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Tracking Solution by using Optimal Control

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Abstract

The PI, PD and PID controllers are widely used and successfully applied controllers to many applications. Their successful application, good performance, easiness of tuning are sufficient rational for their use, although their structure is justifiedby heuristics. In this paper by the use of optimal control theory we formulate a tracking problem and show those cases when their solution gives the PI, PD and PID controllers, thus avoiding heuristics and giving a systematic approach to explanation for excellent performance. It is shown that the PI controller is optimal for first order system, the PID controlleris optimal for a 2nd order system with nonzero.

Keywords: PID controller, tracking problem, optimal control.

1. Introduction

The PD, PI and PID controllers are successfully applied controllers to many applications, almost from the beginning of controls applications [1,2] . The facts of their successful application, good performance, and easiness of tuning are speaking for themselves and are sufficient rational for their use, although their structure is justified by heuristics:

"The PID controller constitutes the heuristic approach to controller design that has found wide acceptance in the process industries."[2].

In this paper we state a problem whose solution leads to PID controller structure, thus avoiding heuristics and giving a systematic approach to explanation for the excellent performance of the PID controllers. Namely, by the use of LQR theory we

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formulate control-tracking problems and show those cases when their solution gives the PID family of controllers. The novelty of results in this paper is that shows for what problems thePID controller are the optimal controller and for which they are not.

The importance of this result is:

1) From theoretical point of view it is important to know that widely used control architecture can be derived from optimal control problem.
2) The solution shows for what kind of systems PID controller are optimal. It is shown that the PI controller is optimal for first order system; the PID controller is optimal for 2nd order systems with no zero. The reference trajectory is generated by system identical to plant. The differences are the initial conditions and the input to reference trajectory generator. The tracking error is the position error, and zero steady state is imposed by integral action on the tracking error.

This is reason that the PID controllers are so well performing in servo application and chemical processes, as there are of this type.

2. Optimal tracking Problem

We assume the n-th order multi input multi-output system

$$x = A\dot{x} + Bu \qquad y = Cx \tag{1}$$

Where x$\in R^n$is the state; u $\in R^m$is the input and y $\in R^p$is the measured output; A$\in R^{n\times n}$, B$\in R^{n\times m}$and C $\in R^{p\times n}$. The reference trajectory generator is

$\dot{x}_r = A_r x_r$

$$y_r = C_r x_r \tag{2}$$

Where $x_r \in R^v$is the state; $u_r \in R^{\mu}$isthe input and y$\in R^p$is the referenceoutput; A$\in R^{v\times v}$, B$\in R^{v\times \mu}$and C $\in R^{p\times v}$.

The integral action is introduced into the control in order to force zero tracking errors for polynomial inputs, and to attenuate disturbances. This is done by introducing the auxiliary variable

$$\dot{\eta} = e = y - y_r \tag{3}$$

The control objective is

$$J = 1/2\{\left(y(t_f) - y_r(t_f)\right))^T G_1 \left(y(t_f) - y_r(t_f)\right)\eta(t_f)^T G_2 \eta(t_f) + \int[(y(t) - y_r(t))^T Q_1 \left(y(t) - y_r(t)\right)\eta(t)^T Q_2 \eta(t) + u(t)^T R u(t)]dt\} \tag{4}$$

The optimal tracking problem is to find an admissible input u(t) such that the tracking objective (4) is minimized subject to dynamic constraints (1,2,4). All vectors and metrics are of the proper dimensions.

3. Solution of the Tracking Problem

In order to solve the optimal Tracking Problem we augment the state variable to the form

$$\bar{x} = \begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} A & 0 & 0 \\ C & 0 & -C_r \\ 0 & 0 & A_r \end{bmatrix} \tag{5}$$

$$\bar{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C} = [C\ 0\ -C_r]$$

Then the problem is minimization of (4) subject to (1,2) is problem ofminimization of

$$J = 1/2\{x(t_f)^T G x(t_f) + \int [\, x(t)^T Q x(t) + u(t)^T R u(t)] dt\}$$

$$P_{22} C_r + P_{23} A_r - P_{12} B R^{-1} B^T P_{13} = 0 \tag{6}$$

Subject to

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \tag{7}$$

Where

$$Q = \bar{C}^T Q_1 \bar{C} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} Q_2 [0\ 1\ 0] \tag{8}$$

$$G = \bar{C}^T G_1 \bar{C} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} G_2 [0\ 1\ 0]$$

The solution is [3]

$$u = -R^{-1} \bar{B} P \bar{x} \tag{9}$$

$$-\dot{P} = P\bar{A} + \bar{A}^T P + Q - P\bar{B}R^{-1}\bar{B}^T P\ , P(t_f) = G$$

If we write P = {[P_{ij}];i, j = 1,2,3}, ijthen

$$u = -R^{-1}[B^T P_{11} B^T P_{12} B^T P_{13}]\bar{x} \tag{10}$$

Where in steady state we have

$$P_{11}A - P_{12}C + A^T P_{11} - C^T P_{12} + C^T Q_1 C - P_{11} B R^{-1} B^T P_{11} = 0 \quad (11.1)$$

$$A^T P_{12} - C^T P_{22} - P_{11} B R^{-1} B^T P_{12} = 0 \quad (11.2)$$

$$P_{12} C_r + P_{12} A_r + A^T P_{12} - C^T P_{23} - C^T Q_1 C - P_{11} B R^{-1} B^T P_{13} = 0 \quad (11.3)$$

$$Q_2 - P_{12} B R^{-1} B^T P_{12} = 0 \quad (11.4)$$

$$P_{23} C_r + P_{33} A_r + C_r^T P_{23} + A_r{}^T P_{33} - C_r{}^T Q_1 C_r - P_{13} B R^{-1} B^T P_{13} = 0 \quad (11.5)$$

4. First-order System

If $A = A_r = 0, C = C_r = 1, B = 1$ then

from (11)

$$-P_{12} - P_{12} + Q_1 - P_{11} R^{-1} P_{11} = 0 \quad (12.1)$$

$$-P_{22} - P_{11} R^{-1} P_{12} = 0 \quad (12.2)$$

$$P_{12} - P_{23} - Q_1 - P_{11} R^{-1} P_{13} = 0 \quad (12.3)$$

$$Q_2 - P_{12} R^{-1} P_{13} = 0 \quad (12.4)$$

$$P_{22} - P_{12} R^{-1} P_{13} = 0 \quad (12.5)$$

$$P_{23} + P_{23} + Q_1 + P_{13} R^{-1} P_{13} = 0 \quad (12.6)$$

from (12.2)

$$P_{12} = -\sqrt{Q_2 R} \quad (13.1)$$

from (12.1) and (13.1)

$$P_{11} = -\sqrt{R(Q_1 - 2P_{12})} = R\sqrt{\frac{Q_1}{R} + 2\sqrt{\frac{Q_2}{R}}}(\quad 13.2)$$

from(12.2)

$$P_{22} = -\frac{P_{12} P_{11}}{R} \quad (13.3)$$

from(12.5) and (13.3)

$$P_{13} = -\frac{P_{22} R}{P_{12}} = -P_{11} \quad (13.4)$$

This means, by the use of (3), that

$$u = -\frac{1}{R}[P_{11}P_{12} - P_{11}]\begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix} = [k_1 k_2 \ -k_1]\begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix} = k_1(x - x_r) + k_2\eta = k_1 e + k_2 \int edt \tag{14}$$

This is PI controller.

5. Second Order System With no Zero

Now for general case $A = A_r$, $C = C_r = I$ and $B = B_r$ i.e the reference trajectory generator is identical to the plant, it can be shown that $P_{13} = -P_{11}$.

Then

$$u = -R^{-1}[B^T P_{11}\ B^T P_{12}\ B^T P_{13}]\begin{bmatrix} x \\ \eta \\ x_r \end{bmatrix} = K_1 e + K_2 \int edt \tag{15}$$

Where

$$K_1 = -R^{-1}B^T P_{11} K_2 = R^{-1}B^T P_{12} \tag{16}$$

$$A = A_r = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ b_2 \end{bmatrix} \tag{17}$$

$$C = C_r = [1\ 0]$$

$$H(s) = H_r(s) = \frac{b_2}{s^2 + a_1 s + a_2}$$

The state of the plant and trajectory generator is denoted $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} x_{1r} \\ x_2 \end{bmatrix}$ respectively. We want to force zero steady state tracking error on the output. Here $y = x_1$, $y_r = x_{1r}$ and $y = x_2$, $y_r = x_{2r}$.

Then, since the tracking error is $e = y - y_r$ we have

$$\eta = e = y - y_r \tag{18}$$

$$u = [k_1 k_2 k_3 \ -k_1 k_2]\begin{bmatrix} y \\ \dot{y} \\ \eta \\ y_r \\ \dot{y}_r \end{bmatrix} = k_1 e + k_2 \dot{e} + k_3 \int edt \tag{19}$$

$$[k_1 k_2 k_3 \ -k_1 k_2]\begin{bmatrix} y \\ \dot{y} \\ \eta \\ y_r \\ \dot{y}_r \end{bmatrix} = k_1 e + k_2 \dot{e} + k_3 \int edt$$

That is, we get PID controller.

6. Conclusions

By the use of linear quadratictracking theory we formulated acontrol-tracking problem and showedthose cases when their solution givesthe PID family of controllers. Thisway we avoided heuristics and gavea systematic approach to explanationfor good performance of the PIDcontrollers. The PID controllerarchitecture is optimal for linearquadratic tracking problem of a 2nd order systems with no zero. Thereference trajectory is generated by asystem identical to the plant.

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