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Approximation Properties of λ-Sz$\acute{a}$sz-Mirakian Operators *

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Abstract:
In this paper, we construct a new kind of λ-Szász-Mirakian operators with parameter $\lambda \in [-1,1]$. The Korovkin type approximation theorem will be investigated. Finally, the Voronovskaja-type asymptotic formula and the $Gr\ddot{u}ss$-Voronovskaja type theorem for the class of λ-Szász-Mirakian operators are obtained.

Keywords: λ-Szász-Mirakian operators; Bézier basis functions; Modulus of continuity; Voronovskaja type theorem

MSC: 41A10; 41A25; 41A36

1. INTRODUCTION

The famous Szász-Mirakian operators are defined by

$$S_n(f;x) = \sum_{k=0}^{\infty} f(\frac{k}{n}) s_{n,k}(x), \tag{1}$$

where $x \in [0,\infty), n = 1, 2, \cdots,$

$$s_{n,k}(x) = e^{-nx}\frac{(nx)^k}{k!}. \tag{2}$$

The classical Szász-Mirakian operators have many approximation properties, we mention here the papers$[1 \sim 5]$.

Recently, Ye etal[6], Cai etal[7], Acu etal[8] introduced and considered a new type Bézier-Bernstein operators. The authors have established a local approximation theorem. In this paper, we introduce a new generalization of Szász polynomials depending on the parameter λ as follows:

$$S_{n,\lambda}(f;x) = \sum_{k=0}^{\infty} f(\frac{k}{n}) \tilde{s}_{n,k}(\lambda;x), \tag{3}$$

where $\lambda \in [-1,1]$,

$$\begin{aligned} \tilde{s}_{n,0}(\lambda;x) &= s_{n,0}(x) - \frac{\lambda}{n+1} s_{n+1,1}(x), \\ \tilde{s}_{n,k}(\lambda;x) &= s_{n,k}(x) + \lambda(\frac{n-2k+1}{n^2-1} s_{n+1,k}(x) \\ &\quad - \frac{n-2k-1}{n^2-1} s_{n+1,k+1}(x)) \quad (1 \le k < \infty), \end{aligned} \tag{4}$$

when λ=0, they reduce to (2).

Through the study and discussion of λ-Szász-Mirakian operators, we will estimate the moments and central moments of these operators(3). We investigate a Korovkin approximation theorem, establish a local approximation theorem, give a convergence theorem for the Lipschitz continuous functions. In Sect. 4, we give a quantitative Voronovskaja type theorem and a Grüss-Voronovskaja type theorem for the λ-Szász-Mirakian operators.

2. LEMMAS

In this section, we give the estimate of moments of the λ-Szász-Mirakian operators.

Lemma 2.1 For λ-Szász-Mirakian operators, we have the following equalities:

$$S_{n,\lambda}(1;x) = 1; \tag{5}$$

$$S_{n,\lambda}(t;x) = x + \left[\frac{1-e^{-(n+1)x}}{n(n-1)} - \frac{2x}{n(n-1)}\right]\lambda; \tag{6}$$

$$S_{n,\lambda}(t^2;x) = x^2 + \frac{x}{n} + \left[\frac{2}{n(n-1)}x - \frac{4(n+1)}{n^2(n-1)}x^2 + \frac{e^{-(n+1)x}-1}{n^2(n-1)}\right]\lambda; \tag{7}$$

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$$S_{n,\lambda}(t^3;x) = x^3+\frac{3}{n}x^2+\frac{x}{n^2}+\left[\frac{-2}{n^3(n-1)}x+\frac{(3n-9)(n+1)}{n^3(n-1)}x^2-\frac{6(n+1)^2}{n^3(n-1)}x^3+\frac{1-e^{-(n+1)x}}{n^3(n-1)}\right]\lambda; \quad (8)$$

$$\begin{aligned} S_{n,\lambda}(t^4;x) &= x^4+\frac{6}{n}x^3+\frac{7}{n^2}x^2+\frac{x}{n^3} \\ &+\left[\frac{2n}{n^4(n-1)}x+\frac{(n+1)(6n-22)}{n^4(n-1)}x^2+\frac{(4n-32)(n+1)^2}{n^4(n-1)}x^3-\frac{8(n+1)^3}{n^4(n-1)}x^4+\frac{e^{-(n+1)x}-1}{n^4(n-1)}\right]\lambda. \quad (9)\end{aligned}$$

Proof : Noting that $\sum_{k=0}^{\infty} s_{n,k}(x)=1$, we can obtain (5). Next,

$$S_{n,\lambda}(t;x)=S_n(t;x)+\lambda\left[\sum_{k=0}^{\infty}\frac{k}{n}\frac{n+1-2k}{n^2-1}s_{n+1,k}(x)-\sum_{k=0}^{\infty}\frac{k}{n}\frac{n-1-2k}{n^2-1}s_{n+1,k+1}(x)\right].$$

We denote the latter two parts in the bracket of the last formula by A and B, then we have

$$S_{n,\lambda}(t;x)=x+\lambda(A+B).$$

First, we will compute A and B respectively.

$$\begin{aligned} A &= \sum_{k=0}^{\infty}\left[\frac{k(n+1)}{n(n^2-1)}-\frac{2k^2}{n(n^2-1)}\right]\frac{e^{-(n+1)x}((n+1)x)^k}{k!} \\ &= \sum_{k=1}^{\infty}\frac{n-1}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^{k-1}}{(k-1)!}(n+1)x-\sum_{k=2}^{\infty}\frac{2}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^{k-2}}{(k-2)!}((n+1)x)^2 \\ &= \frac{x}{n}-\frac{2(n+1)}{n(n-1)}x^2. \end{aligned}$$

$$\begin{aligned} B &= \sum_{k=0}^{\infty}\frac{k}{n}\cdot\frac{n-2k-1}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^{k+1}}{(k+1)!} \\ &= \sum_{k=0}^{\infty}\frac{-2(k+1)}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^k}{k!}(n+1)x \\ &\quad +\sum_{k=0}^{\infty}\frac{n+3}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^k}{k!}(n+1)x-\sum_{k=0}^{\infty}\frac{n+1}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^{k+1}}{(k+1)!} \\ &= \sum_{k=1}^{\infty}\frac{-2}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^{k-1}}{(k-1)!}((n+1)x)^2-\sum_{k=0}^{\infty}\frac{2}{n(n^2-1)}\cdot\frac{e^{-(n+1)x}((n+1)x)^k}{k!}(n+1)x \\ &\quad +\frac{(n+3)(n+1)}{n(n^2-1)}x-\frac{n+1}{n(n^2-1)}(1-e^{-(n+1)x}) \\ &= \frac{-2(n+1)}{n(n-1)}x^2+\frac{n+1}{n(n-1)}x-\frac{1-e^{-(n+1)x}}{n(n-1)}. \end{aligned}$$

Combining A and B, we have

$$S_{n,\lambda}(t;x)=x+\lambda\left(\frac{1-e^{-(n+1)x}}{n(n-1)}-\frac{2x}{n(n-1)}\right),$$

therefore, we get (6).

Similarly, we can obtain (7), (8) and (9) by some computations, here we omit the details.

Lemma 2.2 Using Lemma 2.1 and some easy computation, for $\lambda\in[-1,1]$, the central moments of λ-Szász-Mirakian operators are given below:

$$S_{n,\lambda}(t-x;x)=\left[\frac{1-e^{-(n+1)x}}{n(n-1)}-\frac{2x}{n(n-1)}\right]\lambda\leq \frac{1+2x+e^{-(n+1)x}}{n(n-1)}=\phi_n(x); \quad (10)$$

$$
\begin{aligned}
S_{n,\lambda}((t-x)^2;x) &= \frac{x}{n} + \left[\frac{2e^{-(n+1)x}}{n(n-1)}x - \frac{4}{n^2(n-1)}x^2 + \frac{e^{-(n+1)x}-1}{n^2(n-1)}\right]\lambda \\
&\leq \frac{x}{n} + \frac{2xe^{-(n+1)}}{n(n-1)} + \frac{4x^2+e^{-(n+1)x}+1}{n^2(n-1)} = \psi_n(x); \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
S_{n,\lambda}((t-x)^4;x) \;=\; & \frac{3}{n^2}x^2 + \frac{x}{n^3} + \left[\left(\frac{-2}{n^3(n-1)} + \frac{4e^{-(n+1)x}}{n^3(n-1)}\right)x + \left(\frac{-8n-22}{n^4(n-1)} + \frac{6e^{-(n+1)x}}{n^2(n-1)}\right)x^2\right. \\
& \left. + \left(\frac{-24n-32}{n^4(n-1)} + \frac{4e^{-(n+1)x}}{n(n-1)}\right)x^3 - \frac{8}{n^4(n-1)}x^4 + \frac{e^{-(n+1)x}-1}{n^4(n-1)}\right]\lambda. \qquad (12)
\end{aligned}
$$

Lemma 2.3 For $x \in [0,\infty)$, $\lambda \in [-1,1]$, we have

$$\lim_{n\to\infty} nS_{n,\lambda}(t-x;x) = 0; \qquad (13)$$

$$\lim_{n\to\infty} nS_{n,\lambda}((t-x)^2;x) = x; \qquad (14)$$

$$\lim_{n\to\infty} n^2S_{n,\lambda}((t-x)^4;x) = 3x^2. \qquad (15)$$

3 DIRECT THEOREMS OF $S_{n,\lambda}$

The space C$[0,\infty)$ of all continuous and bounded functions on $[0,\infty)$ is a Banach space with sup-norm $\|f\| := \sup_{x\in[0,\infty)}|f(x)|$, and we estimate the rate of convergence by modulus of continuity.

Theorem 3.1 For $f \in C[0,\infty)$, $\lambda \in [-1,1]$, λ-Szász-Mirakian operators $S_{n,\lambda}(f;x)$ converge to $f(x)$.

Proof: By the Korovkin theorem it suffices to show that

$$\lim_{n\to\infty} \|S_{n,\lambda}(t^k;x) - x^k\| = 0; \qquad k = 0,1,2.$$

We can obtain these conditions easily by (5), (6) and (7) of Lemma 2.1, thus the proof is completed.

The Peetre K-functional is defined by $K_2(f;\delta) := \inf_{g\in C^2[0,\infty)}\{\|f-g\| + \delta\|g''\|\}$, where $\delta > 0$ and $C^2[0,\infty) := \{g \in C[0,\infty) : g', g'' \in C[0,\infty)\}$. By [1], these exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\sqrt{\delta}), \qquad (16)$$

where $\omega_2(f;\delta) := \sup_{0<h\leq\delta}\sup_{x,x+h,x+2h\in[0,\infty)}|f(x+2h) - 2f(x+h) + f(x)|$ is the second order modulus of smoothness of $f \in C[0,\infty)$. We also denote the usual modulus of continuity of $f \in C[0,\infty)$ by $\omega(f;\delta) := \sup_{0<h\leq\delta}\sup_{x,x+h\in[0,\infty)}|f(x+h) - f(x)|$.

Theorem 3.2 For $f \in C[0,\infty)$, $\lambda \in [-1,1]$, we have

$$|S_{n,\lambda}(f;x) - f(x)| \leq C\omega_2(f;\sqrt{\phi_n(x)+\psi_n(x)}/2) + \omega(f;\phi_n(x)), \qquad (17)$$

where C is a positive constant $\phi_n(x)$ and $\psi_n(x)$ are defined in (10) and (11).

Proof: For $S_{n,\lambda}(t-x;x)$=0, we define the auxiliary operators

$$\tilde{S}_{n,\lambda}(f;x) = S_{n,\lambda}(f;x) - f\left(x + \frac{1-2x-e^{-(n+1)x}}{n(n-1)}\lambda\right) + f(x). \qquad (18)$$

Using (5), (6), we know that the operators $\tilde{S}_{n,\lambda}(f;x)$ are linear and preserve the linear functions:

$$\tilde{S}_{n,\lambda}(t-x;x) = S_{n,\lambda}(t-x;x) - (t-x) + (x-x) = 0. \qquad (19)$$

Let $g \in C^2[0,\infty)$, $t, x \in [0,\infty)$ by the Taylor's expansion, we have

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t (t-y)g''(y)dy,$$

by (19), we get

$$\tilde{S}_{n,\lambda}(g;x) = g(x) + \tilde{S}_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy;x\right),$$

Hence, by (18) and (11), we get

$$\begin{aligned}
& |\tilde{S}_{n,\lambda}(g;x) - g(x)| \\
\leq\ & \left|S_{n,\lambda}\left(\int_x^t (t-y)g''(y)dy;x\right)\right| + \int_x^{x+\frac{1-2x-e^{-(n+1)x}}{n(n-1)}\lambda} \left|x + \frac{1-2x-e^{-(n+1)x}}{n(n-1)}\lambda - y\right| |g''(y)|dy \\
\leq\ & \left[S_{n,\lambda}((t-x)^2;x) + \frac{1+2x+e^{-(n+1)x}}{n(n-1)}\right] \|g''\| \\
\leq\ & [\psi_n(x) + \phi_n(x)]\|g''\|. \qquad (20)
\end{aligned}$$

On the other hand, by (18), (5), (4), we have

$$|\tilde{S}_{n,\lambda}(f;x)| \leq |S_{n,\lambda}(f;x)| + 2\|f\| \leq 3\|f\|. \qquad (21)$$

By (18) and (21), we get

$$\begin{aligned}
|S_{n,\lambda}(f;x) - f(x)| &= \left|\tilde{S}_{n,\lambda}(f;x) + f\left(x + \frac{1-2x-e^{-(n+1)x}}{n(n-1)}\lambda\right) - 2f(x)\right| \\
&= \left|\tilde{S}_{n,\lambda}(f-g+g;x) + f\left(x + \frac{1-2x-e^{-(n+1)x}}{n(n-1)}\lambda\right) - 2f(x)\right| \\
&\leq \left|\tilde{S}_{n,\lambda}(f-g;x) - (f-g)(x)| + |\tilde{S}_{n,\lambda}(g;x) - g(x)\right| \\
&\quad + \left|f\left(x + \frac{1-2x-e^{-(n+1)x}}{n(n-1)}\lambda\right) - f(x)\right|
\end{aligned}$$

Since

$$f\left(x + \frac{1-2x-e^{-(n+1)x}}{n(n-1)}\lambda\right) - f(x) \leq f(x+\phi_n(x)) - f(x) \leq \omega(f;\phi_n(x)). \qquad (22)$$

By (20), (21) and (22), we get

$$|S_{n,\lambda}(f;x) - f(x)| \leq 4\|f-g\| + [\psi_n(x) + \phi_n(x)]\|g''\| + \omega(f;\phi_n(x)).$$

Hence, taking infimum on the right hand side over all g$\in C^2[0,\infty)$, we have

$$|S_{n,\lambda}(f;x) - f(x)| \leq 4K_2\left(f;\frac{\phi_n(x)+\psi_n(x)}{4}\right) + \omega(f;\phi_n(x)).$$

By (16), we have

$$|S_{n,\lambda}(f;x) - f(x)| \leq C\omega_2(f;\sqrt{\phi_n(x)+\psi_n(x)}/2) + \omega(f;\phi_n(x)),$$

where $\phi_n(x)$ and $\psi_n(x)$ are defined in (10) and (11). Thus the proof is completed.

Theorem 3.3 If $f \in C^1[0,\infty)$:=$\{f : f'$ is continous and bounded on $[0,\infty)\}$, then

$$|S_{n,\lambda}(f;x) - f(x)| \leq \phi_n(x)|f'(x)| + 2\sqrt{\psi_n(x)}\omega(f';\sqrt{\psi_n(x)}).$$

Proof: For any $x, t \in [0,\infty)$, $f \in C^1[0,\infty)$, we have

$$f(t) - f(x) = (t-x)f'(x) + \int_x^t (f'(y) - f'(x))dy.$$

Applying $S_{n,\lambda}(\cdot;x)$ to both side of the above relation, we get

$$S_{n,\lambda}(f(t) - f(x);x) = f'(x)S_{n,\lambda}(t-x,x) + S_{n,\lambda}(\int_x^t (f'(y) - f'(x))dy;x).$$

Using the following property of modulus of continuity

$$|f(y) - f(x)| \leq \omega(f;|y-x|) \leq (\frac{|y-x|}{\delta} + 1)\omega(f;\delta), \delta > 0,$$

we get

$$\left|\int_x^t |f'(y)-f'(x)|dy\right| \leq \omega(f';\delta)\left[\frac{(t-x)^2}{\delta}+|t-x|\right],$$

Therefore

$$\begin{aligned}|S_{n,\lambda}(f;x)-f(x)| &\leq |f'(x)|\cdot|S_{n,\lambda}(t-x;x)|+S_{n,\lambda}\left|\int_x^t |f'(y)-f'(x)|dy\right| \\ &\leq |f'(x)|\cdot|S_{n,\lambda}(t-x;x)|+\omega(f';\delta)\left[\frac{S_{n,\lambda}((t-x)^2;x)}{\delta}+S_{n,\lambda}(|t-x|;x)\right].\end{aligned}$$

Using the $H\ddot{o}$lder's inequality to the right hand side, we obtain

$$\begin{aligned}|S_{n,\lambda}(f;x)-f(x)| &\leq |f'(x)|\cdot\phi_n(x)+\omega(f';\delta)\cdot\sqrt{\psi_n(x)}[\frac{\sqrt{\psi_n(x)}}{\delta}+1] \\ &\leq |f'(x)|\cdot\phi_n(x)+2\sqrt{\psi_n(x)}\omega(f';\sqrt{\psi_n(x)}),\end{aligned}$$

where $\delta=\sqrt{\psi_n(x)}$, we get the result.
A function f belongs to $Lip_M(\alpha)$, here $0<\alpha\leq 1$, if the inequality

$$|f(t)-f(x)|\leq M|t-x|^\alpha \tag{23}$$

holds for all t$\in[0,\infty)$.
Theorem 3.4 Let $f\in Lip_M(\alpha)$, $x\in[0,\infty)$ and $\lambda\in[-1,1]$, then we have

$$|S_{n,\lambda}(f;x)-f(x)|\leq M[\psi_n(x)]^{\frac{\alpha}{2}},$$

where $\psi_n(x)$ is given by the relation (11).
Proof: Since $S_{n,\lambda}(f;x)$ are linear positive operators and $f\in Lip_M(\alpha)$, we have

$$|S_{n,\lambda}(f;x)-f(x)|\leq M\sum_{k=0}^{\infty}\tilde{s}_{n,k}(\lambda;x)\left|\frac{k}{n}-x\right|^\alpha.$$

Applying the $H\ddot{o}lder^{'}s$ inequality for the sum in the right side of the above relation, we obtain

$$|S_{n,\lambda}(f;x)-f(x)|\leq M\sum_{k=0}^{\infty}\left[\tilde{s}_{n,k}(\lambda;x)(\frac{k}{n}-x)^2\right]^{\frac{\alpha}{2}}\cdot\left[\sum_{k=0}^{\infty}\tilde{s}_{n,k}(\lambda;x)\right]^{\frac{2-\alpha}{2}}\leq M\left[\psi_n(x)\right]^{\frac{\alpha}{2}}.$$

4. VORONOVSKAJA-TYPE THEOREMS

In this section, we will prove the Voronovskaja type theorems for the operator $S_{n,\lambda}$ by means of the Ditzian-Totik modulus of smoothness defined as follows:

$$\omega_\varphi(f;t)=\sup_{0<h\leq t}\left\{\left|f(x+\frac{h\varphi(x)}{2})-f(x-\frac{h\varphi(x)}{2})\right|, x\pm\frac{h\varphi(x)}{2}\in[0,\infty)\right\}, \tag{24}$$

where $\varphi(x)=\sqrt{x}$ and $f\in C[0,\infty)$. The corresponding K-functional is given by

$$K_\varphi(f;t)=\inf_{h\in W_\varphi[0,\infty)}\{||f-h||+t||\varphi h^{'}||\}, \qquad (t>0), \tag{25}$$

here $W_\varphi[0,\infty)=\{h: h\in AC_{loc}[0,\infty), ||\varphi h^{'}||<\infty\}$ and $AC_{loc}[0,\infty)$ is the class of absolutely continuous functions on every interval $[a,b]\subset[0,\infty)$. The K-functional and the Ditzian-Totik first order modulus of smoothness has the following relation:

$$K_\varphi(f;t)\leq C\omega_\varphi(f;t), \tag{26}$$

where $C>0$ is a constant.
Theorem 4.1 For any $f\in C^2[0,\infty)$:=$\{f: f^{''}$ is continous and bounded on $[0,\infty)\}$, and n sufficiently large, the following inequality holds:

$$|S_{n,\lambda}(f;x)-f(x)-E_n(x;\lambda)f^{'}(x)-\frac{1}{2}F_n(x;\lambda)f^{''}(x)|\leq\frac{1}{n}C\varphi^2(x)\omega_\varphi(f^{''};n^{-\frac{1}{2}}),$$

where

$$E_n(x;\lambda) = \frac{1-e^{-(n+1)x}-2x}{n(n-1)}\lambda = S_{n,\lambda}(t-x;x);$$
$$F_n(x;\lambda) = \frac{x}{n}+\left[\frac{2e^{-(n+1)x}}{n(n-1)}x-\frac{4}{n^2(n-1)}x^2+\frac{e^{-(n+1)x}-1}{n^2(n-1)}\right]\lambda = S_{n,\lambda}((t-x)^2;x),$$

and C is a positive constant.

Proof: For $f\in C^2[0,\infty)$, $t,x\in[0,\infty)$, by the Taylor's expansion, we have

$$f(t)=f(x)+f'(x)(t-x)+\int_x^t (t-y)f''(y)dy.$$

Noting

$$\int_x^t (t-y)f''(x)dy=\frac{1}{2}(t-x)^2f''(x),$$

hence

$$f(t)-f(x)-f'(x)(t-x)-\frac{1}{2}(t-x)^2f''(x)=\int_x^t (t-y)[f''(y)-f''(x)]dy.$$

Applying $S_{n,\lambda}(f;x)$ to both side of the above relation, we get

$$|S_{n,\lambda}(f;x)-f(x)-E_n(x;\lambda)f^{'}(x)-\frac{1}{2}F_n(x;\lambda)f^{''}(x)|\le S_{n,\lambda}\left(\left|\int_x^t |t-y|\cdot|f''(y)-f''(x)|dy\right|;x\right). \tag{27}$$

The quantity $\left|\int_x^t |t-y||f''(y)-f''(x)|dy\right|$ was estimated in $[15, p.337]$ as follows:

$$\left|\int_x^t |t-y||f''(y)-f''(x)|dy\right|\le 2||g''-h||(t-x)^2+2||\varphi h'||\varphi^{-1}(x)|t-x|^3, \tag{28}$$

where $h\in W_\varphi[0,\infty)$.
Using Lemma 2.3 it follows that there exists a constant $C>0$ such that, for n sufficiently large,

$$S_{n,\lambda}((t-x)^2;x)\le\frac{C}{2n}\varphi^2(x) \quad and \quad S_{n,\lambda}((t-x)^4;x)\le\frac{C}{2n^2}\varphi^4(x). \tag{29}$$

From (27)-(29) and applying the Cauchy-Schwarz inequality, we get

$$|S_{n,\lambda}(f;x)-f(x)-E_n(x;\lambda)f^{'}(x)-\tfrac{1}{2}F_n(x;\lambda)f^{''}(x)|$$
$$\le 2\|f''-h\|S_{n,\lambda}((t-x)^2;x)+2\|\varphi h'\|\varphi^{-1}(x)S_{n,\lambda}(|t-x|^3;x)$$
$$\le \tfrac{C}{n}\varphi^2(x)\|f''-h\|+2\|\varphi h'\|\varphi^{-1}(x)\left(S_{n,\lambda}((t-x)^2;x)\right)^{\frac{1}{2}}\left(S_{n,\lambda}((t-x)^4;x)\right)^{\frac{1}{2}}$$
$$\le \tfrac{C}{n}\varphi^2(x)\{\|f''-h\|+n^{-\frac{1}{2}}\|\varphi h'\|\}$$
$$\le \tfrac{1}{n}C\varphi^2(x)\omega_\varphi(f^{''};n^{-\frac{1}{2}}).$$

Corollary 4.1 If $f\in C^2[0,\infty)$, then

$$\lim_{n\to\infty} n\{S_{n,\lambda}(f;x)-f(x)-E_n(x;\lambda)f^{'}(x)-\frac{1}{2}F_n(x;\lambda)f^{''}(x)\}=0,$$

where $E_n(x;\lambda)$ and $F_n(x;\lambda)$ are defined in Theorem4.1.

The $Gr\ddot{u}ss$ type approximation theorem have been made by many authors $[9 \sim 14]$. Next, we will provide a $Gr\ddot{u}ss$-Voronovskaja type theorem for λ-Sz$\acute{a}$sz-Mirakian operators.

Theorem 4.2 If $f,g\in C^2[0,\infty)$, for each $x\in[0,\infty)$, we have

$$\lim_{n\to\infty} n\{S_{n,\lambda}((fg);x)-S_{n,\lambda}(f;x)S_{n,\lambda}(g;x)\}=xf'(x)g'(x).$$

Proof: Since

$$
\begin{aligned}
& S_{n,\lambda}((fg);x) - S_{n,\lambda}(f;x)S_{n,\lambda}(g;x) \\
= \quad & S_{n,\lambda}((fg);x) - f(x)g(x) - E_n(x;\lambda)(fg)^{'}(x) - \frac{1}{2}F_n(x;\lambda)(fg)^{''} \\
& -S_{n,\lambda}(f;x)[S_{n,\lambda}(g;x) - g(x) - E_n(x;\lambda)g^{'}(x) - \frac{1}{2}F_n(x;\lambda)g^{''}(x)] \\
& -g(x)[S_{n,\lambda}(f;x) - f(x) - E_n(x;\lambda)f^{'}(x) - \frac{1}{2}F_n(x;\lambda)f^{''}(x)] \\
& +\frac{1}{2}F_n(x;\lambda)[(fg)''(x) - g''(x)S_{n,\lambda}(f;x) - g(x)f''(x)] \\
& +E_n(x;\lambda)[(fg)'(x) - g'(x)S_{n,\lambda}(f;x) - g(x)f'(x)],
\end{aligned}
$$

and

$$
(fg)'' = (f'g + fg')' = f''g + 2f'g' + fg'', \quad (fg)' = f'g + fg'.
$$

Hence

$$
\begin{aligned}
& \frac{1}{2}F_n(x;\lambda)[(fg)''(x) - g''(x)S_{n,\lambda}(f;x) - g(x)f''(x)] \\
& = \frac{1}{2}F_n(x;\lambda)[2f'(x)g'(x) + f(x)g''(x) - g''(x)S_{n,\lambda}(f;x)]; \\
& E_n(x;\lambda)[(fg)'(x) - g'(x)S_{n,\lambda}(f;x) - g(x)f'(x)] \\
& = E_n(x;\lambda)[f(x)g'(x) - g'(x)S_{n,\lambda}(f;x)].
\end{aligned}
$$

Using Theorem 3.1 and Corollary 4.1, we get

$$
\begin{aligned}
& \lim_{n\to\infty} n\{S_{n,\lambda}((fg);x) - S_{n,\lambda}(f;x)S_{n,\lambda}(g;x)\} \\
= \quad & 0 + 0 + 0 + \lim_{n\to\infty} nf'(x)g'(x)F_n(x;\lambda) + \lim_{n\to\infty} \frac{n}{2}F_n(x;\lambda)g''(x)[f(x) - S_{n,\lambda}(f;x)] \\
& + \lim_{n\to\infty} nE_n(x;\lambda)g'(x)[f(x) - S_{n,\lambda}(f;x)] \\
= \quad & \lim_{n\to\infty} nf'(x)g'(x) \cdot F_n(x;\lambda) \\
= \quad & f'(x)g'(x)x.
\end{aligned}
$$

Thus, Theorem 4.2 is proved.

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REFERENCES

[1] Ditzian Z., Totik V., Moduli of Smoothness, Spring-Verlag, New York, (1987).

[2] Totik V., Uniform approximation by Szász-Mirakian operators, Acta Math. Acad. Sci. Hungar., 41, 291-307 (1983).

[3] Zhou D., Weighted approximation by Szász-Mirakian operators, J. Approx. Theory, 76, 393-402 (1994).

[4] Guo S., Li C., Yang G. and Yue S., Pointwise estimate for Szász operators, J. Approx. Theory, 94, 160-171 (1998).

[5] Gal S. G., Approximation with an arbitrary order by generalized Szász-Mirakian operators, Studia Univ. Babes-Bolyai Math., 59(1), 77-81 (2014).

[6] Ye Z., Long X., Zeng X. M.: Adjustment algorithms for Bézier curve and surface. In: International Conference on Computer Science and Education, pp.1712-1716 (2010).

International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 12, Number 5 (2019), pp. 662-669
©International Research Publication House. http://www.irphouse.com

[7] Cai Q., Lian B. and Zhou G., Approximation properties of λ-Bernstein operators, Journal of Inequalities and Applications, 61(2018).
https://doi.org/10.1186/s13660-018-1653-7

[8] Acu A., Manav N. and Sofonea D., Approximation properties of λ-Kantorovich operators, Journal of Inequalities and Applications, 202(2018).
https://doi.org/10.1186/s13660-018-1795-7

[9] Acar T., Quantitative q-Voronovskaya and q-$Grüss$-Voronovskaya-type results for q-$Szász$ operators. Georgian Math. J., 23(4), 459-468 (2016).
https://doi.org/10.1515/gmj-2016-0007

[10] Acar T., Aral A., Rasa I., The new foms of Voronovskaya's theorem in weighted spaces. Positivity, 20(1), 25-40 (2016).
https://doi.org/10.1007/s11117-015-0338-4

[11] Acu A. M., Gonska H., Rasa I., $Grüss$-type and Ostrowski-type inequalities in approximation theory. Ukr. Math. J., 63(6), 843-864 (2011).
https://doi.org/10.1007/s11253-011-0548-2

[12] Agratini O., Properties of discrete non-multiplicative operators. Anal. Math. Phys., (2017).
https://doi.org/10.1007/s13324-017-0186-4

[13] Erencin A., Rasa I., Voronovskaya type theorems in weighted spaces. Numer. Funct. Anal. Optim.37(12), 1517-1528 (2016).
https://doi.org/10.1080/01630563.2016.1219743

[14] Ulusay G., Acar T., q-Voronovskaya type theorems for q-Baskakov operators. Math. Methods Appl. Sci., 39(12), 3391-3401 (2016).
https://doi.org/10.1002/mma.3784

[15] Z. Finta, Remark on Voronovskaja theorem for q-Bernstein operators, Stud. Univ. Babeş-Bolyai Math., 56(2)335-339 (2011).