

Solving Boundary Value Problem in 2D Using Finite Element and Finite Difference Method

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Abstract

This paper presents a simple and powerful approach to introducing boundary-value problems arising in electrostatics. In this paper Finite Element and Finite difference numerical method has been used to solve two dimensional steady heat flow problem with Dirichlet boundary conditions in a rectangular domain. Finite Difference solution with rectangular grid and Finite Element solution with triangular grid using spreadsheets is implemented here. Spreadsheets are used for solving electrostatic boundary-value problems. Finally comparisons are made between the solution obtained from the Finite Difference and Finite Element Method.

Key words: Laplace Equation, Finite Difference Method, Finite Element method.

I. INTRODUCTION

In science and engineering different phenomena are modeled by partial differential equations (PDEs). PDE is a function that depends on more than one variable (typically, space variables x, y, z and the temporal variable t). The heat equation, wave equation, and Laplace's equation are among the most prominent PDEs [1, 2]. This paper emphasizes numerical solutions to PDEs and suggests implementations through spreadsheets. This paper focuses on certain numerical methods for solving PDEs; in particular, the finite difference and the finite element methods. The examples presented in this paper include geometries that are sufficiently nontrivial for hand calculation or analytical solution, but reasonably manageable by using spreadsheets. Although specialized software is available for this purpose, oftentimes such sophistication tends to obscure the inner workings of the numerical methods employed in the solution of PDEs. Spreadsheets offer a transparent alternative perhaps proximate to hand calculation for students to better appreciate the numerical methods for solving PDEs and BVPs [1, 3, 4].

II. MATERIALS AND METHODS

II.1. The Finite Difference Method

The Finite Difference Method (FDM) is conceptually simple. The problems to which the method applies are specified by a PDE, geometry, and boundary conditions.

The Finite Difference Method follow three basic steps [5]:

- (1) Divide the solution region (geometry) into a grid of nodes. Grid points are typically arranged in a rectangular array of nodes.
- (2) Approximate the PDE and boundary conditions by a set of linear algebraic equations (the finite difference equations) on grid points within the solution region.
- (3) Solve this set of linear algebraic equations.

Consider the charge-free region depicted in Figure 1. The region has prescribed potentials along its boundaries. The region is divided into a rectangular grid of nodes, with the numbering of free nodes as indicated in the figure.

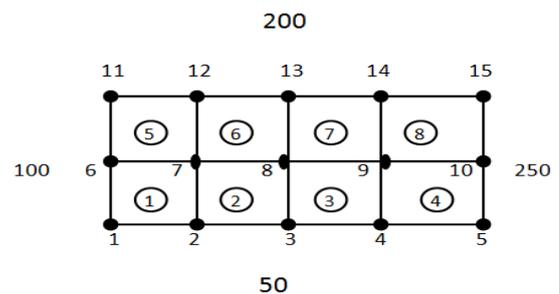


Figure 1: Charge-free region showing prescribed potentials at the boundaries and rectangular grid of free nodes to illustrate the finite difference method.

The potential $V = V(x,y)$ at an interior point (x,y) within the region is governed by the two-dimensional Laplace's equation

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad (1)$$

For $x=[0, a]$ and $y=[0, b]$ with $a=4, b=2$

Where $V(x,y)$ is the steady state temperature distribution in the domain.

Let the location of an interior grid point be identified by a pair of integers (i, j) , where i and j represent the position along the horizontal and vertical directions, respectively. For a grid having equal horizontal and vertical step sizes, the potential is given by the finite difference equation

$$V_{i,j} = \frac{1}{4} (V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}) \quad (2)$$

This system of equations may be solved by a variety of methods. In this section the Gauss-Seidel method is implemented in a spreadsheet to solve this system of equations. The Gauss-Seidel method is a relatively simple iterative method for solving systems such as those encountered in the finite difference formulation.

Boundary Potential

lower side	50
right side	250
left side	100
upper side	200

Figure 2: Input section of spreadsheet implementation of the finite difference method.

Gauss-Seidel Iteration

iteration	v1(=7)	v2(=8)	v3(=9)
0	0	0	0
1	87.5	62.5	125
2	103.125	115.625	140.625
3	116.4063	123.4375	153.9063
4	118.3594	130.0781	155.8594
5	120.0195	131.0547	157.5195
6	120.2637	131.8848	157.7637
7	120.4712	132.0068	157.9712
8	120.5017	132.1106	158.0017
9	120.5276	132.1259	158.0276
10	120.5315	132.1388	158.0315

Figure 3: The Gauss-Seidel iterations of the difference equations.

II.II. The Finite Element Method

The Finite Element Method (FEM) is a numerical technique for solving PDEs. FEM was originally applied to problems in structural mechanics. Unlike FDM, FEM is better suited for solution regions having irregularly shaped boundaries. The finite element analysis involves four basic steps [4, 5]:

- (1) Divide the solution region into a finite number of elements. The most common elements have triangular or quadrilateral shapes. The collection of all elements should resemble the original region as closely as possible.
- (2) Derive governing equations for a typical element. This step will determine the element coefficient matrix.

- (3) Assemble all elements in the solution region to obtain the global coefficient matrix.
- (4) Solve the resulting system of equations.

Consider the same charge-free region shown in Figure 1. The region is divided into 16 equal triangular elements as indicated in Figure 4. The elements are identified by encircled numbers 1 through 16. In this discretization there are 15 global nodes numbered 1 through 15.

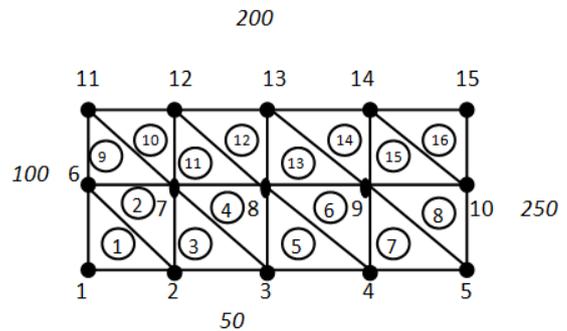


Figure 4: Finite element arrangement for electrostatic problem.

For each element *e* the following quantities are computed

$$P_1 = y_2 - y_3, P_2 = y_3 - y_1, P_3 = y_1 - y_2, \\ Q_1 = x_3 - x_2, Q_2 = x_1 - x_3, Q_3 = x_2 - x_1, \quad (3)$$

Where the subscripts refer to the local node numbers 1, 2, and 3 of element *e*. For example, in Figure 4, element 6 has global nodes 4, 9, and 8, which correspond, respectively to local nodes 1, 2, and 3.

With P_i and Q_i ($i = 1, 2, 3$) for element *e* thus computed, the entries of the 3 x 3 element coefficient matrix are then given by

$$C_{ij}^{(e)} = \frac{1}{4A} [P_i P_j + Q_i Q_j] \quad (i, j = 1, 2, 3) \quad (4)$$

Where

$$A = \frac{1}{2} [P_2 Q_3 - P_3 Q_2] \quad (5)$$

The element coefficient matrix is shown in Figure 6. The global coefficient matrix is then assembled from the element coefficient matrices. Since there are 15 nodes, the global coefficient matrix will be a 15 x 15 matrix shown in Figure 7. For example, node 7, which corresponds to the $C_{7,7}$ entry in the global coefficient matrix *C*, belongs to elements 2, 3, 4, 9, 10 and 11; since node 7 is assigned local node number 1 in elements 10 and 11, and local node number 2 in elements 2 and 9, and local number 3 in element 3 and 4 (as seen in the middle table of Figure 5), the corresponding global coefficient is

$$C_{7,7} = C_{1,1}^{(10)} + C_{1,1}^{(11)} + C_{2,2}^{(2)} + C_{2,2}^{(9)} + C_{3,3}^{(3)} + C_{3,3}^{(4)} \dots (6) \\ = 0.5 + 1 + 1 + 0.5 + 0.5 + 0.5 \\ = 4$$

Defining the vector of potentials v_f and v_p , where the subscripts f and p refer to nodes with free (unknown) potentials and prescribed potentials, respectively, the global coefficient matrix is then partitioned accordingly and the unknown potentials are obtained from

$$v_f = -C_{ff}^{-1} C_{fp} v_p \quad (7)$$

The matrices C_{fp} (free and prescribed node matrix) and C_{ff} (free node matrix) are formed by extracting the appropriate rows and columns from the global coefficient matrix C. In this case node 7, 8, 9 are the free node while 1, 2, 3, 4, 5, 6, 10, 11,12, 13, 14, and 15 are the nodes with prescribed potential, The result is shown in figure 8.

Microsoft Excell function MINVERSE and MMULT are used in the implementation of equation (7), the final results are shown in Figure 9.

C_{ff} matrix (free nodes)

Node	7	8	9
7	4	-1	0
8	-1	4	-1
9	0	-1	4

C_{fp} matrix (free and prescribed nodes)

Node	1	2	3	4	5	6	10	11	12	13	14	15
7	0	-1	0	0	0	-1	0	0	-1	0	0	0
8	0	0	-1	0	0	0	0	0	0	-1	0	0
9	0	0	0	-1	0	0	-1	0	0	0	-1	0

Figure 8: The matrix C_{ff} and C_{fp} are computed from Global Coefficient matrix.

$A^{-1} = C_{ff}^{-1}$			
Node	Inverse		
7	0.267857	0.071429	0.017857
8	0.071429	0.285714	0.071429
9	0.017857	0.071429	0.267857

$b = -C_{fp} * v_p$ (V_p =vector of prescribed potential)	
Node	Vector (b)
7	350
8	250
9	500

$v_f = A^{-1} * b$ (V_f = potential at free node)	
Node	Potential
7	120.5357
8	132.1429
9	158.0357

Figure 9: Final calculations section of finite element solution to electrostatic problem.

III. RESULTS AND DISCUSSION

In Sections II.I and II.II spreadsheet implementations of FDM and FEM were presented. As indicated in Table 1, the potentials at the free nodes computed by both methods compared fairly well. The node numbers in the table for FDM correspond to those in Figure 1, while those for FEM correspond to the node numbers shown in Figure 4.

Table 1: Comparison of results obtained from FDM and FEM.

Finite Difference		Finite Element	
Node	Potential	Node	Potential
7	120.5315	7	120.5357
8	132.1388	8	132.1429
9	158.0315	9	158.0357

IV. CONCLUSION

This paper presented spreadsheet implementations of two numerical methods for solving electrostatics problems. As indicated in above table the potentials at the free nodes were computed by using spreadsheet. Numerical solutions compared fairly well. The better agreement should be obtained between the finite element numerical solution results by using spreadsheet with a triangular grid and finite difference solution with a rectangular grid.

Spreadsheets may be considered as a viable alternative to enhancing education in other subjects and engineering fields.

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