International Journal of Engineering Research and Technology.
ISSN 0974-3154 Volume 8, Number 1 (2015), pp. 35-42

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The Equitable Dominating Graph

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Abstract

The equitable dominating graph $ED(G)$ of a graph G is a graph with $V(ED(G)) = V(G) \cup D(G)$ where $D(G)$ is the set of all minimal equitable dominating sets of G and $u, v \in V(ED(G))$ are adjacent to each other if $u \in V(G)$ and v is a minimal equitable dominating set of G containing u. In this paper we characterize the equitable dominating graphs which are either connected or complete.

Keywords: Minimum equitable dominating set; Equitable dominating graph; Minimum equitable domination number.

Mathematics Subject Classification (2000): 05C

1. Introduction

All the graphs are simple, undirected without loops and multiple edges. Let $G = (V, E)$ be a graph. A subset D of V is said to be a equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|d(u) - d(v)| \le 1$. The minimum cardinality of such a dominating set D is called the equitable domination number of G and is denoted by $\gamma^e(G)$. An equitable dominating set D is said to be minimal equitable dominating set if no proper subset of D is an equitable dominating set. Kulli and Janakiram [5] introduced a new class of intersection graphs. Motivated by this we introduce a new class of graphs in the field of domination theory. Throughout this paper, the graph G is of p vertices and q edges. The terms used in this paper are in the sense of Harary[4].

Definition 1.1[III]: A vertex $u \in V$ is said to be degree equitable with a vertex $v \in V$

if $|d(u)-d(v)|\leq 1$. A vertex $u\in V$ is said to be an equitable isolate if $|d(u)-d(v)|\geq 2, \forall v\in V$.

Definition 1.2[III]: A minimal equitable dominating set of maximum cardinality is called Γ^e $-$set and its cardinality is denoted by $\Gamma^e(G)$.

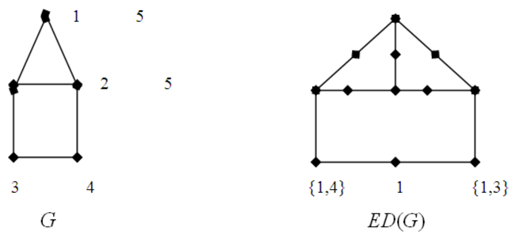
Definition 1.3[III]: Let $u\in V$. The equitable neighbourhood of u denoted by $N^e(u)$ is defined as $N^e(u)=\{v\in V / v\in N(u), |d(u)-d(v)|\leq 1\}$.

Definition 1.4[III]: A subset S of V is called an equitable independent set, if for any $u\in S$, $v\notin N^e(u)$ for all $v\in S-\{u\}$. The maximum cardinality of S is called equitable independence number of G and is denoted $\beta^e(G)$.

Definition 1.5[III]: The maximum order of a partition of V into equitable dominating sets is called equitable domatic number of G and is denoted by $d^e(G)$.

Definition 1.6: The equitable dominating graph $ED(G)$ of a graph G is a graph with $V(ED(G))=V(G)\cup D(G)$ where $D(G)$ is the set of all minimal equitable dominating sets of G and $u,v\in V(ED(G))$ are adjacent to each other if $u\in V(G)$ and v is a minimal equitable dominating set of G containing u.

An example of the equitable dominating graph $ED(G)$ of a graph G is given below:



2. Results

In this section we prove the main results of this paper. First we obtain the necessary and sufficient condition for a given graph G to be connected and followed by some results on completeness, equitable domatic partition and the equitable domination number of $ED(G)$.

Theorem 2.1[III]: Let G be a graph without equitable isolated vertices. If D is a minimal equitable dominating set, then $V-D$ is an equitable dominating set.

Theorem 2.2[I]: A graph G is Eulerian if and only every of vertex of G is of even degree.

Theorem 2.3: For any graph G with $p \geq 2$ and without equitable isolated vertices, the equitable dominating graph $ED(G)$ of G is connected if and only if $\Delta(G) < p-1$.

Proof: Let $\Delta(G) < p-1$. Let D_1 and D_2 be two minimal equitable dominating sets of G. We consider the following cases:-

Case i): Suppose there exists two vertices $u \in D_1$ and $v \in D_2$ such that *u* and *v* are not adjacent to each other. Then, there exists a maximal equitable independent set D_3 containing *u* and *v*. Since every maximal equitable independent set is a minimal equitable dominating set, D_3 is a minimal equitable dominating set joining D_1 and D_2. Hence there is a path in $ED(G)$ joining the vertices of $V(G)$ together with the minimal equitable dominating sets of G. Thus, $ED(G)$ is connected.

Case ii): Suppose for any two vertices $u \in D_1$ and $v \in D_2$, there exists a vertex $w \notin D_1 \cup D_2$ such that *w* is adjacent to neither *u* not *v*. Then, there exists two maximal equitable independent sets D_3 and D_4 containing *u,w* and *w,v* respectively. Thus, the vertices *u,v,w* and the minimal equitable dominating sets D_1, D_2, D_3, D_4 are connected by the path $D_1 - u - D_3 - w - D_4 - v - D_2$. Thus, $ED(G)$ is connected.

Conversely, suppose that $ED(G)$ is connected. Let us assume that $\Delta(G) = p-1$ and let $\{u\}$ be a vertex of degree $p-1$. Then, $\{u\}$ is a minimal equitable dominating set of G and by theorem2.1, $V-D$ has a minimal equitable dominating set say D'. This implies that $ED(G)$ has at least two components, a contradiction. Hence, $\Delta(G) < p-1$.

Hence the result.

Remark 2.4: In $ED(G)$, any two vertices *u* and *v* of $V(G)$ are connected by a path of length at most four.

Theorem 2.5: For any graph G with $\Delta(G) < p-1$ and without equitable isolated vertices, $diam(ED(G)) \leq 5$.

Proof: As $\Delta(G) < p-1$, by theorem 2.4, G is connected. Let $ED(G) = V \cup Y, E$, where Y is the set of all minimal equitable dominating sets of G. Let $u, v \in V \cup Y$. Then, by above theorem 2.4, $diam(ED(G)) \leq 4$ if $u, v \in V$ or $u, v \in Y$. On the other hand, if $u \in V$ and $v \in Y$ then $v = D$ is a minimal equitable dominating set of G. If

$u \in D$, then $d(u,v) \leq 4$; otherwise, there exists a vertex $w \in D$ such that $d(u,v) \leq d(u,w) + d(w,v) \leq 4+1 = 5$. This proves the result.

Theorem 2.6: For any graph G without equitable isolated vertices, $ED(G)$ is a complete bipartite graph if and only if $\overline{K}_p$.

Proof: Suppose that $ED(G)$ is not a complete bipartite graph with $G \approx \overline{K}_p$. As $G \approx \overline{K}_p$ the minimal equitable dominating set of G is $V(G)$, every isolated vertex in $ED(G)$ is adjacent to the vertex $V(G)$. This implies that $ED(G)$ is $K_{1,p}$, which is a contradiction. Thus, $ED(G)$ is complete bipartite graph. Conversely, suppose that $ED(G)$ is complete bipartite graph and $G \neq \overline{K}_p$. Thus G contains a nontrivial subgraph G_1. Then, by theorem2.1, for some vertex $u \in G_1$, there exists a minimal equitable dominating sets D and $\overline{D}$ with $u \in D$ and $u \notin \overline{D}$, which is a contradiction to the fact that G is complete bipartite graph with $u \in G_1$. Hence $G \approx \overline{K}_p$. This completes the proof.

Theorem 2.7: For any graph G without equitable isolated vertices, $d^e(G) \leq \beta^e(ED(G))$. Further, the equality holds if and only if $V(G)$ can be partitioned into union of disjoint minimal equitable dominating sets of cardinality one.

Proof: Let S be the maximum order of equitable domatic partition of $V(G)$. If every equitable dominating set is minimal and S consists of all minimal equitable dominating sets of G, then S is a maximum equitable independent sets of $ED(G)$. Hence $d^e(G) = \beta^e(ED(G))$. Otherwise, let D be a maximum equitable independent set with $D \notin S$. Hence, D is a minimal equitable dominating set of G. Let $u \in D$. Then, there are two following cases:-

Case i): If $u \in D'$, where $D' \in S$. Then, clearly $S \cup \{u\}$ is a equitable independent set in $ED(G)$. Hence the result holds.

Case ii): If $u \notin D'$, where $D' \in S$. Then, there exists a vertex $w \in V(G)$ such that $S \cup \{u, w\}$ is an equitable independent set. Hence the result.

Clearly, the equality condition follows as every component of $ED(G)$ is K_2 as $V(G)$ is the union of disjoint minimal equitable dominating sets of cardinality one. This completes the proof.

Corollary 2.8: For any graph G, $|V(ED(G)| \geq d^e(G)$.

Proof: Follows from theorem2.8 and the fact that for any graph G, $|V(G)| \geq \beta^e(G)$.

Theorem 2.9: For any graph G without equitable isolated vertices $p + d^e(G) \leq p' \leq p(\beta^e(G)+1)$, where p' is the number of vertices of $ED(G)$.

Further the lower bound is attained if and only if every minimal equitable dominating set of G is independent and the upper bound is attained if and only if every maximum equitable independent set is of cardinality one.

Proof: The graph $ED(G)$ has the vertex set $V(G) \cup D(G)$ and it has at least $d^e(G)$ number of minimal equitable dominating sets, hence the lower bound follows.
Clearly upper bound follows as every maximal equitable independent set is a minimal equitable dominating set and every vertex is present in at most $(p-1)$ minimal equitable dominating sets.

Further, suppose that $p + d^e(G) = p'$. As there are $d^e(G)$ number of minimal equitable dominating sets and each vertex is present in exactly one of the minimal equitable dominating set and hence these minimal equitable dominating sets are independent.

Also, suppose that every maximum equitable independent set is of cardinality one then, these are minimal equitable dominating sets of G and are independent and as every vertex is present in at most $(p-1)$ minimal equitable dominating set, the equality holds. This implies the necessary condition. Converse of the result trivially holds.

Theorem 2.10: For any graph G without equitable isolated vertices $\left\lfloor \frac{p + d^e(G)}{2} \right\rfloor \leq q' \leq p(p-1)$, where q' is the number of edges of $ED(G)$.

Further, the lower bound is attained if and only if every minimal equitable dominating set is independent and the upper bound is attained if and only if G is $(p-2)-$regular.

Proof: The proof of the lower bound follows by the same lines of theorem 2.10.
Suppose the lower bound is attained. As every vertex must be in exactly one of the dominating set, Cleary every minimal equitable dominating set is independent. As every vertex is in at most $(p-1)$ minimal equitable dominating set, upper bound follows.

Suppose the upper bound is attained. Then, each vertex is in exactly $(p-1)$ minimal equitable dominating sets and hence G is $(p-2)-$regular. This completes the proof.

Theorem 2.11: For any graph G with $p \geq 3$, $d^e(ED(G)) = 1$ if and only if $G = \overline{K}_p$, where $\overline{K}_p$ is the complement of K_p or $ED(G)$ has an equitable isolated vertex.

Proof: Suppose that $d^e(ED(G))=1$. Then, $ED(G)$ has a vertex D with $D=V(G)$. Thus $ED(G)$ is $K_{1,p}$ and hence $G=\overline{K}_p$. Otherwise, suppose assume that $ED(G)$ has no equitable isolated vertex and $V(ED(G))=p'$. Then, $\gamma^e(ED(G))\le\frac{p'}{2}$. If D is an equitable dominating set, then $V-D$ is an equitable dominating set and hence $d^e(ED(G))\ge 2$, a contradiction. Hence $ED(G)$ has an equitable isolated vertex.

The converse is obvious.

Theorem 2.12: If a graph G is connected, $(p-1)-$regular and without equitable isolated vertices then, $\gamma^e(ED(G))=p$.

Proof: As G is connected and $\Delta(G)=p-1$, by theorem2.4, $ED(G)$ is disconnected. Also, we know that every vertex is present in at most $(p-1)$ minimal equitable dominating sets. Thus, $ED(G)$ is a disconnected graph with each of the component being K_2, there are p number of components. Hence $\gamma^e(ED(G))=p$.

Theorem 2.13: For any graph G of order $p\ge 2$, without equitable isolated vertices and $\Delta(G)<p-1$, the equitable dominating graph $ED(G)$ of a graph G is a tree if and only if $G=\overline{K}_p$.

Proof: As G is a graph of order $p\ge 2$,without equitable isolated vertices and $\Delta(G)<p-1$, by theorem2.4, $ED(G)$ is connected. Suppose assume that $ED(G)$ of G is a tree. Then, clearly G has no cycle. On the contrary assume that $G\ne\overline{K}_p$. Then, by theorem2.12, $d^e(ED(G))\ne 1$. Hence there exists at least two minimal equitable dominating sets containing where u and v are any two vertices in G. If u and v are in the same minimal equitable dominating set D then, $u-D-v-u$ is a cycle in $ED(G)$, a contradiction. On the other hand, if u and v are in different minimal equitable dominating set. Then, there exists vertices u_1, v_1 and the minimal equitable dominating sets D_1, D_2 and D_3 such that $uu_1\in D_1, u_1v_1\in D_2$ and $v_1v\in D_3$. Thus, u and v are connected by two paths in $ED(G)$, a contradiction. Hence $G=\overline{K}_p$

Conversely, suppose that $G=\overline{K}_p$ and $\Delta(G)<p-1$. Then, by theorem2.4, $ED(G)$ is connected. Also, by theorem2.12, $d^e(ED(G))=1$. i.e., there exists a minimal equitable dominating set D with $D=V(G)$. Thus, $ED(G)$ is connected, $K_{1,p}$ and has no cycle. Hence $ED(G)$ is a tree. This completes the proof.

Theorem 2.15: For any graph G, $ED(G)$ is either connected or has at most one component that is not K_2.

Proof: We consider the following cases:-

Case i): If $\Delta(G) < p-1$, then by theorem2.4, $ED(G)$ is connected.

Case ii): If $\delta(G) = \Delta(G) = p-1$, then $G = K_p$. Hence each of the vertex $v \in V(G)$ is a minimal equitable dominating set of G and hence each of the component of $ED(G)$ is K_2.

Case iii): If $\delta(G) < \Delta(G) = p-1$. Let $v_1, v_2, ..., v_n$ be n vertices of G of degree $p-1$. Let $H = G/\{v_1, v_2, ..., v_n\}$ then $\Delta(H) < V(H)-1$. Hence by theorem2.4, $ED(H)$ is connected. Since $ED(G) = \Omega(V(ED(H) \cup V(G_1) \cup V(G_2) \cup ... \cup V(G_n))$ where $G_1, G_2, ..., G_n$ are the graphs joining $v_1, v_2, ..., v_n$ with $\{v_1\}, \{v_2\}, ..., \{v_n\}$ respectively. Then, exactly one of the component of $ED(G)$ is not K_2. Hence the proof.

Theorem 2.17: If G is a $r-$regular graph with $\Gamma^e(G) = 2$ and every vertex is in exactly even number of minimal equitable dominating sets then $ED(G)$ is eulerian.

Proof: Let G is a $r-$regular graph. Since each of the vertex of G is in even number of minimal equitable dominating sets, each of them contributes even number to the degree of the vertex in $ED(G)$ and as $\Gamma(G) = 2$, each of the minimal equitable dominating set of G is a vertex of degree two in $ED(G)$. Thus, by theorem2.2, $ED(G)$ is eulerian.

Theorem 2.18: Let G be a graph with $\Delta(G) < p-1$ and $\Gamma^e(G) = 2$. If every vertex is present in exactly two minimal equitable dominating sets then, $ED(G)$ is Hamiltonian.

Proof: As $\Delta(G) < p-1$, G is connected by theorem2.4. Also, since every vertex is present in exactly two minimal equitable dominating sets, $\gamma^e(G) = \Gamma^e(G)$ and also $\deg(u) = \deg(D) = 2$ in $ED(G)$, where D is a minimal equitable dominating set in G. Thus, $ED(G)$ is connected and 2-regular. Hence $ED(G)$ is Hamiltonian.

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