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Related Fixed Point Theorems on Two Complete and Compact G-Metric Spaces

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Abstract
New results concerning the related fixed point theorems on two complete G-metric spaces are proved and deduced some corollaries. We prove also a related fixed point theorems on two compact G-metric spaces.
Keyword: Fixed point, Complete G-metric spaces, Compact G-metric spaces

1. INTRODUCTION

In [7],[8], Fisher proved some related fixed point theorems in two complete metric spaces which is as follows:

Theorem 1.1. *Let (X, d) and (Y, ρ) be a complete metric spaces. If T is a mapping from X into Y and S is a mapping from Y into X satisfying the following conditions:*

$$\rho(Tx, Tsy) \leq c \max\{d(x, Sy), \rho(y, Tx), \rho(y, STy)\}$$

$$d(Sy, STx) \leq c \max\{\rho(y, Tx), d(x, Sy), d(x, STx)\}$$

for all $x, y \in X$ where $c \in [0, 1)$, then ST have a unique fixed point w in Y. Further $Tz = w$ and $Sw = z$.

In [30], Popa extended the results of Fisher. Besides, Cho [6], extended and improved the results of Fisher [7],[8], and Popa [30]. Recently, related fixed point theorems on three complete metric spaces have been studied by Fisher and Rao [28-30], Nung [24], Jain and Rao[10-12], Jain and Dixit[9].

In 2006 Mustafa, and Sims, introduced the notion of generalized metric space called G-metric space [15]. In this generalization to every triplet of elements in the space assigned a non-negative real number. An analysis of the structure of these spaces was done in details in [15]. Subsequently, several authors proved many kind of fixed point theorems for contractive type mapping and expansive mapping in generalized metric spaces (see [1]-[3],[4-5],[13-14],[16-23],[25],[27],[31]). On the other hand, Rao [31], obtained the related fixed point theorems on three complete G-metric spaces.

In the first part of this paper, we prove some results concerning the related fixed point theorems on two complete G-metric spaces and deduce some corollaries. In the second part, we prove also a related fixed point theorems on two compact G-metric spaces. The results of this paper are new in G-metric spaces.

2. PRELIMINARIES

We recall some basic definitions and results which are important in the sequel. We refer to [19], for details on the following notions. Throughout this paper, $\mathbb{R}$ denotes the set of all real numbers, $\mathbb{R}^+$ denotes the set of nonnegative reals and $\mathbb{N}$ denotes the set of natural numbers.

Definition 2.1. *Let X be a non empty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following axioms:*

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$, for all x, $y \in X$, with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$, for all x, y, $z \in X$, with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all x, y, z, $a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Example 2.1. Define $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ by $G(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in X$. Then it is clear that $(\mathbb{R}, G)$ is a G-metric space.

Proposition 2.1. *Let (X, G)be a G-metric space. Then for any x, y, z and $a \in X$, it follows that:*

(1) if $G(x, y, z) = 0$ then $x = y = z$,

(2) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,

(3) $G(x, y, y) \leq 2G(y, x, x)$.

Definition 2.2. *Let (X, G) be a G-metric space, and (x_n) be a sequence of points of X, we say that (x_n) is G-convergent*

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to $x \in X$ if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 2.2. *Let (X, G) be a G-metric space. Then the following are equivalent:*

(1) (x_n) is G-convergent to x,

(2) $G(x_n, x_n, x) \longrightarrow 0$, as $n \longrightarrow \infty$,

(3) $G(x_n, x, x) \longrightarrow 0$, as $n \longrightarrow \infty$.

Definition 2.3. *Let (X, G) be a G-metric space, a sequence (x_n) is called G-Cauchy if given $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$.*

Definition 2.4. *A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).*

Definition 2.5. *A G-metric space (X, G) is said to be a compact G-metric space if it is G-complete and G-totally bounded.*

Definition 2.6. *Let (X, G_1) and (Y, G_2) be complete G-metric spaces, and let $f : (X, G_1) \longrightarrow (Y, G_2)$ be a function, then f is said to be G-continuous at a point $a \in X$, if given $\varepsilon > 0$, there exists $\delta > 0$, such that $x_1, x_2 \in X, G_1(a, x_1, x_2) < \delta$ implies $G_2(f(a), f(x_1), f(x_2)) < \varepsilon$.*

A function f is G-continuous on X if and only if, it is G-continuous at all $a \in X$.

Proposition 2.3. *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is continuous in all variables.*

3. RELATED FIXED POINT THEOREMS ON COMPLETE G-METRIC SPACES

Our main result follows:

Let $\Im$ be the set of all continuous real functions $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that:

(i) $g(0, 0, 0) = 0$

(ii) If $u^2 \leq g(uv, 0, 0)$ or $u^2 \leq g(0, uv, 0)$ or $u^2 \leq g(0, 0, uv)$, for all $u, v \in \mathbb{R}^+$, then there exists $0 \leq c < 1$ such that $u \leq \frac{1}{4}cv$.

Example 3.1. *If we define a function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$* by the following:

(a) $g(u, v, w) = \frac{1}{4}c \max\{uw, vu, wv\}$, for all $u, v, w \in \mathbb{R}^+$, where $0 \leq c < 1$,

(b) $g(u, v, w) = \frac{1}{4}(auw + bvu + cwv)$, for all $u, v, w, a, b, c \in \mathbb{R}^+$.

Then $g \in \Im$.

Theorem 3.1. *Let (X, G_1) and (Y, G_2) be complete G- metric spaces, and T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities:*

$$\begin{aligned}G_2^2(Tx, TSy_1, TSy_2) \leq g(&G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx),\\ &G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2),\\ &G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2))\end{aligned} \tag{3.1}$$

$$\begin{aligned}G_1^2(Sy_1, Sy_2, STx) \leq g(&G_1(x, x, STx)G_1(x, Sy_1, Sy_2),\\ &G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx),\\ &G_2(y_1, y_2, Tx)G_1(x, x, STx))\end{aligned} \tag{3.2}$$

for all x in X and y_1, y_2 in Y, where $g \in \Im$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. We define the sequences (x_n) in X, and (y_n) in Y by $x_n = (ST)^n x, y_n = T(ST)^{n-1}x$, for $n = 1, 2, ...$We will assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n. Applying the inequality (3.1) and using property (ii), we have

$$\begin{aligned}G_2^2(y_n, y_{n+1}, y_{n+1}) &= G_2^2(Tx_{n-1}, TSy_n, TSy_n) \leq\\ &g(G_2(y_n, TSy_n, TSy_n)G_2(y_n, y_n, Tx_{n-1}),\\ &G_2(y_n, y_n, Tx_{n-1})G_1(x_{n-1}, Sy_n, Sy_n),\\ &G_1(x_{n-1}, Sy_n, Sy_n)G_2(y_n, TSy_n, TSy_n))\\ &\leq g(0, 0, G_1(x_{n-1}, x_n, x_n)G_2(y_n, y_{n+1}, y_{n+1})),\end{aligned}$$

and it follows that

$$G_2^2(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4}cG_1(x_{n-1}, x_n, x_n)G_2(y_n, y_{n+1}, y_{n+1})$$

$$G_2(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4}cG_1(x_{n-1}, x_n, x_n) \tag{3.3}$$

Similarly, applying the inequality (3.2),

$$\begin{aligned}G_1^2(x_n, x_n, x_{n+1}) = G_1^2(Sy_n, Sy_n, STx_n)&\\ \leq g(&G_1(x_n, x_n, x_{n+1})G_1(x_n, Sy_n, Sy_n),\\ &G_1(x_n, Sy_n, Sy_n)G_2(y_n, y_n, Tx_n),\\ &G_2(y_n, y_n, Tx_n)G_1(x_n, x_n, x_{n+1}))\\ \leq g(&G_1(x_n, x_n, x_{n+1})G_1(x_n, x_n, x_n),\\ &G_1(x_n, x_n, x_n)G_2(y_n, y_n, y_{n+1}),\\ &G_2(y_n, y_n, y_{n+1})G_1(x_n, x_n, x_{n+1}))\end{aligned}$$

Using property (ii) and the Proposition(2.2), we have

$$G_1^2(x_n, x_n, x_{n+1}) \leq \frac{1}{4}cG_2(y_n, y_n, y_{n+1})G_1(x_n, x_n, x_{n+1})$$

$$\begin{aligned}&\frac{1}{2}G_1(x_n, x_{n+1}, x_{n+1}) \leq G_1(x_n, x_n, x_{n+1})\\ &\leq \frac{1}{4}cG_2(y_n, y_n, y_{n+1}) \leq \frac{1}{2}\\ &cG_2(y_n, y_{n+1}, y_{n+1})\end{aligned}$$

$$G_1(x_n, x_{n+1}, x_{n+1}) \leq cG_2(y_n, y_{n+1}, y_{n+1}) \tag{3.4}$$

Now it follows from the inequalities (3.3) and (3.4) that

$$G_1(x_n, x_{n+1}, x_{n+1}) \leq \frac{1}{4}c^2 G_1(x_{n-1}, x_n, x_n).$$

Hence, by induction we get

$$G_1(x_n, x_{n+1}, x_{n+1}) \leq (\frac{1}{4})^n c^{2n} G_1(x, x_1, x_1), \text{ for } n = 1, 2, \cdots \tag{3.5}$$

So (x_n) and (y_n) are G-Cauchy sequences with limits z in X and w in Y. Using the inequality (3.1), we have

$$\begin{aligned}
&G_2^2(Tz, y_n, y_n)\\
&\quad = G_2^2(Tz, TSy_{n-1}, TSy_{n-1})\\
&\quad \leq g(G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1})G_2(y_{n-1}, y_{n-1}, Tz),\\
&\quad G_2(y_{n-1}, y_{n-1}, Tz)G_1(z, Sy_{n-1}, Sy_{n-1}), G_1(z, Sy_{n-1},\\
&\quad Sy_{n-1})G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1}))\\
&\quad \leq g\ (G_2(y_{n-1}, y_n, y_n)G_2(y_{n-1}, y_{n-1}, Tz),\\
&\quad G_2(y_{n-1}, y_{n-1}, Tz)G_1(z, x_{n-1}, x_{n-1}),\\
&\quad G_1(z, x_{n-1}, x_{n-1})G_2(y_{n-1}, y_n, y_n)).\\
&\quad G_2^2(Tz, w, w) \leq g(0, 0, 0) = 0,
\end{aligned}$$

it follows that $G_2(Tz, w, w) = 0$, hence $w = Tz$. Using the inequality (3.2), we have

$$\begin{aligned}
&G_1^2(Sw, Sw, x_n) = G_1^2(Sw, Sw, STx_{n-1})\\
&\quad \leq g(G_1(x_{n-1}, x_{n-1}, STx_{n-1})G_1(x_{n-1}, Sw, Sw),\\
&\qquad G_1(x_{n-1}, Sw, Sw)G_2(w, w, Tx_{n-1}),\\
&\qquad G_2(w, w, Tx_{n-1})G_1(x_{n-1}, x_{n-1}, STx_{n-1})).
\end{aligned}$$

Letting n tends to infinity and using (i), we have $G_1^2(Sw, Sw, x_n) \leq g(0, 0, 0) = 0$, and it follows that $z = Sw$. Thus $STz = Sw = z, TSw = Tz = w$, and so ST has a fixed point z and TS has a fixed point w. To prove uniqueness, suppose that ST has a second fixed point z_1 and TS has a second fixed point w_1. Then applying the inequality (3.1) and using property (ii), we have

$$\begin{aligned}
&G_2^2(w, w_1, w_1)\\
&\quad = G_2^2(TSw, TSw_1, TSw_1)\\
&\quad = G_2^2(Tz, TSw_1, TSw_1)\\
&\quad \leq g(G_2(w_1, TSw_1, TSw_1)G_2(w_1, w_1, Tz),\\
&\qquad G_2(w_1, w_1, Tz)G_1(z, Sw_1, Sw_1),\\
&\qquad G_1(z, Sw_1, Sw_1)G_2(w_1, TSw_1, TSw_1))\\
&\quad \leq g(0, G_2(w_1, w_1, w)G_1(Sw, Sw_1, Sw_1), 0),\\
&it\ follows\ that\\
&\quad G_2^2(w, w_1, w_1) \leq \tfrac{1}{4}cG_1(Sw, Sw_1, Sw_1)G_2(w_1, w_1, w),\\
&\quad G_2(w, w_1, w_1) \leq \tfrac{1}{4}cG_1(Sw, Sw_1, Sw_1).
\end{aligned} \tag{3.6}$$

Further, applying the inequality (3.2) and using property (ii), we have

$$\begin{aligned}
&G_1^2(Sw, Sw, Sw_1) = G_1^2(STSw, STSw, STSw_1) \leq\\
&g(G_1(Sw_1, Sw_1, STSw_1)G_1(Sw_1, STSw, STSw),\\
&G_1(Sw_1, STSw, STSw)G_2(TSw, TSw, TSw_1),\\
&G_2(TSw, TSw, TSw_1)G_1(Sw_1, Sw_1, STSw_1))\\
&\leq g(0, G_1(Sw_1, Sw, Sw)G_2(w, w, w_1), 0)
\end{aligned}$$

which implies that

$$\begin{aligned}
&G_1^2(Sw, Sw, Sw_1) \leq \tfrac{1}{4}cG_2(w, w, w_1)G_1(Sw, Sw, Sw_1)\\
&G_1(Sw, Sw, Sw_1) \leq \tfrac{1}{4}cG_2(w, w, w_1)\\
&G_1(Sw, Sw, Sw_1) \leq \tfrac{1}{4}cG_2(w, w, w_1),
\end{aligned} \tag{3.7}$$

again by using the Proposition (2.2), we get,

$$\begin{aligned}
&\frac{1}{2}G_1(Sw, Sw_1, Sw_1)\\
&\leq G_1(Sw, Sw, Sw_1) \leq \frac{1}{4}cG_2(w, w, w_1) \leq \frac{1}{2}cG_2(w, w_1, w_1)
\end{aligned}$$

$$G_1(Sw, Sw_1, Sw_1) \leq cG_2(w, w_1, w_1). \tag{3.8}$$

Now it follows from the inequalities (3.6) and (3.8) that

$$\begin{aligned}
G_2(w, w_1, w_1) &\leq \frac{1}{4}cG_1(Sw, Sw_1, Sw_1)\\
&< \frac{1}{4}c^2 G_2(w, w_1, w_1) < G_2(w, w_1, w_1)
\end{aligned}$$

and so $w = w_1$ since $c < 1$. The fixed point w of TS must be a unique. Now $TSz_1 = z_1$ implies $TSTz_1 = Tz_1$ and so $Tz_1 = w$. Thus $z = STz = Sw = STz_1 = z_1$, proving that z is a unique fixed point of ST. Thus the proof of the Theorem is completes. □

We have the following Corollaries:

Corollary 3.2. *Let (X, G_1) and (Y, G_2) be complete G-metric spaces, and T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities:*

$$\begin{aligned}
&G_2^2(Tx, TSy_1, TSy_2)\ \leq \frac{1}{4}c\max\{G_2(y_1, TSy_1, TSy_2)\\
&G_2(\ y_1, y_2, Tx\ ), G_2(y_1, y_2, Tx)G_1(x\ , Sy_1, Sy_2),\\
&G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2)\}
\end{aligned}$$

$$\begin{aligned}
&G_1^2(Sy_1, Sy_2, STx\ ) \leq \frac{1}{4}c\max\{G_1(x, x, STx)G_1(x, Sy_1, Sy_2),\\
&G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx), G_2(y_1, y_2, Tx)G_1(x, x, STx)\}
\end{aligned}$$

for all x in X and y_1, y_2 in Y, $0 \leq c < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. It is immediate to see that, if we take a function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$, by $g(u, v, w) = \frac{1}{4}c\max\{uw, vu, wv\}$, for

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all $u, v, w \in \mathbb{R}^+$, $where\ 0 \leq c < 1$, then from Example (3.1)(a) it follows that $g \in \Im$ and by the Theorem (3.1), the Corollary follows. □

Corollary 3.3. *Let (X, G_1) and (Y, G_2) be complete G- metric spaces, and T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities:*

$$G_2^2(Tx, TSy_1, TSy_2) \leq \frac{1}{4}(a_1G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx) + b_1G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2)+$$

$$c_1G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2))$$

$$G_1^2(Sy_1, Sy_2, STx) \leq \frac{1}{4}(a_2G_1(x, x, STx)G_1(x, Sy_1, Sy_2) + b_2G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx)+$$

$$c_2G_2(y_1, y_2, Tx)G_1(x, x, STx))$$

for all x in X and y_1, y_2 in Y, $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+$ with $(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Now, we give an example to illustrate Theorem(3.1).

Example 3.2. Let $X = Y = [1, \infty)$, we define on X and Y the G_1-metric space and the G_2-metric space as follows:

$$G_1(x_1, x_2, x_3) = \max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}, \text{ with } x_1, x_2, x_3 \in X$$

$$G_2(y_1, y_2, y_3) = \frac{\sqrt{2}}{16}\max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\}, \text{ with } y_1, y_2, y_3 \in Y.$$

Let T and S defined by $Tx = 2x - 1$ and $Sy = y$, we have

$$G_2^2(Tx, TSy, TSy) = G_2^2(Tx, Ty, Ty) = (\frac{\sqrt{2}}{16})^2\,|Tx - Ty|\,|Tx - Ty| = \frac{1}{4}\frac{\sqrt{2}}{2}G_1(x, Sy, Sy)G_2(y, Ty, Ty)$$

$$= \frac{1}{4}c\max\{0, 0, G_1(x, Sy, Sy)G_2(y, Ty, Ty)\} = g(0, 0, G_1(x, Sy, Sy)G_2(y, Ty, Ty))$$

then ST and TS have the unique fixed point 1.

Theorem 3.4. *Let (X, G_1) and (Y, G_2) be complete G- metric spaces, and T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities:*

$$G_2^3(Tx, TSy_1, TSy_2) \leq \frac{1}{4}c_1\max\{G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2)G_2(y_1, TSy_1, TSy_2), \tag{3.9}$$

$$G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx), G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx)G_2(y_1, y_2, Tx)\}$$

$$G_1^3(Sy_1, Sy_2, STx) \leq \frac{1}{4}c_2\max\{G_2(y_1, y_2, Tx)G_1(x, x, STx)G_1(x, x, STx), \tag{3.10}$$

$$G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2), G_1(x, x, STx)G_1(x, Sy_1, Sy_2)G_1(x, Sy_1, Sy_2)\}$$

for all x in X and y_1, y_2 in Y, where $0 \leq c_1c_2 < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. We define the sequences (x_n) in X, and (y_n) in Y, by $x_n = (ST)^n x$, $y_n = T(ST)^{n-1}x$, for $n = 1, 2, ...$We will assume that $x_n \neq x_{n+1}$ and $y_n \neq y_{n+1}$ for all n. Applying the inequality (3.9), we have

$$G_2^3(y_n, y_{n+1}, y_{n+1}) = G_2^3(Tx_{n-1}, TSy_n, TSy_n) \leq \frac{1}{4}c_1\max\{G_1(x_{n-1}, Sy_n, Sy_n)G_2(y_n, TSy_n, TSy_n)G_2(y_n, TSy_n, TSy_n),$$

$$G_2(y_n, y_n, Tx_{n-1})G_1(x_{n-1}, Sy_n, Sy_n)G_2(y_n, y_n, Tx_{n-1}), G_2(y_n, TSy_n, TSy_n)G_2(y_n, y_n, Tx_{n-1})G_2(y_n, y_n, Tx_{n-1})\}$$

$$\leq \frac{1}{4}c_1\max\{G_1(x_{n-1}, x_n, x_n)G_2(y_n, y_{n+1}, y_{n+1})G_2(y_n, y_{n+1}, y_{n+1}), 0, 0\}.$$

It follows that

$$G_2^3(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4}c_1G_1(x_{n-1}, x_n, x_n)G_2(y_n, y_{n+1}, y_{n+1})G_2(y_n, y_{n+1}, y_{n+1})$$

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$$G_2(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4}cG_1(x_{n-1}, x_n, x_n) \tag{3.11}$$

Applying the inequality (3.10), and using the Proposition (2.2), we get

$$G_1^3(x_n, x_n, x_{n+1}) = G_1^3(Sy_n, Sy_n, STx_n) \leq \frac{1}{4}c_2 \max\{G_2(y_n, y_n, Tx_n)G_1(x_n, x_n, x_{n+1})G_1(x_n, x_n, x_{n+1}),$$

$$G_1(x_n, Sy_n, Sy_n)G_2(y_n, y_n, Tx_n)G_1(x_n, Sy_n, Sy_n), G_1(x_n, x_n, x_{n+1})G_1(x_n, Sy_n, Sy_n)G_1(x_n, Sy_n, Sy_n)\}$$

$$\leq \frac{1}{4}c_2 \max\{G_2(y_n, y_n, y_{n+1})G_1(x_n, x_n, x_{n+1})G_1(x_n, x_n, x_{n+1}),$$

$$G_1(x_n, x_n, x_n)G_2(y_n, y_n, y_{n+1})G_1(x_n, x_n, x_n), G_1(x_n, x_n, x_{n+1})G_1(x_n, x_n, x_n)G_1(x_n, x_n, x_n)\}$$

$$G_1^3(x_n, x_n, x_{n+1}) \leq \frac{1}{4}c_2G_2(y_n, y_n, y_{n+1})G_1(x_n, x_n, x_{n+1})G_1(x_n, x_n, x_{n+1})$$

$$\frac{1}{2}G_1(x_n, x_{n+1}, x_{n+1}) \leq G_1(x_n, x_n, x_{n+1}) \leq \frac{1}{4}c_2G_2(y_n, y_n, y_{n+1}) \leq \frac{1}{2}c_2G_2(y_n, y_{n+1}, y_{n+1}). \tag{3.12}$$

Now it follows from the inequalities (3.11) and (3.12) that

$$G_1(x_n, x_{n+1}, x_{n+1}) \leq c_2G_2(y_n, y_{n+1}, y_{n+1}) \leq \frac{1}{4}c_1c_2G_1(x_{n-1}, x_n, x_n). \tag{3.13}$$

Hence, by induction we get

$$G_1(x_n, x_{n+1}, x_{n+1}) \leq (\frac{1}{4})^n(c_2c_1)^nG_1(x, x_1, x_1),\ for\ n = 1, 2, \cdots$$

Since $c_2c_1 < 1$, it follows that x_n and y_n are G-Cauchy sequences with limits z in X and w in Y. Using the inequality (3.9), we have

$$G_2^3(Tz, y_n, y_n) = G_2^3(Tz, TSy_{n-1}, TSy_{n-1})$$

$$\frac{1}{4}c_1 \max\{G_1(z, Sy_{n-1}, Sy_{n-1})G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1})G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1}),$$

$$G_2(y_{n-1}, y_{n-1}, Tz)G_1(z, Sy_{n-1}, Sy_{n-1})G_2(y_{n-1}, y_{n-1}, Tz),$$

$$G_2(y_{n-1}, TSy_{n-1}, TSy_{n-1})G_2(y_{n-1}, y_{n-1}, Tz)G_2(y_{n-1}, y_{n-1}, Tz)\}$$

$$\leq \frac{1}{4}c_1 \max\{G_1(z, x_{n-1}, x_{n-1})G_2(y_{n-1}, y_n, y_n)G_2(y_{n-1}, y_n, y_n), G_2(y_{n-1}, y_{n-1}, Tz)G_1(z, x_{n-1}, x_{n-1})G_2(y_{n-1}, y_{n-1}, Tz),$$

$$G_2(y_{n-1}, y_n, y_n)G_2(y_{n-1}, y_{n-1}, Tz)G_2(y_{n-1}, y_{n-1}, Tz)\}$$

Letting $n \to \infty$, we have $G_2^3(Tz, w, w) \leq 0$, it follows that $G_2(Tz, w, w) = 0$, hence $w = Tz$. Using the inequality (3.10), we obtain

$$G_1^3(Sw, Sw, x_n) = G_1^3(Sw, Sw, STx_{n-1}) \leq$$

$$\frac{1}{4}c_2 \max\{G_2(w, w, Tx_{n-1})G_1(x_{n-1}, x_{n-1}, STx_{n-1})G_1(x_{n-1}, x_{n-1}, STx_{n-1}),$$

$$G_1(x_{n-1}, Sw, Sw)G_2(w, w, Tx_{n-1})G_1(x_{n-1}, Sw, Sw), G_1(x_{n-1}, x_{n-1}, STx_{n-1})G_1(x_{n-1}, Sw, Sw)G_1(x_{n-1}, Sw, Sw)\}.$$

Letting n tends to infinity, we have $G_1^3(Sw, Sw, x_n) \leq 0$, and it follows that $z = Sw$. Thus $STz = Sw = z, TSw = Tz = w$, and so ST has a fixed point z and TS has a fixed point w. Now suppose that ST has a second fixed point z_1 and TS has a second fixed point w_1. Then using the inequality (3.9) and property (ii), we have

$$G_2^3(w, w_1, w_1) = G_2^3(TSw, TSw_1, TSw_1) = G_2^3(Tz, TSw_1, TSw_1) \leq$$

$$\frac{1}{4}c_1 \max\{G_1(z, Sw_1, Sw_1)G_2(w_1, TSw_1, TSw_1)G_2(w_1, TSw_1, TSw_1),$$

$$G_2(w_1, w_1, Tz)G_2(w_1, w_1, Tz)G_1(z, Sw_1, Sw_1),$$

$$G_2(w_1, TSw_1, TSw_1)G_2(w_1, w_1, Tz)G_2(w_1, w_1, Tz)\}$$

$$\leq \frac{1}{4}c_1 \max\{0, G_2(w_1, w_1, w)G_1(Sw, Sw_1, Sw_1)G_2(w_1, w_1, w), 0\},$$

and so $G_2^3(w, w_1, w_1) \leq \frac{1}{4}cG_1(Sw, Sw_1, Sw_1)G_2(w_1, w_1, w)G_2(w_1, w_1, w)$

$$G_2(w, w_1, w_1) \leq \frac{1}{4}cG_1(Sw, Sw_1, Sw_1). \tag{3.14}$$

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Applying the inequality (3.10), Proposition (2.2) we have

$$G_1^3(Sw, Sw, Sw_1) = G_1^3(STSw, STSw, STSw_1) \leq$$

$$\frac{1}{4}c_2 \max\{G_2(TSw, TSw, TSw_1)G_1(Sw_1, Sw_1, STSw_1)G_1(Sw, Sw, STSw),$$

$$G_1(Sw_1, STSw, STSw)G_2(TSw, TSw, TSw_1)G_1(Sw_1, STSw, STSw),$$
$$G_1(Sw_1, Sw_1, STSw_1)G_1(Sw_1, STSw, STSw)G_1(Sw_1, STSw, STSw)\}$$

$$\leq \frac{1}{4}c_2 \max\{0, G_1(Sw_1, Sw, Sw)G_1(Sw_1, Sw, Sw)G_2(w, w, w_1), 0\}$$

$$G_1^3(Sw, Sw, Sw_1) \leq \frac{1}{4}c_2 G_2(w, w, w_1)G_1(Sw, Sw, Sw_1)G_1(Sw, Sw, Sw_1)$$

$$\frac{1}{2}G_1(Sw, Sw_1, Sw_1) \leq G_1(Sw, Sw, Sw_1) \leq \frac{1}{4}c_2 G_2(w, w, w_1) \leq \frac{1}{2}c_2 G_2(w, w_1, w_1) \tag{3.15}$$

$$G_1(Sw, Sw_1, Sw_1) \leq c_2 G_2(w, w_1, w_1) \tag{3.16}$$

Now it follows from the inequalities (3.14) and (3.16) that

$$G_2(w, w_1, w_1) \leq \frac{1}{4}c_1 G_1(Sw, Sw_1, Sw_1) < \frac{1}{4}c_1 c_2 G_2(w, w_1, w_1) < G_2(w, w_1, w_1)$$

and so $w = w_1$ since $c_1 c_2 < 1$. The fixed point w of TS must be a unique. Now $TSz_1 = z_1$ implies $TSTz_1 = Tz_1$ and so $Tz_1 = w$. Thus $z = STz = Sw = STz_1 = z_1$, proving that z is the unique fixed point of ST. This completes the proof of the Theorem. □

Corollary 3.5. *Let (X, G_1) and (Y, G_2) be complete G- metric spaces, and T be a mapping of X into Y and let S be a mapping of Y into X satisfying the inequalities:*

$$G_2^3(Tx, TSy_1, TSy_2) \leq \frac{1}{4}(a_1 G_1(x\ , Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2)G_2(y_1, TSy_1, TSy_2)+$$

$$b_1 G_2(y_1, y_2, Tx)G_1(x\ , Sy_1, Sy_2)G_2(y_1, y_2, Tx) + c_1 G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx\ )G_2(y_1, y_2, Tx))$$

$$G_1^3(Sy_1, Sy_2, STx\ ) \ \leq \frac{1}{4}\ (a_2 G_2(y_1, y_2, Tx)G_1(x, x, STx)G_1(x, x, STx){+}b_2 G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2)+$$

$$c_2 G_1(x, x, STx)G_1(x, Sy_1, Sy_2)G_1(x, Sy_1, Sy_2))$$

for all x in X and y_1, y_2 in Y , $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+$ with $(a_1 + b_1 + c_1)(a_2 + b_2 + c_2) < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Example 3.3. Let $X = Y = [1, \infty)$, we define on X and Y the G_1-metric space and the G_2-metric space as follows:

$$G_1(x_1, x_2, x_3) = \max\{|x_1 - x_2|, |x_2 - x_3|, |x_3 - x_1|\}, \text{ with } x_1, x_2, x_3 \in X$$

$$G_2(y_1, y_2, y_3) = \frac{\sqrt{4}}{48}\max\{|y_1 - y_2|, |y_2 - y_3|, |y_3 - y_1|\}, \text{ with } y_1, y_2, y_3 \in Y.$$

Let T and S defined by $Tx = 3x - 2$ and $Sy = y$, we have

$$G_2^3(Tx, TSy, TSy) = G_2^3(Tx, Ty, Ty) = 3(\frac{\sqrt{4}}{48})^2\,|x - y|\,|Tx - Ty| = \frac{1}{4}\frac{\sqrt{4}}{4}G_1(x, Sy, Sy)G_2(y, Ty, Ty)G_2(y, Ty, Ty)$$

$$= \frac{1}{4}c \max\{G_1(x, Sy, Sy)G_2(y, Ty, Ty)G_2(y, Ty, Ty), 0, 0\}$$

then ST and TS have a unique fixed point 1.

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4. RELATED FIXED POINT THEOREMS ON COMPACT G-METRIC SPACES

In this section, we prove an analogous results for compact G-metric spaces.

Let $\Im^*$ denotes the set of all real functions $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that :

(i)* If $u^2 < g(uv, 0, 0)$ or $u^2 < g(0, uv, 0)$ or $u^2 < g(0, 0, uv)$,for all $u, v \in R^+$, then $u < \frac{1}{2}v$.

Theorem 4.1. *Let (X, G_1) and (Y, G_2) be compact G- metric spaces, and T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities:*

$$G_2^2(Tx, TSy_1, TSy_2) < g(G_2(y_1, TSy_1, TSy_2)G_2(y_1, y_2, Tx\ ), G_2(y_1, y_2, Tx)G_1(x, Sy_1, Sy_2), \tag{4.1}$$

$$G_1(x, Sy_1, Sy_2)G_2(y_1, TSy_1, TSy_2))$$

for all x in X and y_1, y_2 in Y with $x \neq Sy_1$, and $x \neq Sy_2$, where $g \in \Im^*$,and

$$G_1^2(Sy_1, Sy_2, STx) < g(G_1(x, x, STx)G_1(x, Sy_1, Sy_2), G_1(x, Sy_1, Sy_2)G_2(y_1, y_2, Tx), \tag{4.2}$$

$$G_2(y_1, y_2, Tx)G_1(x, x, STx))$$

for all x in X and y_1, y_2 in Y, where $g \in \Im^*$with $y_1 \neq Tx, y_2 \neq Tx$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in$Y$. Further, $Tz = w$ and $Sw = z$.

Proof. Let $\psi : X \to R^+$ defined by $\psi(x) = G_1(x, STx, STx)$ is G-continuous on X. Since X is compact, there exists a point u in X such that $\psi(u) = G_1(u, STu, STu) = \min\{G_1(x, STx, STx); x \in X\}$. Now suppose that $Tu \neq TSTu$, then $u \neq STu$. Put $y_1 = y_2 = Tu, x = Sy = STu$ in the inequality (4.2), we have

$$G_1^2(STu, STu, STSTu) < g(G_1(STu, STu, STSTu)G_1(STu, STu, STu),$$

$$G_1(STu, STu, STu)G_2(Tu, Tu, TSTu), G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu))$$

$$< g(0, 0, G_2(Tu, Tu, TSTu)G_1(STu\ , STu, STSTu)).$$

Using condition (i)* and Proposition(2.2) we have

$$G_1^2(STu, STu, STSTu\ ) < \frac{1}{2}G_2(Tu, Tu, TSTu)G_1(STu, STu, STSTu)$$

$$G_1(STu, STu, STSTu\ ) < \frac{1}{2}G_2(Tu, Tu, TSTu) < G_2(Tu, TSTu, TSTu)$$

Put $y_1 = y_2 = Tu, x = u$ in the inequality (4.1), we have

$$G_2^2(Tu, TSTu, TSTu) < g(G_2(Tu, TSTu, TSTu)G_2(Tu, Tu, Tu\ ),$$

$$G_2(Tu, Tu, Tu\ )G_1(u, STu, STu), G_1(u, STu, STu)G_2(Tu, TSTu, TSTu))$$

$$< g(0, 0, G_1(u, STu, STu)G_2(Tu, TSTu, TSTu))$$

But using condition (i)*, we get

$$G_2^2(Tu, TSTu, TSTu) < \frac{1}{2}G_1(u, STu, STu)G_2(Tu, TSTu, TSTu),$$

$$G_2(Tu, TSTu, TSTu) < \frac{1}{2}G_1(u, STu, STu)$$

$$\frac{1}{2}G_1(STu, STSTu, STSTu) \leq G_1(STu, STu, STSTu\ ) < \frac{1}{2}G_1(u, STu, STu)$$

$$G_1(STu, STSTu, STSTu) < G_1(u, STu, STu).$$

Hence $\psi(STu) < \psi(u)$, and this gives us a contradiction. So $TSTu = Tu$. If putting $Tu = w$ and $Sw = z$, then we get $ST(STu) = S(TSTu) = STu = Sw = z$, and $w = Tu = TS(Tu) = T(STu) = Tz$. Thus, $Sw = z$ is a fixed point of ST and $Tz = w$ is a fixed point of TS. To prove uniqueness, suppose that ST has a second distinct fixed point z_1. Then applying the inequality (4.2) and using condition (i)*, we have

$$G_1^2(z, z, z_1) = G_1^2(STz, STz, STz_1) < g(G_1(z_1, z_1, STz_1)G_1(z_1, z, z),$$

$$G_1(z_1,z,z)G_2(Tz,Tz,Tz_1),G_2(Tz,Tz,Tz_1)G_1(z_1,z_1,STz_1)).$$

It follows that

$$G_1^2(z,z,z_1) < \frac{1}{2}G_2(Tz,Tz,Tz_1)G_1(z_1,z,z)$$

$$G_1(z,z,z_1) < \frac{1}{2}G_2(Tz,Tz,Tz_1)$$

Further, applying the inequality (4.1) and using condition (i)*we have,

$$G_2^2(Tz,Tz_1,Tz_1) = G_2^2(Tz,TSTz_1,TSTz_1) <$$
$$g(G_2(T\,z_1,TSTz_1,TSTz_1)G_2(Tz_1,Tz_1,Tz\,),G_2(Tz_1,Tz_1,Tz\,)G_1(z,STz_1,STz_1),$$
$$G_1(z,STz_1,STz_1)G_2(T\,z_1,TSTz_1,TSTz_1)) = g(0,G_2(Tz_1,Tz_1,Tz)G_1(z,z_1,z_1),0),$$
$$G_2^2(Tz,Tz_1,Tz_1) < \frac{1}{2}G_1(z,z_1,z_1)G_2(Tz_1,Tz_1,Tz),$$
$$\frac{1}{2}G_2(Tz_1,Tz,Tz) \le G_2(Tz,Tz_1,Tz_1) < \frac{1}{2}G_1(z,z_1,z_1).$$

Now, it follows that $G_1(\ z,\ z,z_1) < \frac{1}{2}G_1(\ z,\ z_1,z_1) \le G_1(\ z,\ z,z_1)$, this is a contradiction and so the fixed point z must be a unique. Similarly, w is a unique fixed point of TS. This completes the proof of the Theorem. □

Corollary 4.2. *Let (X,G_1) and (Y,G_2) be compact G- metric spaces, and T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities:*

$$G_2^2(Tx,TSy_1,TSy_2) \ < \ \frac{1}{2}\,max\{G_2(y_1,TSy_1,TSy_2)G_2(y_1,y_2Tx\,),G_2(y_1,y_2,Tx\,)G_1(x,Sy_1,Sy_2),$$
$$G_1(x,Sy_1,Sy_2)G_2(y_1,TSy_1,TSy_2)\},$$

for all x in X and y_1,y_2 in Y with $x\neq Sy_1$, and $x\neq Sy_2$, and

$$G_1^2(Sy_1,Sy_2,STx) \ < \ \frac{1}{2}\,max\{G_1(x,x,STx)G_1(x,Sy_1,Sy_2),G_1(x,Sy_1,Sy_2)G_2(y_1,y_2,Tx),$$
$$G_2(y_1,y_2,Tx)G_1(x,x,STx)\},$$

for all x in X and y_1,y_2 in Y, with $y_1\neq Tx,y_2\neq Tx$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz=w$ and $Sw=z$.

Corollary 4.3. *Let (X,G_1) and (Y,G_2) be compact G- metric spaces, and T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities:*

$$G_2^2(Tx,TSy_1,TSy_2) < \ \frac{1}{2}(a_1G_2(y_1,TSy_1,TSy_2)G_2(y_1,y_2,Tx\,)+b_1G_2(y_1,y_2,Tx)G_1(x,Sy_1,Sy_2)+$$
$$c_1G_1(x,Sy_1,Sy_2)G_2(y_1,TSy_1,TSy_2))$$

for all x in X and y_1,y_2 in Y with $x\neq Sy_1$, and $x\neq Sy_2$, and

$$G_1^2(Sy_1,Sy_2,STx\,) < \frac{1}{2}\,(a_2G_1(x,x,STx)G_1(x,Sy_1,Sy_2)+b_2G_1(x,Sy_1,Sy_2)G_2(y_1,y_2,Tx)+$$
$$c_2G_2(y_1,y_2,Tx)G_1(x,x,STx))$$

for all x in X and y_1,y_2 in Y, with $y_1\neq Tx$, $y_2\neq Tx$, and $a_1,a_2,b_1,b_2,c_1,c_2\in\mathbb{R}^+$ with $(a_1+b_1+c_1)(a_2+b_2+c_2)<1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz=w$ and $Sw=z$.

We give an example to support Theorem(4.1).

Example 4.1. Let $X=Y=[0,1]$, we define on X and Y the G_1-metric space and the G_2-metric space as follows:

$$G_1(x_1,x_2,x_3)=\max\{|x_1-x_2|,|x_2-x_3|,|x_3-x_1|\},\text{ with }x_1,x_2,x_3\in X$$

$$G_2(y_1,y_2,y_3)=\frac{\sqrt{3}}{9}\max\{|y_1-y_2|,|y_2-y_3|,|y_3-y_1|\},\text{ with }y_1,y_2,y_3\in Y.$$

Let T and S defined by $Tx = \frac{3}{4}x^2$ and $Sy = y$, we have

$$G_2^2(Tx,TSy,TSy) = G_2^2(Tx,Ty,Ty) \le \frac{3}{2}(\frac{\sqrt{3}}{9})^2\,|x-y|\,|Tx-Ty| = \frac{1}{2}\frac{\sqrt{3}}{3}G_1(x,Sy,Sy)G_2(y,Ty,Ty)$$

$$< \frac{1}{2}\max\{0,0,G_1(x,Sy,Sy)G_2(y,Ty,Ty)\} = g(0,0,G_1(x,Sy,Sy)G_2(y,Ty,Ty)),$$

then ST and TS have the unique fixed point 0.

Theorem 4.4. *Let (X,G_1) and (Y,G_2) be compact G- metric spaces, and T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities:*

$$G_2^3(Tx,TSy_1,TSy_2) < \frac{1}{2}\max\{G_1(x,Sy_1,Sy_2)G_2(y_1,TSy_1,TSy_2)G_2(y_1,TSy_1,TSy_2), \tag{4.3}$$

$$G_2(y_1,y_2,Tx)G_1(x,Sy_1,Sy_2)G_2(y_1,y_2,Tx), G_2(y_1,TSy_1,TSy_2)G_2(y_1,y_2,Tx)G_2(y_1,y_2,Tx)\}$$

for all x in X and y_1, y_2 in Y, with $x \ne Sy_1$, and $x \ne Sy_2$,and

$$G_1^3(Sy_1,Sy_2,STx) < \frac{1}{2}\max\ \{G_2(y_1,y_2,Tx)G_1(x,x,STx)G_1(x,x,STx), \tag{4.4}$$

$$G_1(x,Sy_1,Sy_2)G_2(y_1,y_2,Tx)G_1(x,Sy_1,Sy_2), G_1(x,x,STx)G_1(x,Sy_1,Sy_2)G_1(x,Sy_1,Sy_2)\}$$

for all x in X and y_1, y_2 in Y, with $y_1 \ne Tx$, $y_2 \ne Tx$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. Let $\psi : X \to R^+$ defined by $\psi(x) = G_1(x,STx,STx)$ is G-continuous on X. Since X is compact, there exists a point u in X such that $\psi(u) = G_1(u,STu,STu) = \min\{G_1(x,STx,STx); x \in X\}$. Now suppose that $Tu \ne TSTu$. Then $u \ne STu$. By the inequality (4.4), we have

$$G_1^3(STu,STu,STSTu) < \frac{1}{2}\max\{G_2(Tu,Tu,TSTu)G_1(STu,STu,STSTu)G_1(STu,STu,STSTu),$$

$$G_1(STu,STu,STu)G_2(Tu,Tu,TSTu)G_1(STu,STu,STu),$$

$$G_1(STu,STu,STSTu)G_1(STu,STu,STu)G_1(STu,STu,STu)\}$$

$$< \frac{1}{2}\max\{G_2(Tu,Tu,TSTu)G_1(STu,STu,STSTu)G_1(STu,STu,STSTu),0,0\},$$

$$G_1^3(STu,STu,STSTu) < \frac{1}{2}G_2(Tu,Tu,TSTu)G_1(STu,STu,STSTu)G_1(STu,STu,STSTu),$$

$$G_1(STu,STu,STSTu) < \frac{1}{2}G_2(Tu,Tu,TSTu) < G_2(Tu,TSTu,TSTu) \tag{4.5}$$

Using the inequality (4.3), we have

$$G_2^3(Tu,TSTu,TSTu) < \frac{1}{2}\max\{G_1(u,STu,STu)G_2(Tu,TSTu,TSTu)G_2(Tu,TSTu,TSTu),$$

$$G_2(Tu,Tu,Tu)G_1(u,STu,STu)G_2(Tu,Tu,Tu), G_2(Tu,TSTu,TSTu)G_2(Tu,Tu,Tu)G_2Tu,Tu,Tu)\}$$

$$< \frac{1}{2}\max\{G_1(u,STu,STu)G_2(Tu,TSTu,TSTu)G_2(Tu,TSTu,TSTu),0,0\}$$

we get $G_2^3(Tu,TSTu,TSTu) < \frac{1}{2}G_1(u,STu,STu)G_2(Tu,TSTu,TSTu)G_2(Tu,TSTu,TSTu)$,

$$G_2(Tu,TSTu,TSTu) < \frac{1}{2}G_1(u,STu,STu) \tag{4.6}$$

from the inequalities (4.5) and (4.6), we have

$$\frac{1}{2}G_1(STu,STSTu,STSTu) \le G_1(STu,STu,STSTu) < \frac{1}{2}G_1(u,STu,STu),$$

$$G_1(STu,STSTu,STSTu) < G_1(u,STu,STu).$$

Then $\psi(STu) < \psi(u)$, and this gives us a contradiction, so $TSTu = Tu$. If putting $Tu = w$ and $Sw = z$, then we get $ST(STu) = S(TSTu) = STu = Sw = z$, and $w = Tu = TS(Tu) = T(STu) = Tz$. Thus, $Sw = z$ is a fixed point of ST and $Tz = w$ is a fixed point of TS. To prove uniqueness, suppose that ST has a second distinct fixed point z$'$. Then applying the inequality (4.4), we have

$$G_1^3(z,z,z') = G_1^3(STz, STz, STz') < \frac{1}{2}\max\{G_2(Tz,Tz,Tz')G_1(z',z',STz')G_1(z',z',STz'),$$

$$G_1(z',z,z)G_2(Tz,Tz,Tz')G_1(z',z,z), G_1(z',z',STz')G_1(z',z,z)G_1(z',z,z)\}$$

and it follows that

$$G_1^3(z,z,z') < \frac{1}{2}G_2(Tz,Tz,Tz')G_1(z',z,z)$$

$$G_1(z,z,z') < \frac{1}{2}G_2(Tz,Tz,Tz') \tag{4.7}$$

Applying the inequality (4.3) we have, since $z \neq z' = STz'$,

$$G_2^3(Tz,Tz',Tz') = G_2^3(Tz, TSTz', TSTz') <$$

$$\frac{1}{2}\max\{G_1(z,\ STz', STz')G_2(Tz', TSTz', TSTz')G_2(Tz', TSTz', TSTz'),$$

$$G_2(Tz',Tz',Tz)G_1(z,STz',STz')G_2(Tz',Tz',Tz), G_2(T\,z', TSTz', TSTz')G_2(Tz',Tz',Tz\,)G_2(Tz',Tz',Tz\,)\}$$

$$< \frac{1}{2}\max\{0, G_2(Tz',Tz',Tz)G_1(z,z',z')G_2(Tz',Tz',Tz), 0\}$$

$$G_2^3(Tz,Tz',Tz') < \frac{1}{2}G_1(\,z,z',z')G_2(Tz',Tz',Tz\,)G_2(Tz',Tz',Tz\,),$$

$$G_2(Tz,Tz',Tz') < \frac{1}{2}G_1(z,z',z'),$$

$$\frac{1}{2}G_2(Tz',Tz,Tz) \le G_2(Tz,Tz',Tz') < \frac{1}{2}G_1(z,z',z'). \tag{4.8}$$

From the inequalities (4.7) and (4.8), we get $G_1(z,z,z') < \frac{1}{2}G_1(z,z',z') \le G_1(z,z,z')$, This is impossible, and so the fixed point z must be a unique, similarly w is a unique fixed point of TS. □

Corollary 4.5. *Let (X, G_1) and (Y, G_2) be compact G- metric spaces, and T be a continuous mapping of X into Y and let S be a continuous mapping of Y into X satisfying the inequalities:*

$$G_2^3(Tx, TSy_1, TSy_2) < \frac{1}{2}(a_1G_1(x,Sy_1,Sy_2)G_2(y_1,TSy_1,TSy_2)G_2(y_1,TSy_1,TSy_2)+$$

$$b_1G_2(y_1,y_2,Tx\,)G_1(x,Sy_1,Sy_2)G_2(y_1,y_2,Tx\,)+$$

$$c_1G_2(y_1,TSy_1,TSy_2)G_2(y_1,y_2,Tx\,)G_2(y_1,y_2,Tx\,))$$

for all x in X and y_1, y_2 in Y, with $x \neq Sy_1$, and $x \neq Sy_2$, and

$$G_1^3(Sy_1,Sy_2,STx\,) < \frac{1}{2}\,(a_2G_2(y_1,y_2,Tx)G_1(x,x,STx)G_1(x,x,STx)+$$

$$b_2G_1(x,Sy_1,Sy_2)G_2(y_1,y_2,Tx)G_1(x,Sy_1,Sy_2) + c_2G_1(x,x,STx)G_1(x,Sy_1,Sy_2)G_1(x,Sy_1,Sy_2))$$

for all x in X and y_1, y_2 in Y, with $y_1 \neq Tx, y_2 \neq Tx$ and $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}^+$ with $(a_1+b_1+c_1)(a_2+b_2+c_2) < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

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