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Estimation of Variance and Skewness of Non-Gaussian Zero mean Color Noise from Measurements of the Atomic Transition Probabilities

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Abstract

When the system under consideration is of the order of the atomic level and the Noise is filtered out to a particular frequency like red or brown or, a range of frequencies where the filter may be linear or, non-linear then it is imperative to analyze the higher order spectra to satisfactorily estimating its power spectrum and bispectrum. In this paper, we propose a new approach in the estimation of Power spectrum and Bispectrum, from the measurement of atomic transition probabilities. Simulated results and comparison between the theoretical values of (Variance) Power Spectrum and (Skewness) Bispectrum and the experimental results have been presented here.

Keywords: Non-Gaussian Noise, Hamiltonian, Skewness, Transition Probability, Perturbation Theory, Eigenstates, Random signal, Normal Distribution, Bispectrum, Power Spectrum, Color Noise.

1. Introduction

As the second-order statistics (correlation) are phase blind, the higher-order statistics (spectra) also known as cumulants, and their corresponding Fourier Transforms, known as polyspectra helps to pour out amplitude as well as phase information of most of the real-world applications which are non-Gaussian in nature.

In nature, most quantum phenomena are governed by time-dependent Hamiltonians [4]. To study the structure of molecular and atomic systems, we need to know how electromagnetic radiation interacts with these systems. Molecular and atomic spectroscopy [3, 5] deals in essence with the absorption and emission of electromagnetic radiation by molecules and atoms. As a system absorbs or emits radiation, it undergoes transitions from one state to another. In order to treat the transitions of quantum systems from one energy level to another, Time-dependent perturbation theory [2,7] is the most useful tool.

A system under the presence of Non-Gaussian zero mean color Noise having Variance, σ^2 and Skewness, γ is considered –

$$H(t) = H_0 + \epsilon N(t)V(t) \tag{1}$$

where H_0 = time-independent Hamiltonian of the system,
$V(t)$ = time-dependent interaction potential [2],
$N(t)$ =Third Order Non-Gaussian color Noise such that

$$\mathbb{E}\big(N(t_1)N(t_2)\big) = R_N(t_1 - t_2)$$
$$\mathbb{E}(N(t_1)N(t_2)N(t_3)) = C_N(t_1 - t_3, t_2 - t_3)$$

where $\mathbb{E}(.) = Expectation$

The system undergoes a transition from its initial state, ψ_i to the final state ψ_f due to the external excitation.

2. Problem At Hand

How does $V(t)$ affect the system in the presence of Noise

The Schrodinger Equation [1] is:

$$ih\frac{d|\psi(t)>}{dt} = (H_0 + \epsilon N(t)V(t))\,|\psi(t)>$$

where $V(t)$characterizes the interaction of the system with the external source of perturbation, non-Gaussian noise in this case. When the system interacts with $V(t)$, it either absorbs or emits energy. This process inevitably causes the system to undergo transitions from one unperturbed eigenstate to another. If the system is initially in an (unperturbed) eigenstate $|\psi_i>$ of H_0 then by applying time-dependent perturbation theory, we can find out the probability that the system will be found at a later time in another unperturbed eigenstate$|\psi_f>$.

Solution of time evolution equation and Dyson Series

The time evolution equation is:

$$|\psi(t) = U_I(t, t_i)\,|\,\psi(t_i)>_I$$

where the time evolution operator is given in the interaction picture by

$$U_I(t, t_i) = e^{itH_0/h}\,U(t, t_i)e^{-it_iH_0/h}$$

The solutions of this equation, with the initial condition

$$U_I(t, t_i) = I$$

It can be expanded up to third order approximation which is a special case of what is known as Dyson Series [2, 7].

3. Measurement of Transition Probabilities

The integral equation from the Dyson Series can be written as:

$$W(t) = I - i\epsilon \int_0^t N(t_1)V'(t_1)dt_1 - \epsilon^2 \int_{0<t_2<t_1<t} V'(t_1)V'(t_2)N(t_1)N(t_2)dt_1 dt_2 \quad (2)$$

where $V'(t) = U_0(-t)VU_0(t)$ and $O(\in^3)$ terms are neglected.

The system, when unperturbed, is described by a time-independent Hamiltonian H_0 whose solutions—the eigenvalues, E_n and eigenstates, $|\psi_n>$ —are known.

$$H_0|\psi_n> = E_n|\psi_n> \text{ where } n = 0,1,2,\ldots\ldots,N$$

For $m \neq n$, transition probability [2, 4] from $|m> \rightarrow |n>$ in time 't' is :

$$P_t(n|m) = <|<n|W(t)|m>|^2>$$

$$= \epsilon^2 \int R_N(t_1 - t_2) < n|V|m > e^{(iE(n,m)(t_1 - t_2))} dt_1 \, dt_2$$

$$+ 2\epsilon^3 \sum_p Im \int_{\substack{0<t_2<t_1<t,\\ 0<t_3<t}} \begin{matrix} (<m|V|n><n|V|p><p|V|m>) \\ C_N(t_1 - t_3, t_2 - t_3) \\ e^{\{i(-E(n,m)t_3 + E(n,p)t_1 + E(p,m)t_2)\}} \\ dt_1 \, dt_2 \, dt_3 \end{matrix}$$

$$= \epsilon^2 < n|V|m > t \int_{|\tau|\le t} \left(1 - \frac{|\tau|}{t}\right) R_N(\tau)\, e^{(iE(n,m)\tau)}\, d\tau + 2\epsilon^3 Im\left(\int \sum_p (<m|V|n><n|V|p><p|V|m>)\ \xi[t,n,m,p]\right)$$

Where $\xi[t,n,m,p] =$

$$\int_{\substack{0<t_2<t_1<t,\\ 0<t_3<t}} C_N(t_1 - t_3, t_2 - t_3)\ e^{\{i(-E(n,m)t_3 + E(n,p)t_1 + E(p,m)t_2)\}}\, dt_1\, dt_2\, dt_3$$

$$= \int_{\substack{0<t_3+\tau_2<t_3+\tau_1<t,\\ 0<t_3<t}} C_N(\tau_1, \tau_2)\ e^{\{i(-E(n,m)t_3 + E(n,p)\,(t_3+\tau_1) + E(p,m)(t_3+\tau_2)\}}\, d\tau_1\, d\tau_2\, d\tau_3$$

$$= \int_{\substack{-t_3<\tau_2<\tau_1<t-t_3,\\ 0<t_3<t}} C_N(\tau_1, \tau_2)\ e^{\{i(E(n,p)\,\tau_1 + E(p,m)\tau_2)\}}\, d\tau_1\, d\tau_2\, d\tau_3$$

The region of integration is the same as

$-t < \tau_2 < t, -t < \tau_2 < \tau_1,$

$\max(0, -\tau_2) < t_3 < \min(t, t - \tau_1)$

$i.e. -t < \tau_2 < \tau_1 < t,$

$\max(0, -\tau_2) < t_3 < \min(t, t - \tau_1)$

The null region is when

$\max(0, -\tau_2) \nless \min(t, t - \tau_1)$

$i.e. when -\tau_2 > t - \tau_1\ or, \tau_1 - \tau_2 > t$

$For\ \tau_1 - \tau_2 < t, the\ region\ for\ t_3\ is:$

$0 < t_3 < t - \tau_1, if\ \tau_1 > 0, \tau_2 > 0,$

$-\tau_2 < t_3 < t, if\ \tau_1 < 0, \tau_2 < 0,$

$-\tau_2 < t_3 < t - \tau_1, if\ \tau_1 > 0, \tau_2 < 0,$

$$0 < t_3 < t, if\ \tau_1 < 0, \tau_2 > 0,$$
$$(does\ not\ occur\ since\ \tau_2 < \tau_1)$$

Thus,

$$\xi[t, n, m, p] = \int_{-t<\tau_2<\tau_1<t} f_t(\tau_1, \tau_2) C_N(\tau_1, \tau_2)\ e^{\{-i(E(m,p)\,\tau_2 + E(n,p)\tau_1)\}}\, d\tau_1\, d\tau_2$$

where $f_t(\tau_1, \tau_2) = \begin{cases} t - \tau_1\ if\ \tau_2 > 0, \\ t + \tau_2\ if\ \tau_1 < 0, \\ t - \tau_1 + \tau_2\ if\ \tau_1 > 0, \tau_2 < 0, \end{cases}$

In particular, as $t \to \infty, \xi[t, n, m, p]$

$$\approx t \int_{-t<\tau_2<\tau_1<t} C_N(\tau_1, \tau_2)\ e^{\{-i(E(m,p)\,\tau_2 + E(p,n)\tau_1)\}}\, d\tau_1\, d\tau_2$$

$$Hence, as\ t \to \infty,$$

$$\frac{dP_t(n|m)}{dt} \approx$$
$$\epsilon^2 \langle n|V|m\rangle\, S_N(E(n,m)) +$$
$$2\epsilon^3 Im\{\langle m|V|n\rangle \sum_p \langle n|V|p\rangle\, \langle p|V|m\rangle\, B_N(E(p,n), E(m,p))\} + O(\epsilon^4)$$

$$where\ S_N(\omega) = \int_{-\infty}^{\infty} R_N(\tau)\ e^{(-i\omega\tau)}\, d\tau\ and, B_N(\omega_1, \omega_2)$$
$$= \int_{-\infty<\tau_2<\tau_1<\infty} C_N(\tau_1, \tau_2)\ e^{\{-i(\omega_1\tau_1 + \omega_2\tau_2)\}}\, d\tau_1\, d\tau_2$$

$$R_N(\tau) = < N(t)N(t + \tau) >,$$

$$C_N(\tau_1, \tau_2) = < N(t + \tau_1)N(t + \tau_2)N(t) >$$

4. Simulation And Results

Simulation work has been accomplished with the help of MATLAB. Non-Gaussian White Noise has been modeled by applying a non-linear transformation on the random normal distributed data.

$$N(t) = w^2(t) - 1 \quad (1)$$

where $w(t)$ = random variable having normal distribution.

Theoretical value of Mean,Varinace and Skewness are: $\mathbb{E}(N(t)) = 0$, $\mathbb{E}(N^2(t)) = 2$ and, $\mathbb{E}(N^3(t)) = 8$

Noise Characteristics like PDF, Autocorrelation, Power Spectral Density and Bispectrum have been plotted in MATLAB. Hamiltonian of the unperturbed system has been taken as a Hermitian Matrix [6] of that of a Harmonic Oscillator [2].

$$H_0 = (n + 1/2)\hbar\omega$$

where $n = 0,1,2, \ldots\ldots, N$ and ω = angular frequency.

Interaction potential has been taken as a random complex Hermitian Matrix. Eigenvalues and Eigenvectors of Hamiltonian have been calculated in order to find the transition probabilities.

In order to generate the Color Noise, we have utilized AR model of the form:

$$y[n] = h[0]N[n] + h[1]N[n-1]$$

where N[n] is zero mean white Noise process which is Non-Gaussian in nature as stated in the equation-(1).

Here, we have assumed that coefficient of the filter is smaller in absolute value than 1, so that it is a wide –sense stationary random process and corresponds to a low pass system.

Power Spectrum in terms of filter coefficients comes out to be:

$$S_N(\omega) = \sigma^2|h[0] + h[1]e^{-j\omega}|^2$$

Similarly, Bispectrum :

$$B_\omega(\omega_1, \omega_2) = \gamma H(\omega_1)H(\omega_2)H^*(\omega_1 + \omega_2)$$

For different pairs of state and filter coefficients which satisfy the theoretical values and estimated values of Power Spectrum and Bispectrum, we have simulated the transition probability rate.

Noise Characteristics (Color Noise, its PDF, Autocorrelation, Power Spectral Density and Bispectrum) are shown below.

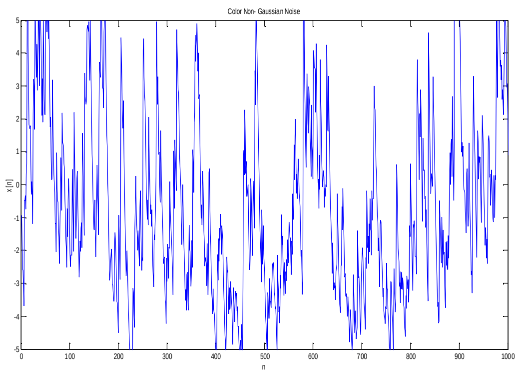

Fig. 1: - Non-Gaussian Red Noise

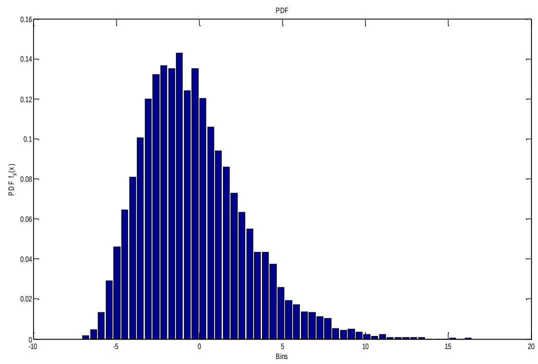

Fig. 2: - PDF

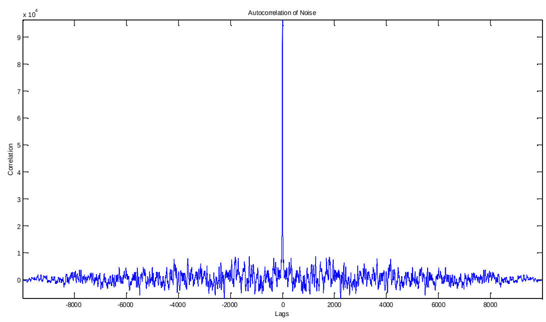

Fig. 3: Auto-Correlation

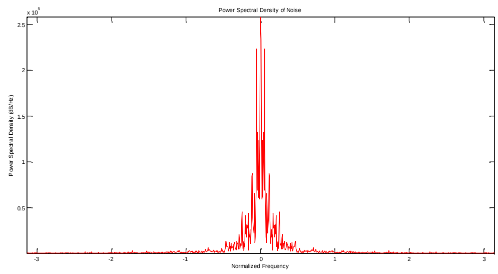

Fig. 4:- Power Spectral Density

Table I

Iteration	**Noise Parameters**			
	σ^2	**% Error**	**ϒ**	**% Error**
1	2.006	0.30	8.028	0.35
2	1.962	1.90	8.006	0.07
3	2.018	0.90	8.101	1.20
4	1.993	0.35	7.972	0.35
5	2.021	1.02	8.099	1.23

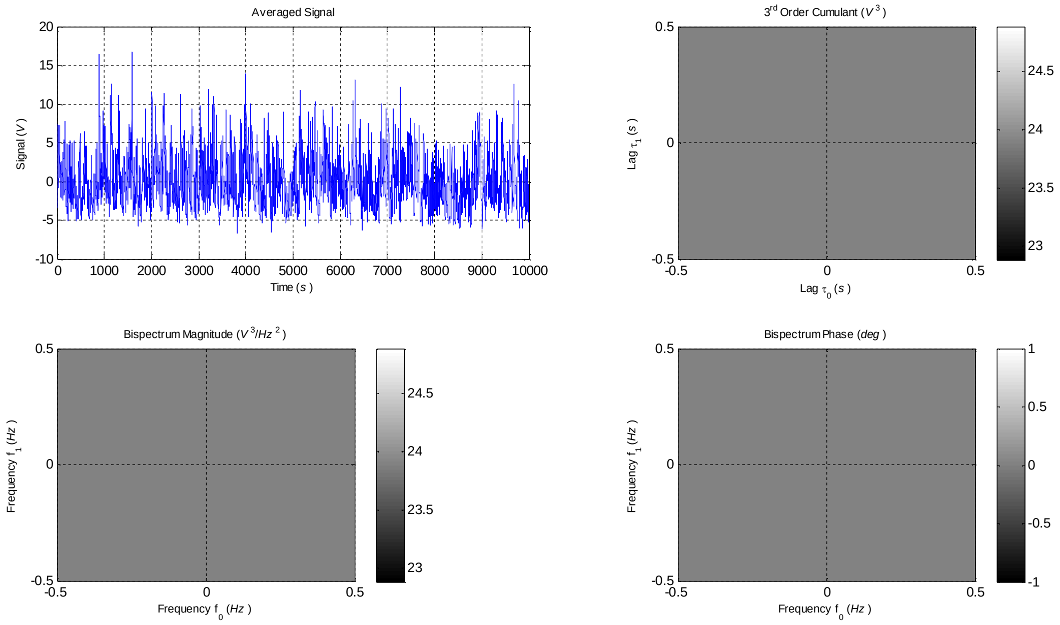

Fig. 4: Bispectrum

5. Conclusion

As suggested by the method in III, we achieved a low error rate (TABLE I) in estimating the Power Spectrum and Bispectrum of Non-Gaussian zero mean Color noise. The higher order spectra estimation helped in analyzing the Noise which is deviating from the Gaussian way which we find in the most of the real world systems of atomic and sub-atomic level.

In the computation of transition probabilities, two kinds of averages are involved. One, a quantum average,

$< |< n|W(t)|m >|^2 >$ which is expressed upto $O(\epsilon^3)$ as the sum of a linear function, a quadratic function and a cubic function of the Noise process, $\{N(t'): 0 \le t' \le t\}$. Two, a classical average of this transition probability over all noise ensembles. The classical average of the linear function yield zero since the noise has zero mean, the classical average of the quadratic function yields a sum over the noise spectral density evaluated at the Bohr Frequencies, E(n,m) and finally the classical average of the cubic function yields a sum over the bispectrum of the noise evaluated at Bohr Frequency pairs.

References

[1] Schrödinger, E. (1926). "An Undulatory Theory of the Mechanics of Atoms and Molecules". *Physical Review* **28**(6): 1049–1070. Bibcode:1926PhRv...28.1049S.doi:10.1103/PhysRev.28.1049

[2] Quantum Mechanics Concepts and Application, Nouredine Zettili, Jacksonville State University, Jacksonville, USA, 2nd Edition, John Wiley and Sons, 2009, pp 574-583.

[3] A. Ben Simon,Y. Paltiel,G. Jung,V. Berger,and H. Schneider, Measurements of non-Gaussian noise in quantum wells, PHYSICAL REVIEW B76, 235308 (2007)

[4] Steven Weinberg, The Quantum Theory of Fields, Volume-1 Foundations, University of Texas, Cambridge University Press (1995). ISBN – 0521550017.

[5] Jean-Philippe Aguilar and Nils Berglund, The effect of classical noise on a quantum two-level system, Citation: J. Math. Phys. 49, 102102 (2008); doi: 10.1063/1.2988180

[6] Athanasios Papoulis, S. Unnikrishna Pillai, Proability, Random Variables and Stochastic Processes,4th Edition, ISBN-10 – 0072817259.

[7] Prof. Michael G. Moore, Time-Independent Perturbation Theory, Phys 852, Quantum mechanics II, Spring 2009, Michigan State University.

[8] Petter Abrahamsen, Gaussian Random Fields and Correlation Functions, April 1997, Norwegian Computing Center, Norway.

[9] Nikias, C.L.; Raghuveer, Mysore R., "Bispectrum estimation: A digital signal processing framework," *Proceedings of the IEEE* , vol.75, no.7, pp.869,891, July 1987

[10] Leonard I.Schiff, Quantum Mechanics, 3rd Edition, Stanford University,McGraw-Hill Book Company.

[11] P.A.M. Dirac, The Principles of Quantum Mechanics, 4th Edition, University of Cambridge, Oxford University Press (1958)

[12] Jerry M.Mendel, Fellow, IEEE, Tutorial on Higher-Order Statistics (Spectra) in Signal Processing and System Theory: Theoretical Results and Some Applications, Proceedings of IEEE, Vol. 79, No. 3, March 1991.

[13] Jerry M. Mendel and Chrysostomos L. Nikias, Signal Processing with Higher-Order Spectra, IEEE Signal Processing Magazine, July 1993.

[14] T.Subba Rao and M.M.Gabr, An Introduction to Bispectral Analysis and Bilinear Time Series Models, Lecture Notes in Statistics. Springer-Verlag, New York, 1984. ISBN – 3-540-96039-2.