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Thermoelastic Model of Ceramic Materials with Fractional Order Strain and Variable Thermal Conductivity

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Abstract

In this work, we present a new thermoelasticity model in the context of the new theory of fractional order strain thermoelasticity with variable thermal conductivity. A new fractional order equation has been used through this model. The thermoelastic properties of a semi-infinite homogeneous isotropic Ceramic material with variable thermal conductivity was investigated. The governing equations are solved using a direct method to obtain the solutions of the field functions in the Laplace domain. The medium is subjected to ramp type thermal loading. To obtain the different inverse field functions numerically we used a complex inversion formula of Laplace transform based on a Fourier expansion. The effects of different parameters on the conductive temperature, the thermodynamical temperature, the displacement, the stress and on the strain distribution are presented graphically. Comparison between each field function with constant and variable thermal conductivity are also presented graphically and discussed.
Keywords: Mathematical model; fractional order equation of motion; variable thermal conductivity; generalized thermoelasticity; numerical inversion of Laplace transform, ceramic materials.

1. Introduction

The theory of thermoelasticity is dealing with the prediction of the thermo-mechanical behavior of elastic medium. Boit [1] and Nowacki[2] while studying the thermo-elastic behavior assumed the propagation of thermal waves to be of infinite speed. This paradox of infinite speed of propagation was removed by converting the parabolic heat equation to hyperbolic equation in generalized thermo-elasticity. Many researchers introduced different relaxation times in the generalized theory. For instance; Lord and Schulman [3] considered one relaxation time while Green and Lindsay [4] considered two relaxation time. The result of [4] is proved to be the explicit version of the theory given by Muller [5], Green and Laws[6] and Suhubi and Eringen [7]. The analytical results about finite speed of wave propagation and second sound have also been reported by Ignaczack [8] and Chandrasekharaiah [9]. Subsequently, Hetnarski and Ignaczack [10] examined and reported five generalizations to the coupled theory. Later Hetnarski and Eslami [11] gave an advanced version of unified general theory and its applications to generalized and classical thermo elasticity, thermodynamics, underlying both the mathematical and mechanical backgrounds as well as the theory of elasticity.

The concept of derivative and integral have been generalized to a non integer order and studied by many researchers. Various physical process and models have been implemented through application of fractional order derivatives [12]-[14].
Due to the non-local behavior of a fractional order differential operator the current state of a system depends on its previous state. This leads to the dependency of the current state of a system on its previous state.
Comparison between the very famous definition of fractional order due to Riemann-Liouville given by [12]:

$$ {}_{RL}D_t^{\beta} f(t) = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1} f(\tau)d\tau\right] \qquad n-1<\beta<n, \quad (1)$$

and the definition of Caputo [15]

$$ {}_{C}D_t^{\beta} f(t) = \frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1} \frac{d^n f(\tau)}{d\tau^n} d\tau \qquad n-1<\beta<n, \quad (2)$$

shows that, Caputo fractional derivative has two major advantages. It allows traditional initial and boundary conditions to be included in the formulation of the problem and in addition the derivative of a constant is zero. In case of $f(0)=0$ the above two definitions becomes the same.
where $\Gamma(\beta)$ is the gamma function. Using the fractional order derivative in the present work permits to the differential equations of the system to take into consideration the effects of the intermediate as well as the previous states to express the present and the next states of the medium.

Applications of the fractional order theory and many other contributions has been published by many researchers [16]-[30].
Thermal conductivity is one of the most important properties of materials explaining different applications of heat. Thermal conductivity is a microstructure sensitive property that measure the ability of materials to transport heat energy from high temperature

region to low temperature one and its value varies for different material.

Ceramic materials have many attractive properties. They are recommended for wide applications and many devices such as semiconductors, capacitors, sensor and transducer applications. In the present work we considered the ceramic material namely; Barium Titanate $BaTiO_3$. Some of the properties of $BaTiO_3$ have been reported in [31] and [32].
The present model is a new model as it uses a new fractional order strain equation together with the form:

$$L_C D_t^{\beta} f(t) = s^{(\beta-n)} L\{f^n(t)\} \qquad n-1<\beta<n, \tag{3}$$

of the fractional order time derivative of Caputo [15] to investigate the thermoelastic properties of an isotropic and homogeneous one dimensional semi infinite elastic medium with variable thermal conductivity subjected to ramp type thermal loading. In equation (3), s denotes to the complex parameter related to Laplace transform.
The presence of the fractional order operator in the present system of equations makes this new model more reliable as the fractional order parameter permits the equations to take into consideration the effects of the intermediate as well as the previous states to express the present and the next states of the medium.
Comparison between the present model and the model [34] shows that the present model is completely different than that model as we used the generalized thermoelasticity with fractional order strain theory which is a different theory of [34]. In the present model; the fractional has been applied on Hook's law of stress-strain relation which gives a different equation of motion with new physical meanings, while in the work of [34], the fractional has been applied on the heat conduction equation. Moreover, the results between the two models are very different and do not give the same conclusion.

2. One Dimensional Formulation

For some details on generalized thermoelectric governing differential the reader may refer to Youssef [30]. In our present model, we assume that all the field functions are initially at rest and the medium coincides with the x- axis. Thus we assume the following form of the displacement:

$$u_x = u(x,t), \qquad u_y = u_z = 0, \tag{4}$$

We will consider the following one dimensional linearized basic equations:
(i) The heat equation:

$$\frac{\partial}{\partial x}\left(K\,\frac{\partial \theta(x,t)}{\partial x}\right) = \left(\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2}\right)\left(\rho C_E\, \theta(x,t) + T_o \gamma (1+\tau^{\beta} D_t^{\beta}) e(x,t)\right) \tag{5}$$

where K is the thermal conductivity, τ_o is the relaxation time, ρ is the density, C_E is the specific heat at constant stain, T_o is the references temperature, $\gamma = \alpha_T(3\lambda + 2\mu)$, α_T is the thermal linear expansion, λ and μ are Lame's constants, $\theta = T - T_o$ is the

temperature increment such that $\theta/T_o \ll 1$ and e is the cubic dilatation.
(ii) The equation of motion:

$$\rho \frac{\partial^2 e(x,t)}{\partial t^2} = (\lambda + 2\mu)(1 + \tau^\beta D_t^\beta)\frac{\partial^2 e(x,t)}{\partial x^2} - \gamma \frac{\partial^2 \theta(x,t)}{\partial x^2} \tag{6}$$

and the constitutive equation can be written in the form:

$$\sigma(x,t) = (1 + \tau^\beta D_t^\beta)(\lambda + 2\mu)e(x,t) - \gamma\theta(x,t) \tag{7}$$

and

$$e(x,t) = \frac{\partial u(x,t)}{\partial x} \tag{8}$$

In most materials, the dependence of K and C_E on the thermodynamical temperature θ is in some range of the temperature. Thus we can consider the following dependence of the thermal conductivity K on temperature:

$$K = K(\theta) = K_o(1 + \kappa_1 \theta) \tag{9}$$

and

$$\rho\, C_E = \frac{K}{\kappa} \tag{10}$$

where K_o is constant and equal to the thermal conductivity of the material when it does not depend on θ , κ_1 is a non-positive small parameter and κ is the thermal diffusivity. The thermal diffusivity of materials is the property that measures how fast the material temperature adapts the surrounding temperature [35]. Using (10) with (5) the heat equation becomes:

$$\frac{\partial}{\partial x}(K\, \frac{\partial \theta(x,t)}{\partial x}) = (\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2})(\frac{K}{\kappa}\, \theta(x,t) + T_o \gamma(1 + \tau^\beta D_t^\beta)e(x,t)) \tag{11}$$

We apply the following mapping:

$$\phi = \frac{1}{K_o}\int_0^\theta K(\vartheta)d\vartheta \tag{12}$$

this gives:

$$\phi = \theta + \frac{\kappa_1}{2}\, \theta^2 \tag{13}$$

Using equations (9), (10) and (13) into (11) we obtain the following heat equation:

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} = (\frac{\partial}{\partial t} + \tau_o \frac{\partial^2}{\partial t^2})(\frac{\phi(x,t)}{\kappa} + \frac{\gamma T_o}{K_o}(1 + \tau^\beta D_t^\beta)e(x,t)) \tag{14}$$

Substituting (9) into the (6) we get:

$$\rho \frac{\partial^2 e(x,t)}{\partial t^2} = (\lambda + 2\mu)(1 + \tau^\beta D_t^\beta)\frac{\partial^2 e(x,t)}{\partial x^2} - \frac{\gamma K_o}{K(\theta)}\frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{15}$$

Neglecting small quantities of the second and higher order; the equation (15) takes the form:

$$\rho \frac{\partial^2 e(x,t)}{\partial t^2} = (\lambda + 2\mu)(1 + \tau^\beta D_t^\beta)\frac{\partial^2 e(x,t)}{\partial x^2} - \gamma \frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{16}$$

Using (9) and applying the same approximation to the normal stress component; the equation (7) becomes:

$$\sigma(x,t) = (1 + \tau^\beta D_t^\beta)(\lambda + 2\mu)e(x,t) - \gamma\phi(x,t) \tag{17}$$

After obtaining the function ϕ, we found the temperature increment θ by solving the equation (13) as

$$\theta = \frac{-1 + \sqrt{1 + 2\kappa_1 \phi}}{\kappa_1} \tag{18}$$

The material is traction free and it is subjected to the following ramp type thermal loading boundary condition:

$$\theta(0,t) = \begin{cases} 0, & t < 0, \\ \frac{t}{t_o}\,\theta_1, & 0 < t < t_o, \\ \theta_1, & t \geq t_o \end{cases} \tag{19}$$

and the other boundary conditions are given by:

$$\sigma(0,t) = 0, \quad \sigma(\infty,t) = 0, \quad \theta(\infty,t) = 0, \quad 0 < t < \infty \tag{20}$$

As the medium is initially at rest we have the following initial conditions:

$$\theta(x,0) = 0, \quad \sigma(x,0) = 0, \quad 0 < t < \infty \tag{21}$$

where θ_1 is constant and t_o is the ramp type parameter (time rise parameter). We use the following dimensionless variable to rewrite the field equations Youssef [30]::

$$\begin{aligned} u' &= c_o \eta u, \quad t' = c_o^2 \eta t, \quad t_o' = c_o^2 \eta t_o, \\ \tau_o' &= c_o^2 \eta \tau_o, \quad \tau'^\beta = c_o^2 \eta \tau^\beta, \quad \tau' = c_o^2 \eta \tau, \\ \sigma' &= \frac{\sigma}{\lambda + 2\mu}, \quad \eta' = \frac{\rho C_E}{\kappa}, \quad \phi' = \frac{\gamma \phi}{\lambda + 2\mu}, \\ \theta' &= \frac{\gamma \theta}{\lambda + 2\mu}, \quad x_o' = c_o \eta x, \quad c_o^2 = \frac{\lambda + 2\mu}{\rho} \end{aligned} \tag{22}$$

Using (22) and dropping the primes for convenience, equations (14) and (16) become:

$$\frac{\partial^2 \phi(x,t)}{\partial x^2} = [\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}](\phi(x,t) + \xi(1 + \tau^\beta D_t^\beta)e(x,t)) \tag{23}$$

$$\frac{\partial^2 e(x,t)}{\partial t^2} = (1 + \tau^\beta D_t^\beta)\frac{\partial^2 e(x,t)}{\partial x^2} - \omega \frac{\partial^2 \phi(x,t)}{\partial x^2} \tag{24}$$

while the constitutive equations (17) and (8) takes the form:

$$\sigma(x,t) = (1+\tau^{\beta} D_t^{\beta})e(x,t) - \omega\phi(x,t) \tag{25}$$

and

$$e(x,t) = \frac{\partial u(x,t)}{\partial x} \tag{26}$$

Combining equations (24) and (25) we get:

$$\frac{\partial^2 \sigma(x,t)}{\partial x^2} = \frac{\partial^2 e(x,t)}{\partial t^2} \tag{27}$$

respectively, where $\xi = \frac{\gamma}{\rho C_E}$ and $\omega = \frac{\gamma\, T_o}{\lambda+2\mu}$ are non dimensional constants.

We use the following definition of Laplace transform

$$Lf(t) = \overline{f}(s) = \int_0^{\infty} e^{-st} f(t)dt, \tag{28}$$

together with the Laplace transform for the fractional order derivative Podlubny [29]:

$$L\{{}_C D_t^{\beta}\ f(t)\} = s^{(\beta\ -n)} L\{f^{(n)}(t)\}, \tag{29}$$

where s denotes the complex parameter related to Laplace transform. Following Youssef [30], we apply the transformations (28) and (29) to both sides of equations (23) - (26) to get the following generalized thermoelasticity system of equations with variable thermal conductivity and fractional order strain equation based on Lord and Schulman theory [3] of generalized thermoelasticity written in the domain of Laplace given in [30]:
the heat equation takes the following form:

$$\frac{\partial^2 \overline{\phi}(x,s)}{\partial x^2} = s(s\tau_o+1)[(\xi\,(\tau^{\beta} s^{\beta}+1)\ \overline{e}(x,s) + \overline{\phi}(x,s)] \tag{30}$$

and the transformed equation of motion assumes the form:

$$\frac{\partial^2 \overline{e}(x,s)}{\partial x^2} = \frac{s^2 \overline{e}(x,s)}{(1+s^{\beta}\tau^{\beta})} + \frac{(1+s)\omega}{(1+s^{\beta}\tau^{\beta})}\ \frac{\partial^2 \overline{\phi}(x,s)}{\partial x^2} \tag{31}$$

while the transformed constitutive equations take the forms:

$$\overline{\sigma}(x,s) = (1+s^{\beta}\tau^{\beta})\ \overline{e}(x,s) - \omega\ \overline{\phi}(x,s) \tag{32}$$

$$\overline{e}(x,s) = \frac{\partial \overline{u}(x,s)}{\partial x} \tag{33}$$

Apply Laplace transform to the equation (27) we get:

$$\frac{\partial^2 \overline{\sigma}(x,s)}{\partial x^2} = s^2 \overline{e}(x,s) \tag{34}$$

Using the dimensionless variables in equation (22) and by applying the Laplace transform (28) to the boundary and initial conditions (19), (20) and (21) respectively yield:

$$\overline{\theta}(0,s)=\frac{\theta_1(1-e^{-st_o})}{t_o\, s^2} \tag{35}$$

and

$$\overline{\theta^2}(0,s)=\frac{2\theta_1^2(1-e^{-st_o})(st_o+1)}{t_o^2\, s^3} \tag{36}$$

Hence (13) leads to the following boundary conditions:

$$\overline{\phi}(0,s)=\frac{\theta_1(1-e^{-st_o})}{t_o\, s^2}+\frac{\kappa_1\theta_1^2(1-e^{-st_o})(st_o+1)}{t_o^2\, s^3}=\varphi_o \tag{37}$$

$$\overline{\sigma}(0,s)=0,\quad \overline{\phi}(\infty,s)=0,\qquad \overline{\sigma}(\infty,s)=0 \tag{38}$$

while the initial conditions (21) take the following forms:

$$\overline{\phi}(x,0)=0,\ \ \overline{\sigma}(x,0)=0, \tag{39}$$

3. Solution in the Laplace Domain

Eliminating $\overline{e}$ between equations (30) and (31) we get the following fourth order differential equation:

$$b\overline{\phi}-a\frac{\partial^2\overline{\phi}}{\partial x^2}+\frac{\partial^4\overline{\phi}}{\partial x^4}=0, \tag{40}$$

where $a=\frac{s^2+mL(1+\xi\omega)}{L}$, $b=\frac{s^2m}{L}$, $m=s+s^2\tau_o$ and $L=1+s^\beta\tau_0^\beta$.

The most general solution of equation (40) is of the form:

$$\overline{\phi}(x,s)=\phi_1 e^{-k_1x}+\phi_2 e^{k_2x}, \tag{41}$$

where $\phi_1$and ϕ_1 are coefficients depending on s which can be determined by using the boundary conditions, while k_1,k_2 are the roots of the characteristic equation;

$$b-ak^2+k^4=0 \tag{42}$$

and are given by:

$$\begin{aligned} k_1&=\pm(\sqrt{a+\sqrt{a^2-4b}})/\sqrt{2},\\ k_2&=\pm(\sqrt{a-\sqrt{a^2-4b}})/\sqrt{2} \end{aligned} \tag{43}$$

Hence, the most general solution of the equations (31)- (33) in the domain of Laplace take the following forms:
the strain distribution:

$$\bar{e}(x,s) = \frac{1}{m\xi L}\sum_{i=1}^{2}\phi_i(k_i^2 - m)e^{-k_i x} \tag{44}$$

the stress distribution:

$$\overline{\sigma}(x,s) = \frac{1}{m\xi}\sum_{i=1}^{2}\phi_i(k_i^2 - m(1+\omega\xi))e^{-k_i x} \tag{45}$$

and the displacement distribution assumes the form:

$$\bar{u}(x,s) = \frac{1}{m\xi s^2}\sum_{i=1}^{2}k_i\phi_i(-k_i^2 + m(1+\omega\xi))e^{-k_i x} \tag{46}$$

Using the boundary conditions (37) - (38), we can determine the coefficients ϕ_1 and ϕ_2 as:

$$\begin{aligned} \phi_1 &= \frac{\varphi_o(-k_2^2 + m(1+\xi\omega))}{k_1^2 - k_2^2}, \\ \phi_2 &= \frac{\varphi_o(k_1^2 - m(1+\xi\omega))}{k_1^2 - k_2^2} \end{aligned} \tag{47}$$

Equations (41) and (44) - (47) represent the complete solution of the system (30) - (33) in the Laplace transform domain.

4. Numerical Inversion of the Laplace Transform

By using the Fourier expansion technique the inverse of the field function in the Laplace domain is computed by approximating the original function f(t) of the Laplace transform $\overline{f(s)}$ by:

$$f(t) = \frac{\exp(ct)}{t_1}[\frac{1}{2}\overline{f}(c) + \Re(\sum_{1}^{N}\overline{f}(c+\frac{ik\pi}{t_1}\exp(\frac{ik\pi}{t_1})], \qquad 0 < t_1 < 2t \tag{48}$$

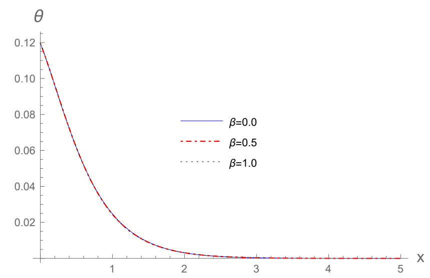
where $\Re$ is the real part, i is imaginary number unit and N is a sufficiently large integer representing the number of terms in the truncated Fourier series, chosen such that:

$$\exp(ct)\Re\left[\overline{f}(c+\frac{iN\pi}{t_1})\exp(\frac{iN\pi t}{t_1})\right] \leq \varepsilon_1, \tag{49}$$

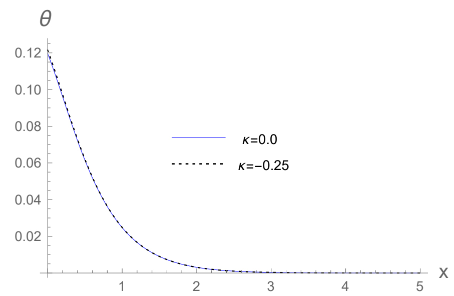
where ε_1 is prescribed small positive number that corresponds to the degree of accuracy required and parameter c is a positive free parameter that must be greater than the real part of all the singularities of $\overline{f(s)}$. The optimal choice of c was obtained according to the criteria described in Honig and Hirdes [37].

5. Results and Discussion

For numerical computation, we use the physical constants of $BaTiO_3$ used before in [36] In all figures the cases were the parameter takes the minimum value, maximum value and between minimum and maximum values are represented by the blue lines, black lines and red lines respectively.
We illustrate the distributions of the field functions; the thermodynamical temperature θ, the normal stress component σ and the distribution of the strain e with the variation of the fractional order β, time t and ramp time t_o parameters. Comparison between each field function with constant and variable conductivity is illustrated for $t = 0.3, t_o = 0.1, \beta = 0.5$ and different values of κ_1.
Figures (1)-(3) represent the first group and it illustrate the effect of different values of the fractional order parameter; $\beta = 0.0, \beta = 0.5$ and $\beta = 1.0$ on the distributions of the field functions for value of time $t = 0.3$, ramp time parameter $t_o = 0.1$ and different values of κ_1.



(a) Thermodynamic temperature distribution for $\kappa_1 = 0.0$

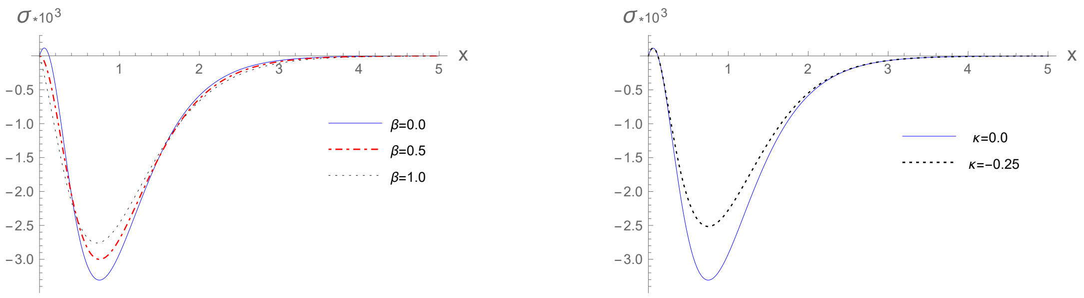


(b) Thermodynamic temperature at different values of κ_1

Figure 1: Variation of temperature θ with the fractional order parameter β at $t = 0.3, \tau_o = 0.1$ and ramp time $t_o = 0.1$

Figures 1(a) and Fig.1(b) reflect the profile of θ along the x- axis. The profiles of the thermodynamical temperature θ are found to be independent of the fractional order parameter for all values of x. Figure 1(b) represents the profile of the thermodynamical temperature θ with constant and variable conductivity for different values κ_1. Very Slight changes has been noticed in the amplitude of θ as the value of κ_1 changes.

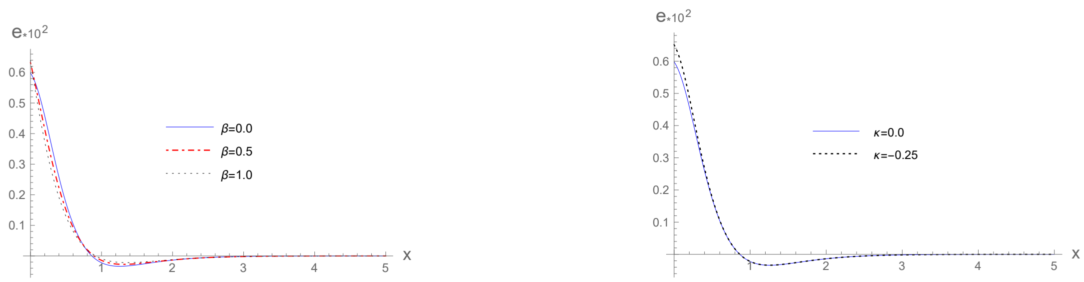
Figures 2(a) and Fig. 2(b) illustrate the changes of the stress component σ with the different values of the fractional order parameter β. It is noticed that the absolute value of the amplitude of the stress component vary inversely with the value of the fractional order parameter. It is also noticed that the peak points of the stress curves



(a) Stress distribution for $\kappa_1 = 0.0$ (b) Stress distribution for $\kappa_1 = -0.25$

Figure 2: Variation of stress component σ with the fractional order parameter β at $t = 0.3$, $\tau_o = 0.1$ and ramp time $t_o = 0.1$

occur at the same point for different values of the fractional order parameter β. The stress component reaches the equilibrium state at values of $x \geq 3.0$ and the point of equilibrium is independent of the value of κ_1 as seen in Fig 2(a) and Fig. 2(b). It is also noticed that the stress component initially increases slightly near the point $x = 0$ and it starts decaying rapidly at the point $x \backsimeq 1.4$ until it reaches the equilibrium state. The amplitude of the stress component has common values for different value of the fractional order parameter β at the points $x = 0$ and $x \backsimeq 0.6$ as seen in Fig. 2(a) and 2(b). In Fig.2(b) the amplitude of the stress component decreases with the value of κ_1.



(a) Strain distribution for $\kappa_1 = 0.0$ (b) Strain distribution for $\kappa_1 = -0.25$

Figure 3: Variation of strain component e with the fractional order parameter β at $t = 0.3, \tau_o = 0.1$ and ramp time $t_o = 0.1$

Figure 3 (a) shows that the strain component affected slightly by the variation of the value of fractional order parameter β.

The absolute value of the amplitude of the strain component increases with variable conductivity as seen in Fig. 3(b).

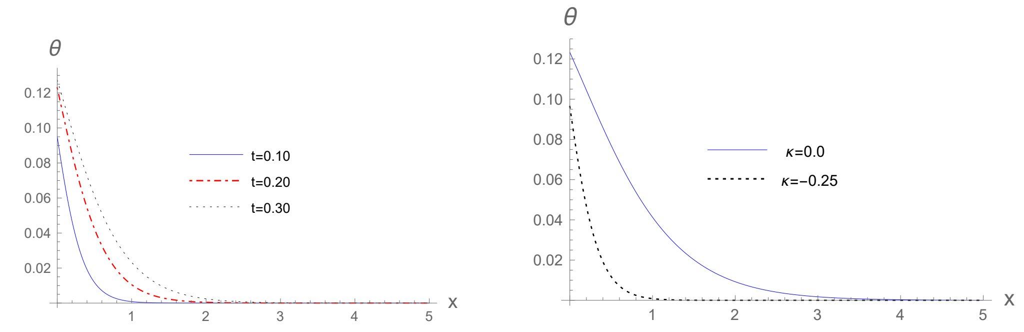
The figures (4) to (6) illustrate the variation of the field functions θ, σ and e for values of time $t = 0.1, t = 0.2$ and $t = 0.3$ with the the values of fractional order parameter $\beta = 0.5$, ramp time $t_o = 0.1$ and different values of κ_1.

Figure 4(a) indicates that the profiles of the temperature distribution depend strongly on t and an inverse proportionality is noticed between the value of the amplitudes of the temperature and the value of time. It is also noticed that the thermodynamical temperature θ attains its equilibrium state at different values of x. Figure 4(b) illustrates that the amplitude of the thermodynamical temperature decreases with increase in the thermal conductivity.

Figures (5) reflect the effect of the time t on the profile of the stress function σ. Figure 5(a) indicates that the stress component depends strongly on the value of time t. It can be seen that the absolute value of the stress component increases as the value of t increases. It is also noticed from Fig. 5(a) that the peak point shifts away from the point $x = 0$ where the thermal loading is applied. It is also noticed that the stress component σ attains its equilibrium state at different values of x. In Fig 5 (b) we notice a slight change in the amplitude of the stress components with variable conductivity.

Figures (6) reflect the profile of the strain components with different values of the time t. Figure 6(a) illustrates strong dependence of the strain components on t. The third group of figures (7)- (9), reflects the effect of ramp time parameter t_o on the field functions. The numerical calculations are carried out for the values of the ramp time parameter $t_o = 0.1,\ 0.3$ and 0.5 with the value of time $t = 0.3$ and the fractional order parameter $\beta = 0.5$ together with different values of κ_1.

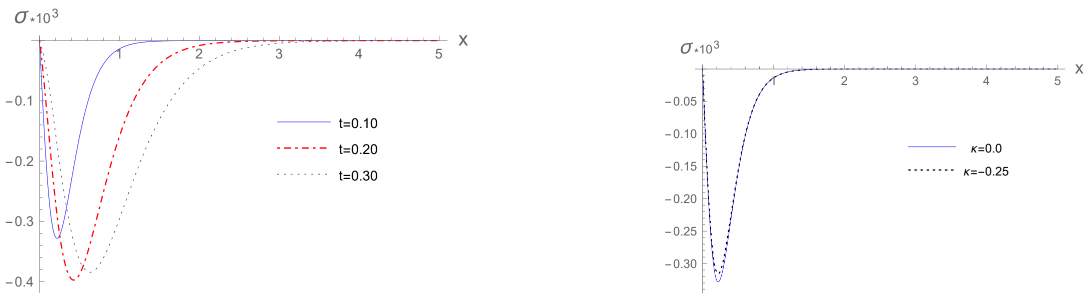
Figure 7(a) illustrates that the amplitude of the temperature decreases with increase in the ramp time this leads to the increase in the wave length and decrease in the energy of the thermal distribution wave. The effect of the ramp time parameter on the stress distribution is presented in Fig.(8). Figure 8(a) indicates an inverse proportionality of the absolute value of the amplitude of the stress component with the ramp time parameter t_o in contrary with its behaviour with the time parameter t as seen in Fig.(5). The absolute value of the amplitude of the stress component decreases if the conductivity is variable as seen in Fig.8(b). Comparison between Fig.(3) and Fig.(9) shows that the components of the strain resemble the same behaviour with the fractional order parameter as well as with the ramp time parameter.



(a) Thermodynamic temperature distribution for $\kappa_1 = 0.0$

(b) Thermodynamic temperature for different values of κ_1

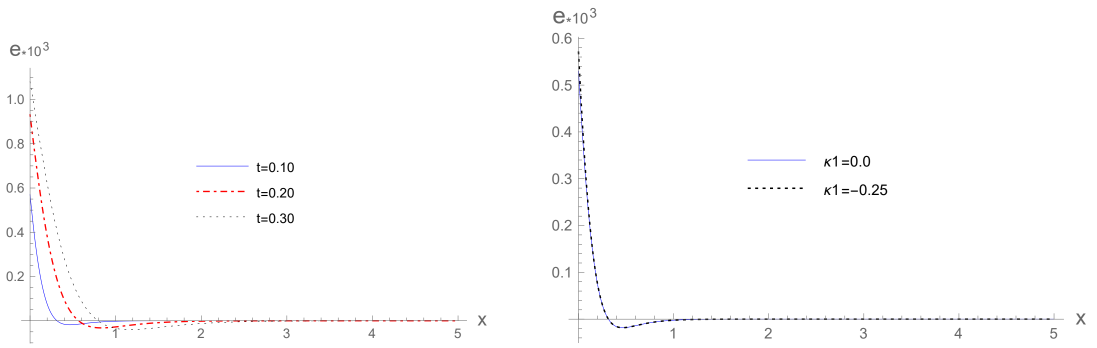
Figure 4: Variation of the thermodynamical temperature θ with the time parameter t at $\tau_o = 0.1$, $t_o = 0.2$ and $\beta = 0.5$



(a) Stress Distribution for $\kappa_1 = 0.0$

(b) Stress Distribution for $\kappa_1 = -0.25$

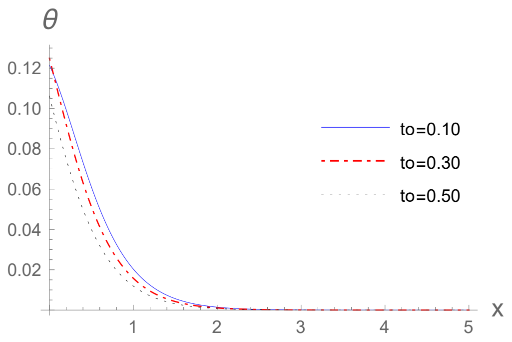
Figure 5: Variation of the stress component σ with the time parameter t at $\tau_o = 0.1$, $\beta = 0.5$ and ramp time $t_o = 0.1$



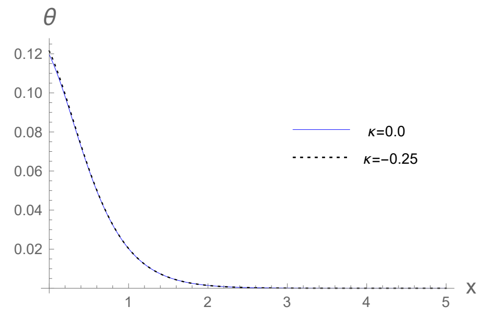
(a) Strain distribution for $\kappa_1 = 0.0$

(b) Strain distribution for different values of κ_1

Figure 6: Variation of the strain distribution e with the time parameter t at $\tau_o = 0.2$, $\beta = 0.5$ and ramp time $t_o = 0.1$

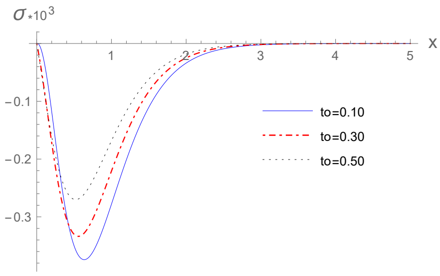


(a) Thermodynamic temperature for $\kappa_1 = 0.0$

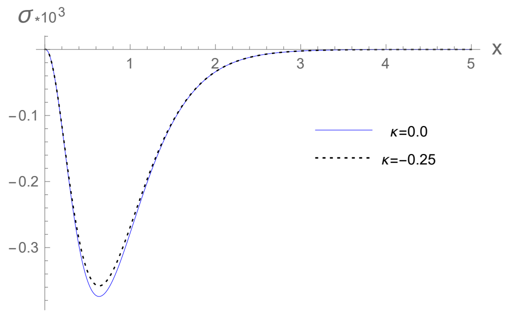


(b) Thermodynamical temperature at different values of κ_1

Figure 7: Variation of thermodynamical temperature θ with the ramp time parameter t_o at $t = 0.3, \tau_o = 0.2$ and $\beta = 0.5$

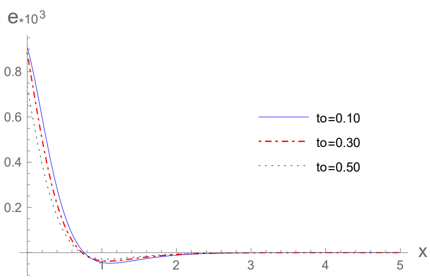


(a) Stress distribution for $\kappa_1 = 0.0$

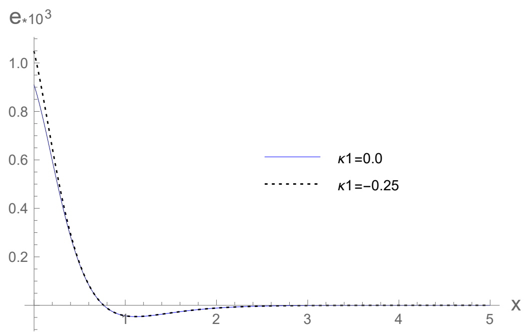


(b) Stress Distribution at different values of κ_1

Figure 8: Variation of stress component σ with the ramp time parameter t_o at $t = 0.3, \tau_o = 0.2$ and $\beta = 0.5$



(a) Strain distribution for $\kappa_1 = 0.0$



(b) Strain distribution at different values of values of κ_1

Figure 9: Variation of strain e with the ramp time parameter t_o at $t = 0.3, \tau_o = 0.2$ and $\beta = 0.5$

6. Conclusion

It is found that the thermodynamical temperature θ is independent of the change of the value of the fractional order parameter which means that there is no difference between the new theory of fractional order generalized thermoelasticity with fractional order strain equation and the generalized theory of thermoelasticity. The change in thermal conductivity is more significant in both the stress and in the thermodynamical temperature with the change of time and the fractional order parameter respectively than the other field functions. The increase in the thermal conductivity leads to slight increase in the absolute value of the amplitude of the strain. The behaviour of the field functions with different values of time in the present work shows that their behaviour within the context of theory of fractional order generalized thermoelasticity with fractional order strain equation is the same as in the generalized theory of thermoelasticity (without fractional order in the strain equation) which means that the present theory maintains the preferences of the generalized thermoelasticity theory that the propagation of waves is finite.

Notation

C_E	Specific heat at constant strain.
c_o	Longitudinal wave speed.
D_i	The components of electric displacement.
E_i	The components of electric field vector.
q_i	The components of the heat flux vector.
T	Absolute temperature.
T_o	Reference temperature
t	Time
t_o	Ramping time parameter.
u_i	Components of displacement vector.
v_i	The electric potential function.
α	Dimensionless thermoelastic coupling constant.
α_T	Coefficientρnt of linear thermal expansion
δ_{ij}	Kronecker delta function
ε	Dimensionless mechanical coupling const.
ζ	The entropy density
η	The thermal viscosity
θ	The dynamical temperature increment
λ, μ	Lame's constants
ρ	Mass density
σ_{ij}	The components of stress tensor.
σ	The principle stress component.
τ_o	One relaxation time parameter.
κ	Thermal diffusivity Coefficient.
ω	Dimensionless two temperature parameter.
β	Fractional order parameter.

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References

[1] M. A. Biot, Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. II. Higher Frequency Range, The Journal of the acoustical Society of america, Vol. 28 No.2 (1956) 168-178.

[2] W. Nowacki, Couple stresses in the theory of thermoelasticity I , Bulletin L' Academie Polonaise des Science, Serie des Sciences Technology, 14(1966) 129-138.

[3] Lord, H Wesley, and Y. Shulman. A generalized dynamical theory of thermoelasticity, Journal of the Mechanics and Physics of Solids 15.5 (1967) 299-309.

[4] Green, A. E., Lindsay, K. A. Thermoelasticity. Journal of Elasticity, 2(1), 1-7(1972)

[5] Müller, I. The coldness, a universal function in thermoelastic bodies. Archive for Rational Mechanics and Analysis, 41(5), 319-332 (1971)

[6] A. E. Green, K.A. Lindsay,Thermoelasticity, Journal of Elasticity, 2(1972) 1-7

[7] Suhubi, E.S., Thermoelastic solids, in: A. C. Eringen(Ed.), Continuum Physics II, Academic Press, New York, 1975(Chapter 2).

[8] J. Ignaczak, Generalized thermoelasticity and its applications. Thermal stresses, 3(1989) 279-354.

[9] D. S. Chandrasekharaiah, Hyperbolic thermoelasticity: a review of recent literature. Applied Mechanics Reviews, 51(12)(1998) 705-729.

[10] R. B. Hetnarski, J. Ignaczak, Generalized thermoelasticity. Journal of Thermal Stresses, 22(4-5), (1999) 451-476.

[11] R. B.Hetnarski, M. R. Eslami, Advanced theory and applications, J. Therm. Stresses,(2008).

[12] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley and Sons, New York (1993)

[13] A. Atangana, A. Secer, The time-fractional coupled-Korteweg-de-Vries equations. In Abstract and Applied Analysis (Vol. 2013) Hindawi (2013)

[14] E.C. De Oliveira , T.M. JA, A review of definitions for fractional derivatives and integral, Mathematical Problems in Engineering (2014)

[15] M. Caputo, Linear models of dissipation whose Q is almost frequency independent—II. Geophysical Journal International, 13(5)(1967) 529-539

[16] M. Caputo, Elasticitá e dissipazione (Elasticity and anelastic dissipation). Zanichelli, Bologna,(1969)

[17] M. Caputo, F. Mainardi, Linear models of dissipation in anelastic solids. La Rivista del Nuovo Cimento (1971-1977), 1(2),(1971) 161-198

[18] M. Caputo, F. Mainardi, A new dissipation model based on memory mechanism, Pure and Applied Geophysics, 91(1) (1971) 134-147

[19] M. Caputo, Vibrations of an infinite viscoelastic layer with a dissipative memory. The Journal of the Acoustical Society of America, 56(3)(1974) 897-904

[20] I. Podlubny, Geometric and Physical Interpretation of Fractional Integration and Fractional Differentiation, Fractional Calculus and Applied Analysis, 5, no.4(2002) 367-386

[21] R. Kimmich, Strange kinetics, porous media, and NMR. Chemical Physics, 284(1-2)(2002) 253-285

[22] Y. Fujita, Integrodifferential equation which interpolates the heat equation and the wave equation. Osaka Journal of Mathematics, 27(2) (1990) 309-321

[23] H. H. Sherief, A. M. A. El-Sayed, A. A. El-Latief, Fractional order theory of thermoelasticity. International Journal of Solids and structures, 47(2)(2010) 269-275

[24] Y. Povstenko, Theories of thermal stresses based on space–time-fractional telegraph equations. Computers Mathematics with Applications, 64(10)(2012) 3321-3328

[25] A. S. El-Karamany, M. A. Ezzat, On fractional thermoelasticity. Mathematics and Mechanics of Solids, 16(3)(2011) 334-346

[26] H. M. Youssef, Theory of fractional order generalized thermoelasticity. Journal of Heat Transfer, 132(6)(2010) 061301

[27] E. Bassiouny, Z. Abouelnaga, H. M. Youssef, One-dimensional thermoelastic problem of a laser pulse under fractional order equation of motion. Canadian Journal of Physics, 95(5)(2017) 464-471

[28] R. Hilfer, Applications of fractional calculus in physics World Scientific Publ. Co., Singapore,(2000)

[29] I. Podlubny, Fractional Differential Equations, Academic Press, New York,(1999)

[30] H. M. Youssef, Theory of generalized thermoelasticity with fractional order strain. Journal of Vibration and Control, 22(18)(2016) 3840-3857

[31] E. Bassiouny, H. M. Sandwich Structure Panel Subjected to Thermal Loading using Fractional Order Equation of Motion and Moving Heat Source, Canadian Journal of Physics,96(2) (2018) 174-182

[32] S. Zhang, F. Yu, Piezoelectric materials for high temperature sensors. Journal of the American Ceramic Society, 94(10)(2011) 3153-3170

[33] Z. M. Zhang, Nano/microscale heat transfer (No. Sirsi) i9780071436748)

[34] H. H. Sherief and A. M. Abd El-Latief, Effect of variable thermal conductivity on a half-space under the fractional order theory of thermoelasticity, Int. J. Mech. Sci. 74(2013) 185–189

[35] R. Hetnarski, , Thermal Stresses I. Amsterdam: North-Holland, (1986)

[36] E. Bassiouny, Thermo-elastic behavior of thin sandwich panel made of piezoelectric layers. Applied Mathematics and Computation, 218(20)(2012) 10009-10021

[37] G. Honig, U. Hirdes, A method for the numerical inversion of Laplace transforms. Journal of Computational and Applied Mathematics, 10(1)(1984) 113-132