001037554299 for k=1 and M=3.

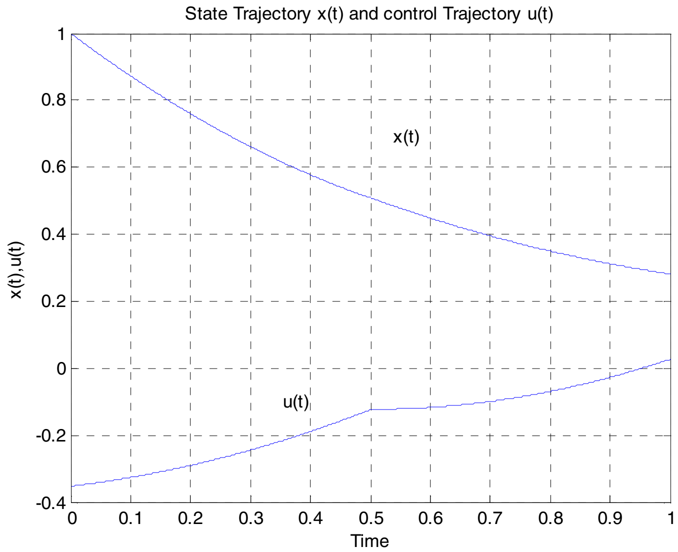

Figure 1: Optimal state and control trajectories $x(t)$ and $u(t)$ $k = 1, M = 3$ $(J = 0.192915719226705)$

$for\ k = 2\ , M = 3$

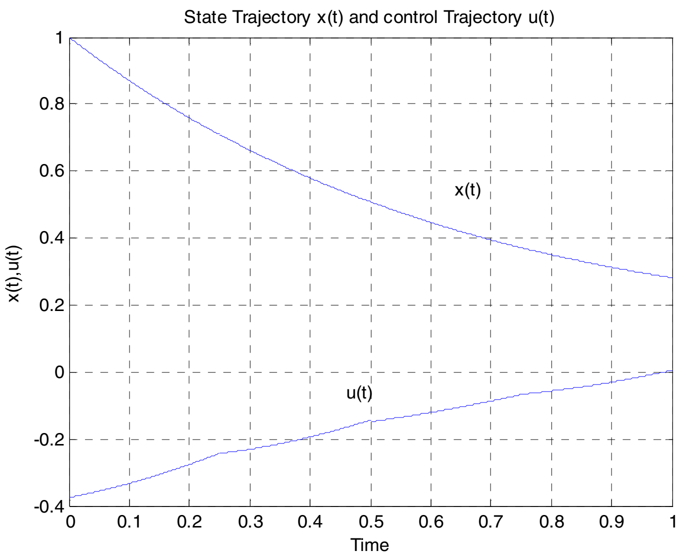

Figure 2: Optimal state and control trajectories $x(t)$ and $u(t)$ $k = 2, M = 3$ $(J = 0.192909783507572)$

$for\ k = 3\ , M = 3$

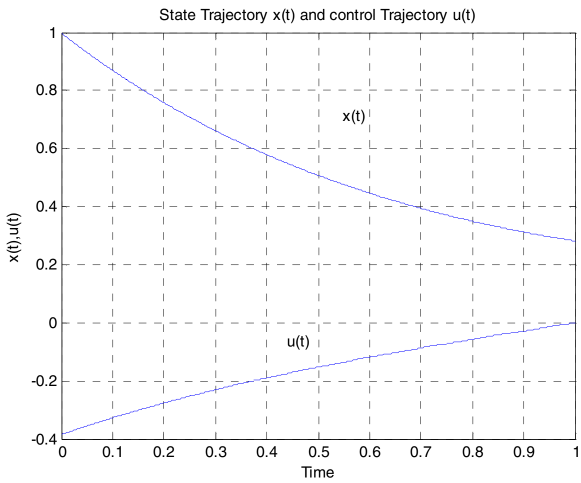

Figure 3: Optimal state and control trajectories x (t) & u(t), k=3, M=3

Table 1: Performance index results

	K=1 M=3	K=2 M=3
J	0.1930010375	0.1929157192
	K=3 M=3	K=3 M=4
J	0.1929097835	0.1929093208
	EXACT VALU	
J	0.1929092981	

Table (1) shows that by increasing k or M, the performance index, J, moves closer to the exact value. Figures (1-3) show plots of the OCP trajectories convergence rate increased as the values of K and M increased.

Numerical Example 2

Find an optimal controller $u(t)$ that minimizes the following performance index

$$J = \frac{1}{2}\int_0^1 (x_1^2 + x_2^2 + 0.005u^2)dt$$

subject to

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0$$
$$\dot{x}_2 = -x_2 + u \qquad x_2(0) = -1$$

The proposed method is applier to this example. The obtained solution is

$$k = 3\,, and\ M = 5 \qquad J = 0.0694046775616713$$
$$k = 3\,, and\ M = 6 \qquad J = 0.0693859107633072$$

By solving the corresponding quadratic programming problem, the obtained optimal value of performance index, J = 0.0693859107633072.

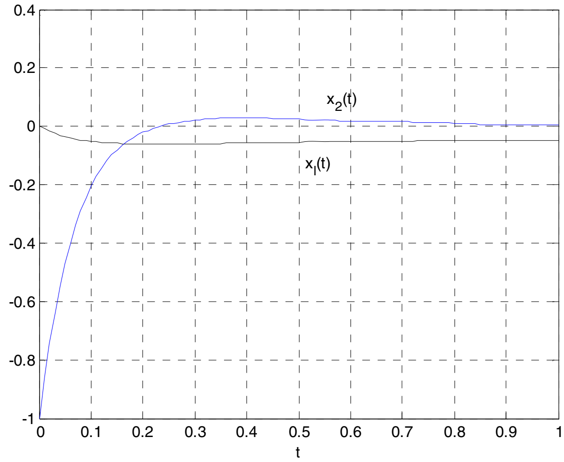

Figure 4: Optimal state trajectories, $x_1(t)$ & $x_2(t)$

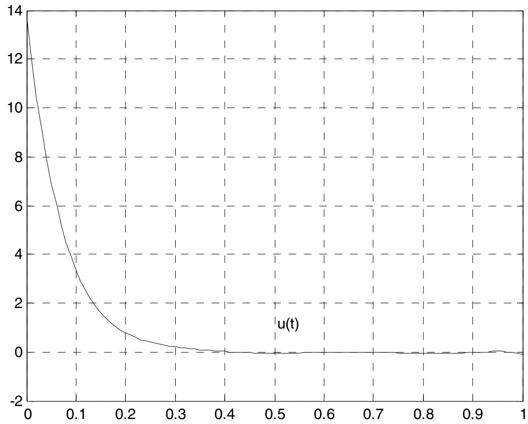

Figure 5: Optimal control trajectory u(t)

Table (2) Comparison between different researches for (J) value

Research	J	Deviation error
Exact value	0.06936094	0
Hsieh [36]	0.0702	8.4×10^{-4}
Neuman and Sen [31]	0.06989	5.3×10^{-4}
Vlassenbroeck [41]	0.069368	7.1×10^{-6}
Jaddu [2]	0.0693689	7.96×10^{-6}
Majdalawi [22]	0.0693668896	7.9562×10^{-6}
This research	0.0693859107	2.49×10^{-5}

Here, a numerical method is proposed for solving linear time in-variant quadratic optimal control problems. A Chebyshev wavelet is used to approximate controls and states of the system using a finite length of Chebyshev wavelet.

Two examples are solved to demonstrate the effectiveness of the proposed method; the first example contains one state and the second example contains two states. A comparison with other researches is performed. This research gives better or comparable results in comparison with others.

The difficult linear quadratic optimal control problem is converted into a quadratic programming problem which was easily solved using MATLAB.

Conclusion

In this paper, a numerical methods to solve optimal control problems for linear time invariant systems was proposed. This method was based on parameterizing the system state and control variables using a finite length Chebyshev wavelet. The aim of the proposed method is the determination of the optimal control and state vector by a direct method of solution based upon Chebyshev wavelet.

An explicit formula for the performance index was presented. In addition, Chebyshev wavelet operational matrix of integration was presented and used to approximate the solution. A product operational matrix of Chebyshev wavelets was also presented and used to solve linear time-varying systems. Thus, the solution of the linear optimal control problem is reduced to a simple matrix-vector multiplication that can be solved easily using MATLAB.

Numerical examples were solved to show the effectiveness and efficiency of the proposed method. The proposed method gave better or comparable results compared to other research.

Future work can deal with using Chebyshev wavelet to solve nonlinear and time varying optimal control problems.

References

[1] D. Kirk, *Optimal Control Theory: An Introduction*, Prentice-Hall, Englewood Cliffs, NJ, 1970.

[2] H. Jaddu, "Numerical methods for solving optimal control problems using Chebyshev polynomials," PHD Thesis, JAIST, Japan, 1998.

[3] M. Tavallaei and B. Tousi, "Closed form solution to an optimal control problem by orthogonal polynomial expansion," American J. of Engineering and Applied Sciences, Vol. 1, no 2, 104-109, 2008.

[4] E. Babolian and F. Fattahzadeh, "Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration," Applied Mathematics and Computation, Vol. 188, no. 1, 417–426, 2007.

[5] M. Ghasemi and M. Tavassoli Kajani, "Numerical solution of time-varying delay systems by Chebyshev wavelets," Applied Mathematical Modelling, Vol. 35, no. 11, 5235–5244, 2011.

[6] J. W. Brewer, "Kronecker product and matrix calculus in system theory," IEEE Transactions on Circuits and Systems, Vol. 25, No. 9, 1978.

[7] H. Jaddu and M. Vlach, "Wavelets-based approach to optimizing linear systems," Proceeding on Control and Applications Conference, CA 2002, Cancun, Mexico, 344-354, May 20-22, 2002.