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Effect of Randomly Distributed Gravitational Dust on GPS Signals in a Gravitational Field

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Abstract

Conventional GPS (Global Positioning System) is based on linear propagation of photons. In conventional Global Positioning Systems, signals are send from the satellite to Earth. These signals contain timing information. Using the timing information in the GPS signal and the time at which these signals are received, position of the receiver can be determined. In this paper, corrections to GPS for range and bearing on curved space-time produced by gravitational field are determined. The aim is to determine the range and bearing (r_f, φ_f) of a target based on the co-ordinate time taken by a photon in propagating from a transmitter on the surface of the earth to the target, taking gravitational effects into account

KEY TERMS: GPS, Null Geodesics, Schwarzchild metric, Euler Lagrangian, Metric perturbation

INTRODUTION

If gravitational effects on photon propagation are ignored, then the photon will propagate along a straight line and the range and bearing of the target relative to the earth's center can be immediately determined by the time taken t_f for the photon to propagate and the angle at which the photon pulse was sent. The space time is altered by presence of mass distributions. Presence of gravitational body causes photons to take a curved path. In the presence of gravitation, according to Einstein's general theory of relativity, the photon propagation trejectory is described by a null geodesic associated with the metric corresponding to the gravitational field. This metric is the Schwarzchild metric. In this paper we determine the proper range and bearing and the proper time of the transmitted signals in curved space time is determined. The change in distance, bearing and time due to perturbations in the transmitted signals by gravitational bodies is determined.

PHOTON TRAJECTORY IN GRAVITATIONAL FIELD

The Schwarzschild metric describes the space time surrounding a spherical non rotating mass. In spherical polar coordinates $[(x^0, x^1, x^2, x^3)=(t, r, \theta, \varphi)]$, the line element can be written as [7]

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 sin^2\, \theta d\phi^2. \quad (1)$$

Where

τ = Proper time, measured by a clock moving along the test particle

c = speed of light

t = Time measured by stationary clock far from body

r = radial coordinate measured as circumference divided by 2π of a sphere centred around the massive body

θ = Co latitude.(Angle from north)

φ = Longitude

$r_s = 2GM/c^2$ Schwarzschild radius

Schwarchild metric is perturbed by the presence of other bodies like comets, planets and other gravitational objects.

Let there be two points $r_2 = (r_2, \theta_2, \varphi_2)$ and $r_1 = (r_1, \theta_1, \varphi_1)$ A photon propagates from r_1 to r_2. The photon follows the null geodesic equation.

$$0 = d\tau^2 = \alpha(r)\, dt^2 - \alpha(r)^{-1}\, dr^2\, /c^2 - \frac{r^2 d\theta^2}{c^2} - \frac{r^2 sin^2(\theta) d\emptyset^2}{c^2} \quad (2)$$

Where τ = proper time measured by a clock moving along the particle

$$\alpha(r) = \frac{1-2m}{r}$$

$$m = \frac{GM}{c^2}$$

A curve in space time which has the property that the infinitesimal interval between any two neighboring points on the curve equals zero is called null geodesic. Let us assume for simplicity that $d\theta = 0$ with specifically $\theta = {}^{\pi}/_{2}$ i.e., the photon trajectory is in the xy plane. Then the null proper time condition is

$$0 = \alpha(r)\, dt^2 - \alpha(r)^{-1} \frac{dr^2}{c^2} - \frac{r^2\, d\phi^2}{c^2} \quad (3)$$

Suppose we assume that $d\tau$ is non-zero but tends to zero. The Euler Lagrangian equations are

$$\frac{\partial \mathcal{L}}{\partial t^2} = K$$

$$\frac{\partial \mathcal{L}}{\partial \emptyset^1} = {}^{-\beta}/_{c^2}$$

Where K, β are constants and λ is a parameter along the geodesic

$$\xi' = \frac{d\xi}{d\lambda}$$

Where ξ the coordinate system and

$$\mathcal{L}\,(r, r^1, t^1, \varphi') = [\alpha(r)\, t^{|2} - \alpha(r)^{-1}\frac{(r^1)^2}{c^2} - r^2\, \phi'^2/_{c^2}]^{\frac{1}{2}}. \quad (4)$$

Thus we get

$$\alpha(r)\frac{dt}{d\tau} = K,\ r^2\frac{d\phi}{d\tau} = \beta$$

Taking the ratio of these two equations we get

$$(r^2 / \alpha(r)\, d\varphi / dt = \beta / K = A \quad (5)$$

Where A is a constant

The other equation is the null proper time condition

$$\alpha(r) - (\alpha(r)^{-1} / c^2)\left(\frac{dr}{dt}\right)^2 - \frac{r^2}{c^2}\left(\frac{d\emptyset}{dt}\right)^2 = 0 \quad (6)$$

Thus we have two different equations for two functions of time r(t), φ(t) which describe the complete photon trajectory in the gravitational field of earth.

TABLE 1: UNPERTURBED DISTANCE

Φ	π/2	π/3	π/4	π/6
R	1.8830e+011	1.7911e+011	1.7074e+011	1.5534e+011

TABLE 3: UNPERTURBED FINAL TIME

Φ	π/2	π/3	π/4	π/6
t_f	5.788	5.7833	5.7781	5.7864

TRAJECTORY OF A PHOTON IN GPS WHEN PERTURBED

Let the perturbation be represented by ϵ then we can write from equation (2),

$$d\tau^2 = [\alpha(r) + \epsilon\, h_1\,(r, \varphi')\, dt^2 - \alpha(r)^{-1} + \epsilon\, h_2\,(r, \varphi')\, dr^2 - r^2\, d\phi^2 \quad (7)$$

Now Lagrangian $\mathcal{L}$ is represented as [2, 7] $\mathcal{L} = (\alpha(r) - \alpha(r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2)^{1/2} = \dot{\tau}$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\varphi} = 0 \rightarrow r^2\frac{d\varphi}{d\tau} = \beta \text{ (a constant)}$$

$$\tilde{\mathcal{L}} = \frac{d\tau}{d\lambda} = \dot{r}\,\frac{dt}{d\lambda} = \left[\alpha(r)\frac{dt}{d\lambda}^2 - \alpha(r)^{-1}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\varphi}{d\lambda}\right)^2\right]^{1/2} \quad (8)$$

$$\frac{d}{d\lambda}\,\frac{\partial\tilde{\mathcal{L}}}{\partial\left(\frac{dt}{d\lambda}\right)} = \frac{\partial\tilde{\mathcal{L}}}{dt} = 0 \quad (9)$$

$$\alpha(r)\frac{dt}{d\tau} = K \text{ (a constant)}$$

Solving the above equations we get the unperturbed trajectory in curved space time as

$$r_0(\phi_0,\beta)^{-1} = \frac{1}{\beta}\sin\left(\phi_0 + \sin^{-1}\frac{\beta}{R}\right) + \frac{2m}{\beta^2}\int_0^{\phi_0} \exp\left(\int_{\phi_0^{|}}^{\phi_0} \tan\left(x + \sin^{-1}\frac{\beta}{R}\right)dx\right)\tan\left(\phi_0' + \sin\frac{\beta}{R}\right)\sin^2\left(\phi_0' + \sin^{-1}\frac{\beta}{R}\right)d\phi_0' \quad (10)$$

The final time can be determined from

$$t_f = \frac{1}{\beta}\int_0^{\phi_f} r^2(\phi_{0,}\beta)d\phi_0 \quad (11)$$

We can now treat the stochastic metric perturbation case along the same lines

$$\tau^{|} = \frac{d\tau}{d\lambda} = [(\alpha(r) + \epsilon\, h_1\,(r, \varphi))t^{|2} - (\alpha(r)^{-1} + \epsilon\, h_2\,(r, \varphi))r^{|2} - r^2\,\varphi^{|2}\,]^{1/2}$$

the equations are $\frac{\partial \mathcal{L}}{\partial t^1} = K$; because $\frac{\partial \mathcal{L}}{\partial t} = 0$

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \phi^1} = \frac{\partial \mathcal{L}}{\partial \phi}, \tau^{|2} = 0 \quad (12)$$

The last condition is to be incorporated at the end. Thus

$$(\alpha(r) + \epsilon\, h_1 \frac{dt}{d\tau} = K - \frac{d}{d\lambda}(r^2\frac{d\phi}{d\tau}) = \frac{\epsilon(h_{1,\phi}t^{|2} - h_{2,\phi}r^{|2}}{2\tau^1}$$

or

$$\frac{d}{d\tau}\left(r^2\frac{d\phi}{d\tau}\right) = \frac{\epsilon}{2}\left[h_{1,\phi}\left(\frac{dt}{d\tau}\right)^2 - h_{2,\phi}\left(\frac{dr}{d\tau}\right)^2\right] \quad (13)$$

Solution to the above expression is

$$r_1 = \frac{\left(F(\phi_0,\beta) - \frac{r_0^3}{(r_0-2m)}\frac{d\phi_1}{d\phi_0}\right)}{\frac{2(r_0-m)\beta}{(r_0-2m)^2}} \quad (14)$$

Where,

$$F(\phi_0,\beta) = \frac{2(r_0-2m)r_0^2}{(r_0-2m)}\frac{\beta r_1}{r_0^2} + \frac{r_0^3}{(r_0-2m)}\frac{d\phi_1}{d\phi_0} - \frac{r_0^2}{2\beta^2}\int_o^{\phi_0} r_0 d\phi_0(h_{1,\phi}(r_0\phi_0) - h_{2,\phi}(r_0\phi_0)d\phi_0$$

and

$$\left[\frac{2m}{r_0^2} + \frac{2m}{(r_0-2m)^2}\frac{\beta^2}{r_0^4}\frac{dr_0}{d\phi_0} - \frac{2\beta^2}{r_0^3}\right]\left[\frac{(r_0-2m)^2}{2(r_0-m)\beta}F(\phi_0,\beta) - \frac{r_0^3(r_0-2m)}{2\beta(r_0-m)}\frac{d\phi_1}{d\phi_0}\right]$$
$$-\frac{2r_0}{2(r_0-m)}\left(\frac{dr_0}{d\phi_0}\right)\frac{\beta}{r_0} - \frac{2\beta^2}{r_0}\frac{d\phi_1}{d\phi_0} = h_2(r_0,\phi_0)\left(\frac{dr_0}{d\phi_0}\right)^2\frac{\beta^2}{r_0^4} - h_1(r_0,\phi_0) \quad (15)$$

We can write the above equation of the form

$$\frac{d\phi_1}{d\phi_0}\{G(\phi_0,\beta)\} = H(\phi_0,\beta)$$

Therefore

$$\frac{d\phi_1}{d\phi_0} = \frac{H(\phi_0{}',\beta)}{G(\phi_0{}',\beta)} \text{ and hence } \phi_1 = \int_0^{\phi_0}\frac{H(\phi_0{}',\beta)}{G(\phi_0{}',\beta)}\,d\phi_0' \quad (16)$$

$$\int_0^{\phi_f} r_0^2(\phi_{0,}\beta)d\phi_0 = \beta t_f \quad (17)$$

TABLE 2: PERTURBED DISTANCE

Φ	DISTANCE 1.0e+011 *
π/2	1.8830 1.8834 1.8839 1.8841 1.8843 1.8840 1.8835 1.8834 1.8835 1.8839
π/3	1.7911 1.7907 1.7932 1.7927 1.7943 1.7951 1.7947 1.7946 1.7944 1.7948
π/4	1.7074 1.7050 1.7079 1.7091 1.7083 1.7081 1.7085 1.7089 1.7107 1.7122
π/2	1.5534 1.5530 1.5552 1.5555 1.5588 1.5568 1.5582 1.5590 1.5630 1.5591

TABLE 4: PERTURBED FINAL TIME

Φ	I	II	III	IV
π/2	5.7833	5.7781	5.7829	5.7864
π/3	3.8683	3.8692	3.8635	3.8692
π/4	2.8978	2.8998	2.9018	2.9097
π/6	1.9477	1.9488	1.9465	1.944

TABLE 5: PERTURBED VALUES OF φ

Φ	PERTURBED VALUE
1.0470	0.3130
0.7854	0.2371
0.5236	0.1645

TABLE 6: RMSD AND NORMALISED RMSD UNDER NOISY CONDITION

Φ	DISTANCE	RMSD	NRMSD
π/3	$1.7911*10^{13}$	$2.4944*10^{8}$	0.1699
π/4	$1.7074*10^{13}$	$4.2809*10^{8}$	0.1586
π/6	$1.5534*10^{13}$	$5.3588*10^{8}$	0.1853

Photon pulse is repeatedly send to target.Angle φ is varied until the target receiver registers a click. The receiver then conveys to the transmitter the time t_f at which the click was registered. The transmitter then computes the target receivers range and bearing based on above equations.

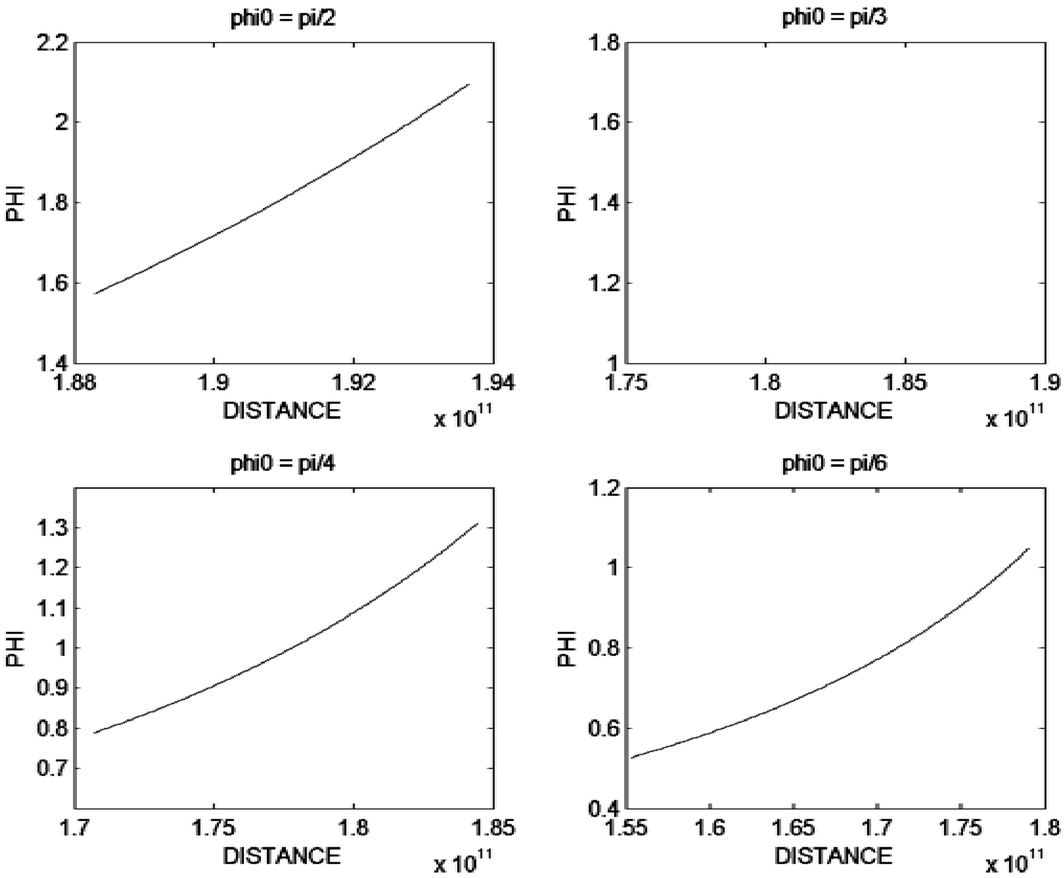

Fig 1: r vs φ curve. As φ is varied distance increases

CONCLUSIONS:

We can determine the position of target more accurately if we take gravitation due to massive objects into account. There is difference in distance and time when we take the gravitational forces into account. Distance and time varies upto one hundredth of their values when random perturbations are taken into account. For short distances this change is very less but for larger distances the errors due to perturbations are also very large and they must be taken into account for accurate position determination.(Table 3 and 4)As the bearing is perturbed, there will be change in the distance and consequently there will be change in time (Fig 1).These factors must be taken into account while determining the position of the target.

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