International Journal of Electronic and Electrical Engineering.
ISSN 0974-2174, Volume 7, Number 5 (2014), pp. 491-496
© International Research Publication House
http://www.irphouse.com

Fractional Order Digital Differentiator with Linear Phase and Low Absolute Error

Narendra Kumar[1] and D.K. Upadhyay[2]

[1] M.Tech, Dept. of Electronics of Communication Engineering, Netaji Subhas Institute of Technology, Sector-3, Dwarka, New Delhi 110075, India
[2]Dept. of Electronics of Communication Engineering, Netaji Subhas Institute of Technology, Sector-3, Dwarka, New Delhi 110075, India

Abstract

This paper is an attempt to design a fractional order digital differentiator s^r where r is a real number, using indirect discretization scheme having very low absolute error and linear phase characteristics. Rational approximation of s^r is done in Laplace domain using continued fraction expansion(CFE), to discretize the approximated function (i.e s to z transformation) digital differentiator designed using Genetic Algorithm (GA) optimization method, is used as an operator. The discretized approximated fractional order digital differentiator is minimum phase, stable and outperforms all the existing fractional order digital differentiator both in magnitude and phase response.
The simulation results are obtained using MATLAB scripts and the result are in close conformity with the ideal response in continuous time as the absolute error plot shows error less than -50db while $\omega < 0.8\pi$ and less than -35db when $\omega > 0.8\pi$

Keywords: Fractional order differentiator, Indirect discretization, Continued fraction expansion (CFE), Genetic algorithm(GA), Rational approximation, Minimum phase, Absolute error.

1. Introduction

Fractional calculus was introduced by Leibniz in 1695 around 300 [1] years ago but its application studied relatively little. Applying it to Dynamic/Automatic control system is recent interest due to wide spread of industrial use of controllers. A small improvement could have relevant impact on the industry. Earlier integer order model are in practice, our main objective to obtain Fractional-order controller (FOC) is to enhance system control performance.

For example in CRONE [2,3] controller the major ingredients are fractional order derivatives, CRONE being the French acronym of “Commande Robuste d’Ordre Non Entier” which means Robust Control of non-integer order, represent the first framework for non-integer order systems application in the automatic control area .

$PI^{\lambda}D^{\mu}$ [4] is another example of fractional order controllers here λ and μ are real number, where PID controllers are integer order model, taking λ=1 and μ=1 in $PI^{\lambda}D^{\mu}$ controller we get PID controller.

Fractional order derivatives and integrators have wide range of applications apart from FOC such as: transmission line theory, chemical analysis of aqueous solutions, design of heat-flux meters, rheology of soils, growth of intergranular grooves on metal surfaces, quantum mechanical calculations, and dissemination of atmospheric pollutants.

Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. This is the main advantage of fractional derivatives in comparison with classical integer-order models, in which such effects are in fact neglected. The advantages of fractional derivatives become apparent in modelling mechanical [5] and electrical properties of real materials, as well as in the description of rheological properties of rocks, and in many other fields.

All these fractional order controllers are infinite dimensional filters, for the band limited implementation of FOD’s and FOI’s finite dimensional approximation of the same is to be done with in the desired range of frequencies

The paper is about approximating the value of $s^{\pm a}$, $+a$ for differentiator and $-a$ for integrator and the value of $'a'$ varies from 0 to 1. s is the Laplacian operator. Numerical methods are used for rational approximating the FOD .Power series expansion (PSE) [6,7] and Continued fraction expansion (CFE) [8,9] are two popular methods for approximating the fractional order function (Section II). s to z transformation is done using discretization [10]. Direct and Indirect discretization are two methods of discretization, we are using the indirect discretization method Section (III). In Indirect discretization first the rational approximation in s domain takes place then discretization is done, where as in Direct discretization first discretization of the fractional order differentiator is done then the rational approximation in z domain takes place. Section (IV) shows the simulation results and Section (V) concludes the paper.

2. Rational Approximation

There are two approaches PSE and CFE, we are using CFE based indirect discretization. PSE leads to approximation in the form of polynomials, that the approximated fractional operator is in the form of FIR filter, which have only zeors.

Continued Fraction Expansion (CFE) of the generating function and then the approximated fractional operator is in the form of a IIR filter, which has poles and zeros. Due to poles in the approximated transfer function it converges more rapidly and have wider domain of convergence.

A.N Khovanskii [11] given CFE for $(1 + x)^a$ as given below

$$(1+x)^a := \cfrac{1}{1 - a * \cfrac{x}{1 + \left(\frac{1}{2}\right) * (a+1) * \cfrac{x}{1 - \left(\frac{1}{6}\right) * (a-1) * \cfrac{x}{1 + \left(\frac{1}{6}\right) * (a+2) * \cfrac{x}{1 - \left(\frac{1}{10}\right) * (a-2) * \cfrac{x}{1 + \ldots}}}}}} \quad (1)$$

Substituting $(1 + x) = s$ in the above CFE expression left hand side becomes s^a, that is the fractional order differentiator . The above series should converge and for band limited implementation of fractional order differentiator (FOD) it should be finite dimensional, therefore first ten terms of the equation (1) is used. The resultant approximated transfer function in s domain is given by

$$s^a := -\frac{\begin{array}{c} a^5x^5+15a^4x^5+30a^4x^4+85a^3x^5+420a^3x^4+225a^2x^5+ \\ 420a^3x^3+2130a^2x^4+274ax^5+5040a^2x^3+ \\ 4620ax^4+120x^5+3360a^2x^2+19740ax^3+3600x^4+30240ax^2+ \\ 25200x^3+15120ax+67200x^2+75600x+30240 \end{array}}{\begin{array}{c} a^5x^5-15a^4x^5-30a^4x^4+85a^3x^5+420a^3x^4-225a^2x^5+ \\ 420a^3x^3-2130a^2x^4+274ax^5-5040a^2x^3+ \\ 4620ax^4-120x^5-3360a^2x^2+19740ax^3-3600x^4+30240ax^2- \\ 25200x^3+15120ax-67200x^2-75600x-30240 \end{array}} \quad (2)$$

In the equation (2) for half order differentiator substituting the value of $a = 1/2$ we get

$$H_{half} := -\frac{\left(\frac{10395}{32}\right)s^5+\left(\frac{155925}{32}\right)s^4+\left(\frac{218295}{16}\right)s^3+\left(\frac{155925}{16}\right)s^2+\left(\frac{51975}{32}\right)s+\frac{945}{32}}{-\left(\frac{945}{32}\right)s^5-\left(\frac{51975}{32}\right)s^4-\left(\frac{155925}{16}\right)s^3-\left(\frac{218295}{16}\right)s^2-\left(\frac{155925}{32}\right)s-\frac{10395}{32}} \quad (3)$$

For different order we can substitute the value of a, order of the equation remains same the coefficient changes. For $a = 1/4$ the equation becomes

$$G_{1/4} := -\frac{\left(\frac{208845}{1024}\right)s^5+\left(\frac{3968055}{1024}\right)s^4+\left(\frac{6613425}{512}\right)s^3+\left(\frac{5595975}{512}\right)s^2+\left(\frac{2304225}{1024}\right)s+\frac{65835}{1024}}{-\left(\frac{65835}{1024}\right)s^5-\left(\frac{2304225}{1024}\right)s^4-\left(\frac{5595975}{512}\right)s^3-\left(\frac{6613425}{512}\right)s^2-\left(\frac{3968055}{1024}\right)s-\frac{208845}{1024}} \quad (4)$$

3. Discretization of Rationalized Transfer Function

The rationalized approximated transfer function in the previous section is discretized using operators that map s to z transformation, these operators are nothing but optimized integer order differentiators. In recent times to many optimization algorithms are used namely Linear Programming (LP) [12], Genetic Algorithms (GA) [13], Simulated Annealing (SA) [14], Pole Zero (PZ) optimization [15], Partical Swarn Optimization (PSO) [19] and many others.

A differentiator optimized using GA is used as an operator to map s to z transformation that has linear phase and magnitude response close to ideal one,

$$s = 1/T \frac{z^2-.4881z-.5107}{.8633(z^2+.6938z+0.05962)} \quad (5)$$

There are many operators on which significant work has been done such as Al-Alaoui 2-segment, Al-Alaoui 3-segment, Alaoui 4-segment,Schineder .Before using the above operator stability of the operator is to be checked both the poles and zeros should lie inside the unit circle. If any of te pole and zeros lie's outside the unit circle then we have to reflect these poles and zeros inside the unit circle by using the method suggested by Steigglitz.K [17] for stability and minimum phase .

The pole zero plot of the operator is given below

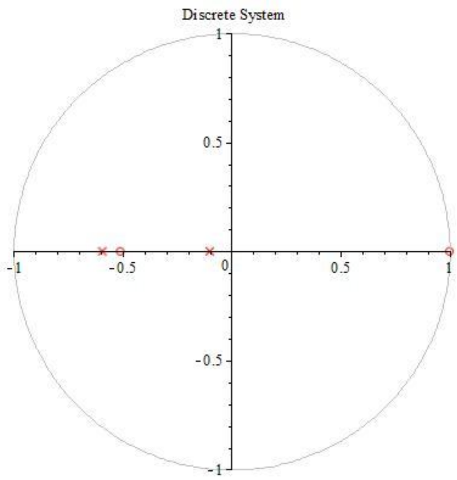
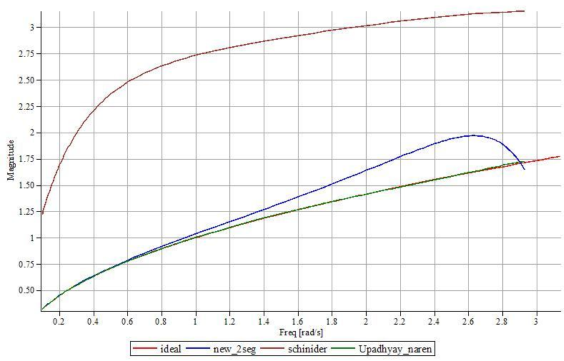


Fig. I

All poles and zeros lie inside the unit circle, therefore the operator is stable .For discretization the value of s in equation (5) is substituted in (3) gives half order digital differentiator with rationalized transfer function of 10th order.

$$G_{\frac{1}{2}}(z) := -\frac{96.45681698z+1.301600227+21886.21454z^{10}+2175.241017z^{9}-10342.17021z^{7}-37212.02691z^{8}+8576.983867z^{5}+20707.66517*z^{6}-2020.937030z^{3}-3691.494533z^{4}+67.7516181z^{2}}{-10.16411407z-1.689900535-20335.34394z^{10}-14038.27362z^{9}+22199.29047z^{7}+25267.15523z^{8}-9576.105266z^{5}-6420.819419z^{6}+1049.474892z^{3}-973.577162z^{4}+227.5417178z^{2}} \quad (6)$$

4. Simulation Results



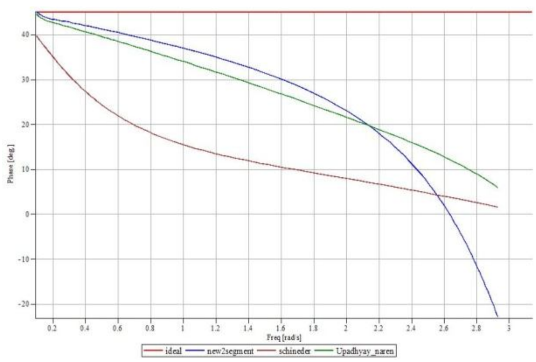


Fig. II **Fig. III**

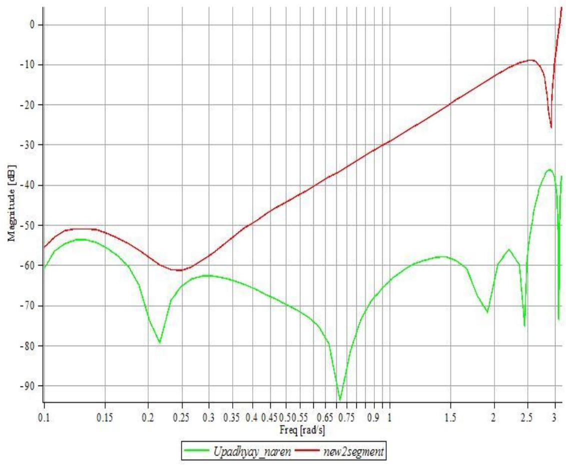


Fig. IV

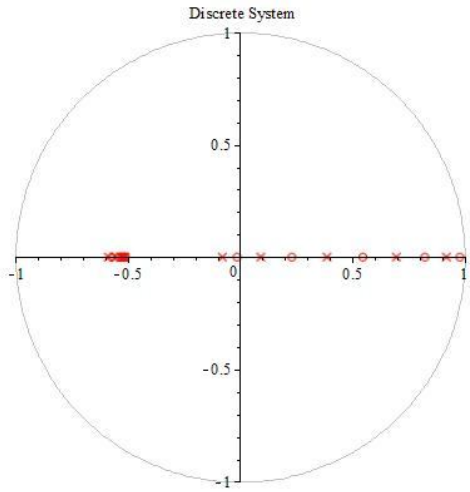


Fig. V

All the simulations and performed using MATLAB Figure II is the magnitude response of approximated 10^{th} order discrete half order differentiator, the response overlaps to the ideal one and outperforms all the existing differentiators. Figure III is the phase response and it linear with the frequency. Figure IV is the Absolute error plot and compare it's performance with the Al-Alaoui-2 segment operator.

5. Conclusion

It is clear from the magnitude, phase and Absolute error plot, the designed Upadhyay_naren ½ order differentiator is stable and minimum phase and outperforms all the differentiators. Absolute error plot shows error less than -50db while $\omega < 0.8\pi$ and less than -35db when $\omega > 0.8\pi$.

References

[1] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations. New York: Wiley, 1993.

[2] A. Oustaloup, B. Mathieu, and P. Lanusse, “The CRONE control of resonant plants: Application to a flexible transmission,” Eur. J. Contr., vol.1, no. 2, 1995.

[3] A. Oustaloup, X. Moreau, and M. Nouillant, “The CRONE suspension,” Control Eng. Pract., vol. 4, no. 8, pp. 1101–1108, 1996.

[4] I. Podlubny, “Fractional-order systems and PI D -controllers,” IEEE Trans. Automat. Contr., vol. 44, no. 1, pp. 208–214, Jan. 1999.

[5] N. M. F. Ferreira, F. B. Duarte, M. F. M. Lima, M. G. Marcos, and J. A. T. Machado, “Application of fractional calculus in the dynamical analysis and control of mechanical manipulators,” Fractional Calculus and Applied Analysis, vol. 11, pp. 91–113, 2008.

[6] Y. Ferdi, “Computation of fractional order derivative and integral via power series expansion and signal modelling,” Nonlinear Dynamics, vol. 46, no. 1-2, pp. 1–15, 2006.

[7] Y. Chen and B. M. Vinagre, “A new IIR-type digital fractional order differentiator,” Signal Processing, vol. 83, no. 11, pp. 2359–2365, 2003.

[8] G. S. Visweswaran, P. Varshney, and M. Gupta, “New approach to realize fractional power in z-domain at low frequency,” IEEE Transactions on Circuits and Systems II, vol. 58, no. 3, pp. 179–183, 2011.

[9] P. Varshney, M. Gupta, and G. S. Visweswaran, “Switched capacitor realizations of fractional-order differentiators and integrators based on an operator with improved performance,” Radioengineering, vol. 20, no. 1, pp. 340–348, 2011.

[10] Y. Q. Chen and K. L. Moore, “Discretization schemes for fractional-order differentiators and integrators,” IEEE Transactions on Circuits and Systems I, vol. 49, no. 3, pp. 363–367, 2002.

[11] A.N. Khovanskii, The Application of Continued Fractions and Their Generalizations to Problems in Approximation Theory (Transl. by Peter Wynn). P. Noordhoff Ltd. (1963).

[12] N. Papamarkos and C. Chamzas, “A new approach for the design of digital integrators,” IEEE Transactions on Circuits and Systems I, vol. 43, no. 9, pp. 785–791, 1996.

[13] C.-C. Hse, W.-Y. Wang, and C.-Y. Yu, “Genetic algorithm-derived digital integrators and their applications in discretization of continuous systems,” in Proceedings of the CEC Congress on Evolutionary Computation, pp. 443–448, Honolulu, Hawaii, USA, 2002

[14] M. A. Al-Alaoui, “Class of digital integrators and differentiators,” IET Signal Processing, vol. 5, no. 2, pp. 251–260, 2011.

[15] [15]D. K. Upadhyay and R. K. Singh, “Recursive wideband digital differentiator and integrator,”Electronics Letters, vol. 47, no. 11, pp. 647–648, 2011.

[16] J. Kennedy and R. Eberhart, “Particle swarm optimization,” in Proceedings of the IEEE International Conference on Neural Networks, pp. 1942–1948, December 1995.

[17] Steigglitz. K., “Computer – aided design of recursive digital filters”, IEEE Trans., AU-18, pp 123-129, 1970.