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Nonlinear Boundary-Value Problem for Differential Equations of Shell Theory of Timoshenko Type

Liliya Sergeevna Kharasova

Kazan Federal University, Senior Lecturer of the Department of Mathematics, Engineering and Construction Department, Naberezhnye Chelny Institute, K(P)FU, ID scopus: 56609224100, ORSID: 0000-0002-4713-3930

Abstract

The paper studies the solvability of one system of nonlinear second-order partial differential equations for given boundary conditions. Currently, there are many works devoted to the derivation of partial differential equations in shell theory and numerical methods for their solution. However, when they derive partial differential equations and boundary conditions for them, the problem of the adequacy of these problems to real processes comes to the fore. The solution to this problem is based on a rigorous mathematical study of boundary value problem solvability for nonlinear partial differential equations. Besides, the existence of theorems makes it easy to prove the convergence of numerical methods to an exact real solution. Therefore, a rigorous study of boundary value problem solvability for partial differential equations, the proof of theorem existence is a very urgent problem. The research method consists in reduction the original system of equations to one nonlinear operator equation in Sobolev space. The research method is based on integral representations for the desired solution containing arbitrary holomorphic functions. Finding holomorphic functions is one of the main and difficult moments of the proposed research. In this work, an arbitrary region is first conformally mapped onto a unit circle. Then, explicit representations of solutions of the Riemann-Hilbert problem for holomorphic functions in the unit disc are used. The integral representations constructed in this way make it possible to reduce the original boundary value problem to one nonlinear equation, the solvability of which is established using the principle of squeezed mappings.

Keywords: system of nonlinear differential equations, integral representations, contracted mapping principle, existence theorem.

I. INTRODUCTION

A system of five nonlinear partial differential equations of the second order of the following form is considered in an arbitrary bounded domain Ω :

$$
\begin{gathered}
w_{1\alpha^1\alpha^1}+\mu_1 w_{1\alpha^2\alpha^2}+\mu_2 w_{2\alpha^1\alpha^2}=f_1,\ \ \mu_1 w_{2\alpha^1\alpha^1}+w_{2\alpha^2\alpha^2}+\mu_2 w_{1\alpha^1\alpha^2}=f_2,\\
k^2\mu_1\left(w_{3\alpha^1\alpha^1}+w_{3\alpha^2\alpha^2}+\psi_{1\alpha^1}+\psi_{2\alpha^2}\right)+k_3 w_{1\alpha^1}+k_4 w_{2\alpha^2}-k_5 w_3+\\
+k_3 w_{3\alpha^1}^2/2+k_4 w_{3\alpha^2}^2/2+\beta_2\left[\left(T^{\lambda\mu}w_{3\alpha^\lambda}\right)_{\alpha^\mu}+R^3\right]=0 \qquad (1)\\
\psi_{1\alpha^1\alpha^1}+\mu_1\psi_{1\alpha^2\alpha^2}+\mu_2\psi_{2\alpha^1\alpha^2}=g_1+k_0\psi_1,\ \ \mu_1\psi_{2\alpha^1\alpha^1}+\psi_{2\alpha^2\alpha^2}+\mu_2\psi_{1\alpha^1\alpha^2}=g_2+k_0\psi_2,
\end{gathered}
$$

under the following conditions

$$w_2=w_3=\psi_2, \qquad (2)$$

$$(w_{1\alpha^1}+\mu w_{2\alpha^2})(t)\,d\alpha^2/ds-\mu_1(w_{1\alpha^2}+w_{2\alpha^1})(t)\,d\alpha^1/ds=\varphi_1(w_3)(t), \qquad (3)$$

$$(\psi_{1\alpha^1}+\mu\psi_{2\alpha^2})(t)\,d\alpha^2/ds-\mu_1(\psi_{1\alpha^2}+\psi_{2\alpha^1})(t)\,d\alpha^1/ds=\tilde{\varphi}_1(w_3)(t) \qquad (4)$$

on its border G.

The following designations are adopted in the formulas (1) - (4):

$$f_1 \equiv f_1(w_3) = k_3 w_{3\alpha^1} - w_{3\alpha^1} w_{3\alpha^1\alpha^1} - \mu_2 w_{3\alpha^2} w_{3\alpha^1\alpha^2} - \mu_1 w_{3\alpha^1} w_{3\alpha^2\alpha^2} - \beta_2 R^1,$$
$$f_2 \equiv f_2(w_3) = k_4 w_{3\alpha^2} - w_{3\alpha^2} w_{3\alpha^2\alpha^2} - \mu_2 w_{3\alpha^1} w_{3\alpha^1\alpha^2} - \mu_1 w_{3\alpha^2} w_{3\alpha^1\alpha^1} - \beta_2 R^2, \qquad (5)$$

$$g_j \equiv g_j(w_3) = k_0 w_{3\alpha^j} - \beta_1 L^j, j = 1,2, \mu_1 = (1-\mu)/2, \mu_2 = (1+\mu)/2,$$
$$\varphi_1(w_3)(t) = \beta_2 P^1(s) - [w_{3\alpha^1}^2 / 2(t) + \mu w_{3\alpha^2}^2 / 2(t)] d\alpha^2/ds + \mu_1 w_{3\alpha^1}(t) w_{3\alpha^2}(t) d\alpha^1/ds,$$
$$\tilde{\varphi}_1(w_3)(t) = \beta_1 N^1(s), t = t(s) = \alpha^1(s) + i\alpha^2(s), k_3 = k_1 + \mu k_2, k_4 = k_2 + \mu k_1,$$
$$k_5 = k_1^2 + k_2^2 + 2\mu k_1 k_2, k_0 = 6k^2(1-\mu)/h^2, \beta_1 = 12(1-\mu^2)/(h^3 E), \beta_2 = (1-\mu^2)/(Eh).$$

The system (1), together with boundary conditions (2) - (4), describes the equilibrium state of an elastic shallow isotropic homogeneous shell with hinged edges within the shear model by S.P. Tymoshenko [1, pp. 168-170, 269]. In this case: $T^{\lambda\mu}$ - the efforts $(\lambda, \mu = \overline{1,3})$; $w_j\,(j=1,2)$ and w_3 – tangential and normal displacement of the points, S_0, $\psi_j\,(j=1,2)$ – the angles of normal section rotation, S_0, $R^j\,(j=\overline{1,3})$, $L^k\,(k=1,2)$, N^l, P^l – the components of external forces acting on the shell, $\mu = const$ – Poisson's ratio, $E = const$ – Young's modulus, $k_1, k_2 = const$ – principal curvatures, $k^2 = const$ – shear coefficient, $h = const$ – shell thickness, $\alpha^1, \alpha^2 = const$ – Cartesian point coordinates of the domain Ω.

Problem A. Find a solution to the system (1) satisfying the boundary conditions (2) - (4).

II. METHODS

Currently, there is a number of works devoted to the study of nonlinear problems in the framework of the Timoshenko shear model [2–9]. The studies in [2–9] are based on integral representations for generalized displacements containing arbitrary holomorphic functions, which are found in such a way that generalized displacements satisfy the given boundary conditions. Two approaches are used to construct them. The first approach is based on the use of explicit representations of solutions to the Riemann - Hilbert problems for holomorphic functions in the unit disc. Therefore, a flat domain homeomorphic to the middle surface of the shell is either assumed from the very beginning to be the unit disk [2–5], or is mapped conformally onto the unit disk [6], [9]. In the second approach, holomorphic functions are sought in the form of Cauchy-type integrals with real densities, which are found as the solutions of a system of one-dimensional singular integral equations [7], [8]. In this paper, the conformal mapping method is used to study a nonlinear boundary value problem for arbitrary shallow shells under other boundary conditions.

We will study the boundary value problem A in a generalized setting. We consider the following conditions to be satisfied: a) Ω is a simply connected domain with the boundary $\Gamma \in C_\beta^1$; *b)* external forces $R^j\,(j=\overline{1,3})$, $L^k\,(k=1,2) \in L_p(\Omega)$, the components of external forces N^l, $P^l \in C_\beta(\Gamma)$; here and further everywhere: $p > 2$, $0 < \beta < 1$.

Definition. A generalized solution of problem A is the vector of generalized displacements $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W_p^{(2)}(\Omega)$, $p > 2$, which satisfies the system (1) and pointwise boundary conditions (2) - (4) almost everywhere.

III. RESULTS AND DISCUSSION

Let's consider a system of the first two equations in (1), in which the deflection w_3 is temporarily assumed to be fixed. The general solution of the system (1) has the form [2]:

$$\omega_0(z) = w_2 + iw_1 = \Phi_2(z) + iTd[\Phi_1 + Tf](z), \quad z = \alpha^1 + i\alpha^2, \quad f = (f_1 + if_2)/2, \qquad (6)$$

where $\Phi_1(z) \in C_\alpha(\bar{\Omega}), \Phi_2(z) \in C^1_\alpha(\bar{\Omega})$ – arbitrary holomorphic functions;

$$Tf = -\frac{1}{\pi}\iint_\Omega \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \zeta = \xi + i\eta, \; d[g] = d_1 g + d_2 \bar{g}, \quad d_j = (\mu_1 + (-1)^j)/(4\mu_1), j = 1, 2.$$

We find the function $\Phi_2(z)$ from the boundary condition $w_2 = 0$ on Γ. We obtain the Riemann - Hilbert problem with the boundary condition for a holomorphic function $\Phi_2(z)$ in the domain Ω:

$$\text{Re}[\Phi_2(t)] = -\text{Re}\, iTd[\omega](t), \quad t \in \Gamma, \qquad (7)$$

$$\omega(z) = \Phi_1(z) + Tf(z) = w_{1\alpha^1} + w_{2\alpha^2} + i\mu_1(w_{2\alpha^1} - w_{1\alpha^2}).$$

Let us denote by the function $z = \varphi(\zeta)$ the conformal mapping of the single circle $\bar{K}: |\zeta| \le 1$ onto the domain $\bar{\Omega}$. Since the condition a) is satisfied for the region Ω, it follows from [10, p. 25] that the function $\varphi(\zeta)$ belongs to the space $C^1_\beta(\bar{K})$. In the boundary condition (7), we make the change $t \to \varphi(t)$, $\Phi_2(\varphi(t)) \to \Phi_2(t)$, leaving the same designations for the new variables. Thus, in the single circle K we arrive at the Riemann - Hilbert problem for a holomorphic function $\Phi_2(z)$ with the boundary condition:

$$\text{Re}[\Phi_2(t)] = -\text{Re}\, iTd[\omega](\varphi(t)), \quad t \in \partial K; |t| = 1, \qquad (8)$$

where $d[g]$, Tf are defined in (6). The solution of the problem (8) is given by the formula [11, p. 253]:

$$\Phi_2(z) = \frac{1}{2\pi}\int_{\partial K} \text{Re}\, Td[\Phi_1 + Tf](\varphi(t)) \frac{t+z}{t-z}\frac{dt}{t} + ic_0, z \in \bar{K}, \qquad (9)$$

where c_0 – an arbitrary real constant.

We find the holomorphic function $\Phi_1(z)$ using the boundary condition (3). The expressions of the functions w_1, w_2 from (6) are introduced into (3). Taking into account the ratios

$$d\alpha^1 / ds = \text{Re}\, t' = (t' + \bar{t}')/2, \quad t \in \Gamma, \qquad (10)$$

$$t' = dt/ds = d(\alpha^1 + i\alpha^2)/ds = -\alpha^2 + i\alpha^1 = i(\alpha^1 + i\alpha^2) = it,$$

the boundary condition (3) can be represented as

$$\text{Re}\{t'\Phi(t)\} = h_2(t), \quad t' = dt/ds, \quad t \in \Gamma, \qquad (11)$$

where

$$h_2(t) = l(w_3)(t) + \text{Re}\{t' Sd[\Phi_1]^+(t)\} - \text{Re}\{\mu_3 \bar{t}'\Phi_1(t)\}/2, \quad \mu_3 = (1+\mu)/(2(1-\mu)), \quad (12)$$

$$l(w_3)(t) = \varphi_1(w_3)(t)/(\mu-1) + \mathrm{Re}\{t' Sd[Tf]^+(t)\} - \mu_3\, d\alpha^1/ds\, \mathrm{Re}\, Tf(t)\} \equiv l[f(w_3); \varphi_1(w_3)];$$

$\varphi_1(t)$ defined in (5). Through $Sd[\Phi_1]^+(t)$ they denote the limit of the function $Sd[\Phi_1](z)$ at $z \to t \in \Gamma$ from inside the region Ω. $\Phi(t)$ – the boundary value of the holomorphic function Ω :

$$\Phi(z) = i\Phi_2'(z) + \mu_3 \Phi_1(z)/2. \qquad (13)$$

Thus, for the function $\Phi(z)$ in an arbitrary domain Ω we have the Riemann - Hilbert problem with the boundary condition (11). We will reduce this problem to the problem in a single circle. Using the conformal mapping of the domain $\overline{\Omega}$ onto the single circle $\overline{K}$, we obtain

$$t' = \frac{dt}{ds} = \frac{d(\varphi(\tau))}{ds} = \frac{d\varphi(\tau)}{d\tau}\frac{d\tau}{ds} = \varphi'(\tau)\tau' \frac{1}{|\varphi'(\tau)|}, \qquad (14)$$

$$t = \varphi(\tau), \quad \tau = \tau(\sigma), \quad \tau \in \partial K : |\tau| = 1, \quad \tau' = d\tau/d\sigma.$$

Therefore, leaving the previous designations for the new variables, taking into account (10) and (14), we arrive at the Riemann - Hilbert problem for a holomorphic function $\Phi(z)\varphi'(z)$ in the single circle $\overline{K}$ with the boundary condition:

$$\mathrm{Re}\{t'\varphi'(t)\Phi(t)\} = h_2(\varphi(t))\,|\varphi'(t)|, \quad t \in \partial K : |t| = 1. \qquad (15)$$

Let us study the problem (15). The index of the problem (15) is –1. Then, following [11, p. 253], the solution of the problem (15) has the following form:

$$\Phi(z) = -\frac{1}{\pi\varphi'(z)} \int_{\partial K} \frac{h_2(\varphi(t))\,|\varphi'(t)|}{t-z}\frac{dt}{t}, \quad z \in \overline{K}, \qquad (16)$$

in this case, they satisfy the condition of this problem solvability

$$\int_{\partial K} \frac{h_2(\varphi(t))\,|\varphi'(t)|}{t} dt = 0. \qquad (17)$$

For $\Phi_1(z)$ from relation (13) we have

$$\Phi_1(z) = (\Phi(z) - i\Phi_2'(z))2/\mu_3, \quad z \in \Omega. \qquad (18)$$

Let's transform the representation (18). To do this, you need to find $Sd[\Phi_1]^+(t)$, $\Phi_2'(z)$, $\Phi(z)$. The representation for $Sd[\Phi_1]^+(t)$ is found in [6]. For the function $\Phi_2'(z)$ $(z \in \overline{K})$ из (9), from (9), using the representation $d[g]$ from (6) and taking into account that $((t+z)/(t-z))' = 2t/(t-z)^2$ we have the following:

$$\Phi_2'(z) = -\frac{d_1}{2\pi\varphi'(z)}\left[\int_{\partial K} \frac{T\Phi_1(\varphi(t))}{(t-z)^2} dt + \int_{\partial K} \frac{\overline{T\Phi_1(\varphi(t))}}{(t-z)^2} dt\right] - \qquad (19)$$

$$-\frac{d_2}{2\pi\varphi'(z)}\left[\int_{\partial K} \frac{T\overline{\Phi_1(\varphi(t))}}{(t-z)^2} dt + \int_{\partial K} \frac{\overline{T}\Phi_1(\varphi(t))}{(t-z)^2} dt\right] - \frac{1}{\pi\varphi'(z)} \int_{\partial K} \mathrm{Re}\, Td[Tf](\varphi(t)) \frac{dt}{(t-z)^2}.$$

Let us calculate all the integrals on the right-hand side of the representation (19). Taking into account the operator T representation (6) and changing the variable $t \to \varphi(t)$, we obtain

$$T\Phi_1(\varphi(t)) = -\frac{1}{\pi}\iint_{\Omega}\frac{\Phi_1(\zeta)}{\zeta-\varphi(t)}d\xi d\eta = -\frac{1}{\pi}\iint_{\Omega}\frac{\Phi_1(\zeta)\,|\varphi'(\zeta)|^2}{\varphi(\zeta)-\varphi(t)}d\xi d\eta, \quad t \in \partial K. \qquad (20)$$

Further, taking into account the representation for the function $\varphi_0(\tau;z) = (\varphi(\tau)-\varphi(z))/(\tau-z)$ and the Cauchy formula, we find the following integrals easily

$$\int_{\partial K}\frac{1/\varphi_0(\tau;t)}{(t-\tau)(t-z)^2}dt = \frac{\pi i}{(z-\tau)^2\varphi'(\tau)} - \frac{2\pi i\varphi'(z)}{[\varphi(z)-\varphi(\tau)]^2}, \qquad (21)$$

$$\int_{\partial k}\frac{1/\varphi_0(\zeta;t)}{(t-\zeta)(t-z)^2}dt = \frac{2\pi i}{(z-\zeta)^2\varphi'(\zeta)} - \frac{2\pi i\varphi'(z)}{[\varphi(z)-\varphi(\zeta)]^2},$$

$$\int_{\partial K}\frac{1/\varphi_0(\zeta;t)}{(t-\zeta)(1-\bar{z}t)^2}dt = \frac{2\pi i}{\varphi'(\zeta)(1-\bar{z}\zeta)^2}, \quad \int_{\partial K}\frac{1/\varphi_0(\tau;t)}{(t-\tau)(1-\bar{z}t)^2}dt = \frac{\pi i}{\varphi'(\tau)(1-\tau\bar{z})^2}.$$

Then the first integral in the formula (19), taking into account (20) and the representation Tf from (6), is transformed to the form

$$\int_{\partial K}\frac{T\Phi_1(\varphi(t))}{(t-z)^2}dt = -\frac{1}{\pi}\int_{\partial K}\left(\iint_{K}\frac{\Phi_1(\zeta)\,|\varphi'(\zeta)|^2}{\varphi(\zeta)-\varphi(t)}d\xi d\eta\right)\frac{dt}{(t-z)^2} =$$

$$= -\frac{1}{\pi}\int_{\partial K}\left(\iint_{K}\frac{\partial}{\partial\bar{\zeta}}\left(\frac{\Phi_1(\zeta)\varphi'(\zeta)\overline{\varphi(\zeta)}}{\varphi_0(\zeta;t)}\right)\frac{d\xi d\eta}{\zeta-t}\right)\frac{dt}{(t-z)^2}.$$

Applying the formula (4.9) [10, p. 29], we have

$$\int_{\partial K}\frac{T\Phi_1(\varphi(t))}{(t-z)^2}dt = \int_{\partial K}\left(\frac{\Phi_1(t)\overline{\varphi(t)}}{2} - \frac{1}{2\pi i}\int_{\partial K}\frac{\Phi_1(\tau)\varphi'(\tau)\overline{\varphi(\tau)}}{\varphi_0(\tau;t)}\frac{d\tau}{\tau-t}\right)\frac{dt}{(t-z)^2} =$$

$$= \frac{1}{2}\int_{\partial K}\frac{\Phi_1(t)\overline{\varphi(t)}}{(t-z)^2}dt + \frac{1}{2\pi i}\int_{\partial K}\left(\int_{\partial K}\frac{1/\varphi_0(\tau;t)}{(t-\tau)(t-z)^2}dt\right)\Phi_1(\tau)\varphi'(\tau)\overline{\varphi(\tau)}d\tau.$$

Then, taking into account (21), we obtain

$$\int_{\partial K}\frac{T\Phi_1(\varphi(t))}{(t-z)^2}dt = \int_{\partial K}\frac{\Phi_1(\tau)\overline{\varphi(\tau)}}{(\tau-z)^2}d\tau - \varphi'(z)\int_{\partial K}\frac{\Phi_1(\tau)\varphi'(\tau)\overline{\varphi(\tau)}}{[\varphi_0(z;\tau)]^2(z-\tau)^2}d\tau.$$

Let us transform the second integral in representation (19). First, for this we use (20), then we apply the formula (4.9) to the integral over the domain K [10, p.29], and then - the last formula in (21), and calculate the integral

$$\int_{\partial K}\frac{T\Phi_1(\varphi(t))}{(\bar t-\bar z)^2}d\bar t=\int_{\partial K}\left(\frac{1}{\pi}\iint_K\frac{\partial}{\partial\bar\zeta}\left(\frac{\Phi_1(\zeta)\varphi'(\zeta)\overline{\varphi(\zeta)}}{\varphi_0(\zeta;t)}\right)\frac{d\xi d\eta}{\zeta-t}\right)\frac{dt}{t^2(\bar t-\bar z)^2}=$$

$$=\int_{\partial K}\left(\frac{1}{2\pi i}\int_{\partial K}\frac{\Phi_1(\tau)\varphi'(\tau)\overline{\varphi(\tau)}}{\varphi_0(\tau;t)}\frac{d\tau}{\tau-t}-\frac{1}{2}\Phi_1(t)\overline{\varphi(t)}\right)\frac{dt}{t^2(\bar t-\bar z)^2}=$$

$$=-\frac{1}{2\pi i}\int_{\partial K}\left(\int_{\partial K}\frac{1/\varphi_0(\tau;t)}{(t-\tau)(1-\bar zt)^2}dt\right)\Phi_1(\tau)\varphi'(\tau)\overline{\varphi(\tau)}d\tau-\frac{1}{2}\int_{\partial K}\frac{\Phi_1(t)\overline{\varphi(t)}}{(1-\bar zt)^2}dt=-\int_{\partial K}\frac{\Phi_1(t)\overline{\varphi(t)}}{(1-\bar zt)^2}dt.$$

It's obvious that

$$\int_{\partial K}\frac{\overline{T\Phi_1(\varphi(t))}}{(t-z)^2}dt=\overline{\left(\int_{\partial K}\frac{T\Phi_1(\varphi(t))}{(\bar t-\bar z)^2}d\bar t\right)}=-\int_{\partial K}\frac{\overline{\Phi_1(t)}\varphi(t)}{(1-\bar tz)^2}d\bar t=\int_{\partial K}\frac{\overline{\Phi_1(t)}\varphi(t)}{(t-z)^2}dt\,.$$

Let us calculate the third integral in representation (19). Applying the formulas (20) and (21), we obtain

$$\int_{\partial K}\frac{T\overline{\Phi_1(\varphi(t))}}{(t-z)^2}dt=-\frac{1}{\pi}\int_{\partial K}\left(\iint_K\frac{\overline{\Phi_1(\zeta)}\,|\varphi'(\zeta)|^2}{\varphi(\zeta)-\varphi(t)}d\xi d\eta\right)\frac{dt}{(t-z)^2}=\frac{1}{\pi}\iint_K\left(\int_{\partial K}\frac{1/\varphi_0(\zeta;t)}{(t-\zeta)(t-z)^2}dt\right)\overline{\Phi_1(\zeta)}\,|\varphi'(\zeta)|^2\,d\xi d\eta=$$

$$=2i\iint_K\frac{\overline{\Phi_1(\zeta)}\,|\varphi'(\zeta)|^2}{(z-\zeta)^2\varphi'(\zeta)}d\xi d\eta-2i\varphi'(z)\iint_K\frac{\overline{\Phi_1(\zeta)}\,|\varphi'(\zeta)|^2}{[\varphi(z)-\varphi(\zeta)]^2}d\xi d\eta\,.$$

Let K_ε – the area obtained from the area by throwing out the circle centered at the point $z:|\zeta-z|\le\varepsilon$ with the boundary $\partial K_\varepsilon=\partial K\cup\gamma_\varepsilon^-$.

Then we have

$$\iint_K\frac{\overline{\Phi_1(\zeta)}\,|\varphi'(\zeta)|^2}{(z-\zeta)^2\varphi'(\zeta)}d\xi d\eta=\iint_K\frac{\overline{\Phi_1(\zeta)}\,\overline{\varphi'(\zeta)}}{(z-\zeta)^2}d\xi d\eta=-\lim_{\varepsilon\to0}\iint_{K_\varepsilon}\frac{\partial}{\partial\zeta}\left(\frac{\overline{\Phi_1(\zeta)}\,\overline{\varphi'(\zeta)}}{\zeta-z}\right)d\xi d\eta.$$

Using the formula (4.7) [10, p. 28], we calculate the following integral:

$$\iint_{K_\varepsilon}\frac{\partial}{\partial\zeta}\left(\frac{\overline{\Phi_1(\zeta)}\,\overline{\varphi'(\zeta)}}{\zeta-z}\right)d\xi d\eta=-\frac{1}{2i}\int_{\partial K\cup\gamma_\varepsilon}\frac{\overline{\Phi_1(\tau)}\,\overline{\varphi'(\tau)}}{\tau-z}d\bar\tau=-\frac{1}{2i}\overline{\left(\int_{\partial K}\frac{\Phi_1(\tau)\varphi'(\tau)}{\bar\tau-\bar z}d\tau\right)}+$$

$$+\frac{1}{2i}\int_{\gamma_\varepsilon}\frac{\overline{\Phi_1(\tau)}\,\overline{\varphi'(\tau)}}{\tau-z}d\bar\tau=-\frac{1}{2i}\overline{\left(\int_{\partial K}\frac{\Phi_1(\tau)\varphi'(\tau)\tau}{1-\bar z\tau}d\tau\right)}+\frac{1}{2i}\int_{\gamma_\varepsilon}\frac{\overline{\Phi_1(\tau)}\,\overline{\varphi'(\tau)}}{\tau-z}d\bar\tau\,.$$

Here the first term is equal to zero, since the function $\Phi_1(\tau)\varphi'(\tau)\tau/(1-\bar{z}\tau)$ is the boundary value of the function $\Phi_1(\zeta)\varphi'(\zeta)\zeta/(1-\bar{z}\zeta)$, holomorphic by ζ in $\overline{K}$. We use the replacement for the second term:

$$\frac{1}{2i}\int_{\gamma_\varepsilon}\frac{\overline{\Phi_1(\tau)}\overline{\varphi'(\tau)}}{\tau-z}d\bar{\tau}=\begin{bmatrix}\tau=z+\varepsilon e^{i\theta}\\ \bar{\tau}=\bar{z}+\varepsilon e^{-i\theta}\\ d\bar{\tau}=-i\varepsilon e^{-i\theta}d\theta\end{bmatrix}=-\frac{1}{2}\int_0^{2\pi}\frac{\overline{\Phi_1(z+\varepsilon e^{i\theta})}\overline{\varphi'(z+\varepsilon e^{i\theta})}}{e^{i2\theta}}d\theta,$$

and passing to the limit at $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{2i}\int_{\gamma_\varepsilon}\frac{\overline{\Phi_1(\tau)}\overline{\varphi'(\tau)}}{\tau-z}d\bar{\tau}=-\frac{1}{2}\overline{\Phi_1(z)}\overline{\varphi'(z)}\int_0^{2\pi}e^{-i2\theta}d\theta=0.$$

Therefore, the integral over the region K

$$\iint_K\frac{\overline{\Phi_1(\zeta)}\,|\,\varphi'(\zeta)\,|^2}{(z-\zeta)^2\varphi'(\zeta)}d\xi d\eta=0.$$

Similarly, using the formula (4.7) [10, p.28] in the case of the domain K_ε, we find the integral

$$\iint_K\frac{\overline{\Phi_1(\zeta)}\,|\,\varphi'(\zeta)\,|^2}{[\varphi(z)-\varphi(\zeta)]^2}d\xi d\eta=-\lim_{\varepsilon\to 0}\iint_{K_\varepsilon}\frac{\partial}{\partial\zeta}\left(\frac{\overline{\Phi_1(\zeta)}\overline{\varphi'(\zeta)}/\varphi_0(\zeta;z)}{\zeta-z}\right)d\xi d\eta=$$

$$=\frac{1}{2i}\int_{\partial K}\frac{\overline{\Phi_1(\tau)}\overline{\varphi'(\tau)}}{\varphi_0(\tau;z)(\tau-z)}d\bar{\tau}-\frac{1}{2i}\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{\overline{\Phi_1(\tau)}\overline{\varphi'(\tau)}}{\varphi_0(\tau;z)(\tau-z)}d\bar{\tau}=-\frac{1}{2i}\int_{\partial K}\frac{\overline{\Phi_1(\tau)}\overline{\varphi'(\tau)}\bar{\tau}^2}{\varphi_0(\tau;z)(\tau-z)}d\tau,$$

where, using the substitution $\tau=z+\varepsilon e^{i\theta}$, it is easy to establish that

$$\lim_{\varepsilon\to 0}\int_{\gamma_\varepsilon}\frac{\overline{\Phi_1(\tau)}\overline{\varphi'(\tau)}}{\varphi_0(\tau;z)(\tau-z)}d\bar{\tau}=0.$$

Thus, for the third integral in representation (19), we have

$$\int_{\partial K}\frac{T\overline{\Phi_1(\varphi(t))}}{(t-z)^2}dt=\varphi'(z)\int_{\partial K}\frac{\overline{\Phi_1(\tau)}\overline{\varphi'(\tau)}\bar{\tau}^2}{\varphi_0(\tau;z)(\tau-z)}d\tau.$$

Now we calculate the fourth integral in the representation (19). Taking into account (20), (21), the formula (4.7) [10, p.28] and Cauchy's formula, we obtain

$$\int_{\partial K}\frac{\overline{T}\Phi_1(\varphi(t))}{(t-z)^2}dt=\int_{\partial K}\left(-\frac{1}{\pi}\iint_K\frac{\Phi_1(\zeta)\,|\,\varphi'(\zeta)\,|^2}{\overline{\varphi(\zeta)}-\overline{\varphi(t)}}d\xi d\eta\right)\frac{dt}{(t-z)^2}=-\frac{1}{\pi}\iint_K\left(\int_{\partial K}\frac{\overline{1/\varphi_0(t;\zeta)\,dt}}{(t-\zeta)(1-\bar{z}t)^2}\right)\Phi_1(\zeta)\,|\,\varphi'(\zeta)\,|^2\,d\xi d\eta=$$

$$=2i\iint_K\frac{\Phi_1(\zeta)\varphi'(\zeta)}{(1-z\bar{\zeta})^2}d\xi d\eta=\frac{2i}{z}\iint_K\frac{\partial}{\partial\bar{\zeta}}\left(\frac{\Phi_1(\zeta)\varphi'(\zeta)}{1-z\bar{\zeta}}\right)d\xi d\eta=\frac{1}{z}\int_{\partial K}\frac{\tau\Phi_1(\tau)\varphi'(\tau)}{\tau-z}d\tau=2\pi i\Phi_1(z)\varphi'(z).$$

Substituting the found expressions of the integrals into the formula (19), we obtain the following representation for $\Phi_2'(z)$:

$$\Phi_2'(z)=\frac{d_1}{2\pi}\int_{\partial K}\frac{\Phi_1(\tau)\varphi'(\tau)\overline{\varphi(\tau)}}{[\varphi_0(z;\tau)]^2(z-\tau)^2}d\tau-\frac{d_1}{2\pi\varphi'(z)}\int_{\partial K}\frac{\Phi_1(\tau)\overline{\varphi(\tau)}}{(\tau-z)^2}d\tau- \quad (22)$$

$$-\frac{d_1}{2\pi\varphi'(z)}\int_{\partial K}\frac{\overline{\Phi_1(\tau)}\varphi(\tau)}{(\tau-z)^2}d\tau-\frac{d_2}{2\pi}\int_{\partial K}\frac{\overline{\Phi_1(\tau)}\,\overline{\varphi'(\tau)}\,\overline{\tau}^{-2}}{\varphi_0(\tau;z)(\tau-z)}d\tau-id_2\Phi_1(z)-\frac{1}{\pi\varphi'(z)}\int_{\partial K}\frac{\operatorname{Re}Td[Tf](\varphi(\tau))d\tau}{(\tau-z)^2},\quad z\in\overline{K}.$$

To find $\Phi(z)$ we substitute in the formula (12) according to formulas (14) and put the resulting expression into the formula (16). After cumbersome calculations for the function $\Phi(z)$ we obtain the following representation

$$\Phi(z)=\frac{id_1}{2\pi}\int_{\partial K}\frac{\Phi_1(\tau)\varphi'(\tau)\overline{\varphi(\tau)}}{[\varphi_0(z;\tau)]^2(\tau-z)^2}d\tau-\frac{id_1}{2\pi\varphi'(z)}\int_{\partial K}\frac{\Phi_1(\tau)\overline{\varphi(\tau)}}{(\tau-z)^2}d\tau- \quad (23)$$

$$-\frac{id_2}{2\pi}\int_{\partial K}\frac{\overline{\Phi_1(\tau)}\,\overline{\varphi'(\tau)}\,\overline{\tau}^{-2}}{\varphi_0(\tau;z)(\tau-z)}d\tau-\frac{id_1}{2\pi\varphi'(z)}\int_{\partial K}\frac{\varphi(\tau)\overline{\Phi_1(\tau)}}{(\tau-z)^2}d\tau-\frac{1}{\pi\varphi'(z)}\int_{\partial K}\frac{l(w_3)(\varphi(t))\,|\,\varphi'(t)\,|}{t(t-z)}dt.$$

We substitute the expressions $\Phi_2'(z)$ from (22) and $\Phi(z)$ from (23) into the formula (18). After cumbersome transformations, we get the following representation for $\Phi_1(z)$:

$$\Phi_1(z)\equiv\Phi_1[l(w_3)](z)=2(\mu-1)S_{\partial K}(\operatorname{Re}Td[Tf](\varphi(\tau)))(z)+\frac{\mu-1}{\pi}\int_{\partial K}\frac{l(w_3)\varphi(t)\,|\,\varphi'(t)\,|}{t(t-z)}dt,\ z\in K,\quad (24)$$

$$S_{\partial K}f(z)=\frac{1}{2\pi i}\int_{\partial K}\frac{f(t)}{(t-z)^2}dt,$$

the operator $l(w_3)$ is introduced in the formula (12).

Let us denote via $\zeta=\psi(z)$ the function inverse to the function $z=\varphi(\zeta)$. From [10, p. 25] it is known that $\psi(z)\in C^1_\beta(\overline{\Omega})$. Then, obviously, $\Phi_1(\psi(z))\in W_p^{(1)}(\Omega)$, $2<p<2/(1-\beta)$.

Substituting the expression of the function $\Phi_1(z)$ from the formula (24) into the formula (9), we obtain the representations for $\Phi_2(\psi(z))$ of the following form

$$\Phi_2(\psi(z))=\Phi_2[l(w_3)](\psi(z))+ic_0,\quad z\in\Omega, \quad (25)$$

$$\Phi_2[l(w_3)](\psi(z))=-\frac{1}{2\pi}\int_{\partial K}(\operatorname{Re}Td[\Phi_1[l(w_3)]](t)+\operatorname{Re}Td[Tf]](t))\frac{t+\psi(z)}{t-\psi(z)}\frac{dt}{t}.$$

Substituting the representations (24), (25) into the formula (6), for the functions w_1, w_2, satisfying the system of the first two equations in (1) and the boundary conditions in (2), (3), under condition (17) we obtain the desired representation

$$\omega_0(z)=H_0w_3(z)+ic_0,\quad z\in\Omega, \quad (26)$$

$$H_0 w_3(z) \equiv H_0[f(w_3); l(w_3)](z) = \Phi_2[l(w_3)](\psi(z)) + iTd[\Phi_1[l(w_3)](\psi(\zeta)) + Tf(w_3)(\zeta)](z).$$

To transform the solvability condition (17), the function $h_2(\varphi(t))$ is replaced by the formula (12). Further, applying the formula (14) and Cauchy's formula, we can easily reduce the condition (17) to the form $\int_{\partial K} \left(l(w_3)(\varphi(t)) \,|\, \varphi'(t) \,|/t \right) dt = 0$, which, in turn, is transformed to the following final form

$$\int_{\Gamma} P^1(s) ds + \iint_{\Omega} R^1 d\alpha^1 d\alpha^2 = 0, \qquad (27)$$

where the functions $P^1(s)$, $R^1(\alpha^1, \alpha^2)$ – the components of the external load.

Further, from the last two equations of the system (1), we find the functions ψ_1, ψ_2, that satisfy the condition $\psi_2 = 0$ and the term (4) on the boundary. Note that the structure of the left-hand sides of the last two equations of the system (1) and boundary condition (4) is the same as for the functions w_1, w_2; они отличаются лишь правыми частями. they differ only in the right-hand sides. Therefore, for the functions ψ_1, ψ_2 with fixed right-hand sides, we find similar representations:

$$\psi = \psi_2 + i\psi_1 = H_0[g + \tilde{\psi}; l[g + \tilde{\psi}; \tilde{\varphi}_1] + ic_1, \qquad (28)$$

$$g \equiv g(w_3) = (g_1 + ig_2)/2, \;\; \tilde{\psi} = k_0(\psi_1 + i\psi_2)/2,$$

where the functions $g_j \equiv g_j(w_3)$ are defined in (5), the operator $H_0[f; g]$ – in (26), c_1 – an arbitrary real constant.

Besides, the solvability condition must be satisfied

$$\beta_1\Big(\int_{\Gamma} N^1(s) ds + \iint_{\Omega} L^1 d\alpha^1 d\alpha^2\Big) - k_0 \iint_{\Omega} \psi_1 d\alpha^1 d\alpha^2 = 0, \qquad (29)$$

where N^1, L^1 – the components of the external load.

Taking into account that the operator $H_0[g + \tilde{\psi}; l[g + \tilde{\psi}; \tilde{\varphi}_1]] = H_0[g; \tilde{\varphi}_1] + H_0[\tilde{\psi}; 0],$ of the function ψ_1, ψ_2 from (28) it can be represented in the form of an operator equation $\psi - K_0\psi = H_0[g; \tilde{\varphi}_1] + ic_1, \; K_0\psi = H_0[\tilde{\psi}; 0].$ Further, proceeding as in [6], we reduce this equation to the following form

$$\psi \equiv \psi(w_3) = (I - K_0)^{-1} H_0[g(w_3); \tilde{\varphi}_1]. \qquad (30)$$

In this case, the solvability condition (29) will be satisfied identically. Thus, we obtain an unambiguous representation for the functions ψ_1, ψ_2 through the deflection w_3 of the form (30).

Problem A will be reduced to a single operator equation for the function w_3. For this, in the third equation of system (1), the functions w_1, w_2, ψ_1, ψ_2 and their first-order derivatives are replaced by expressions from (26), (30). Then we arrive at a nonlinear second-order partial differential equation with respect to w_3 of the form:

$$w_{3\alpha^1\alpha^1} + w_{3\alpha^2\alpha^2} + K_1 w_3 + G_1 w_3 = 0, \qquad (31)$$

where

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$$K_1 w_3 = \psi_{11\alpha^1}(w_3) + \psi_{21\alpha^2}(w_3) + \{k_3 w_{11\alpha^1}(w_3) + k_4 w_{21\alpha^2}(w_3) - k_5 w_3\}/(k^2\mu_1),$$

$$G_1 w_3 = \psi_{10\alpha^1} + \psi_{20\alpha^2} + \{k_3 w_{12\alpha^1}(w_3) + k_4 w_{22\alpha^2}(w_3) + (k_3 w_{3\alpha^1}^2 + k_4 w_{3\alpha^2}^2)/2 +$$
$$+\beta_2[(T^{\lambda\mu} w_{3\alpha^\lambda})_{\alpha^\mu} + R^3]\}/(k^2\mu_1).$$

Thus, finding the solution to the problem A was reduced to the equation (31) solution for the function w_3 with the condition $w_3 = 0$ on the boundary Γ. Next, we reduce the equation (31) with the term $w_3 = 0$ on the boundary Γ to the equivalent equation

$$w_3 + G_* w_3 = 0, \qquad (32)$$

$G_* w_3 = (I+K)^{-1} G w_3$, $G w_3 = \iint_\Omega H(\zeta, z) G_1 w_3(\zeta)\, d\xi d\eta$, $H(\zeta, z) = \dfrac{1}{2\pi} \ln \dfrac{|\psi(z) - \psi(\zeta)|}{|1 - \psi(z)\overline{\psi(\zeta)}|}$ – the harmonic Green's function for the domain Ω, the operator $G_1 w_3$ is defined in the formula (31).

Moreover, for any two values of $w_3^j\ (j=1,2) \in W_p^{(2)}(\Omega)$, belonging to the ball $\|w_3\|_{W_p^{(2)}(\Omega)} < r$, the following estimate is fair

$$\| G_* w_3^1 - G_* w_3^2 \|_{W_p^{(2)}(\Omega)} \le q_* \| w_3^1 - w_3^2 \|_{W_p^{(2)}(\Omega)}, \quad q_* = c \| (I+K_0)^{-1} \|_{W_p^{(2)}(\Omega)} [q_0 + (1+r)r],$$

$$q_0 = \sum_{\lambda,\mu=1}^{2} \| T^{\lambda\mu}(0) \|_{C(\bar{\Omega})} + \sum_{\lambda=1}^{2} \| R^\lambda \|_{L_p(\Omega)}.$$

IV. CONCLUSIONS

Let us assume that the external forces acting on the shell and the radius r of the sphere are such that the following conditions are satisfied:

$$q_* < 1, \quad \|G_*(0)\|_{W_p^{(2)}(\Omega)} < (1 - q_*) r. \qquad (33)$$

Then the equation (32) in the ball $\|w_3\|_{W_p^{(2)}} < r$ has a unique solution $w_3 \in W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$, according to [12]. Substituting the found solution $w_3 \in W_p^{(2)}(\Omega)$ into the formulas (26), (30), we find the functions $w_1, w_2, \psi_1, \psi_2 \in W_p^{(2)}(\Omega)$. Moreover, the functions w_2, ψ_1, ψ_2 are determined uniquely, and the function w_1 – up to a constant term c_0. The condition (27) is not only sufficient, but also a necessary condition for the solvability of the problem A.

Thus, the following main theorem is true.

Theorem. Let conditions a), b) from the problem A, and the inequality (33) be satisfied. Then, to solve the problem A, it is necessary and sufficient that the condition (27) be satisfied. In the case of its fulfillment, the problem has a generalized solution $a = (w_1, w_2, w_3, \psi_1, \psi_2) \in W_p^{(2)}(\Omega)$, $2 < p < 2/(1-\beta)$, *in which the components* w_2, w_3, ψ_1, ψ_2 are determined uniquely, and the component w_1 – up to a constant term.

V. SUMMARY

This work is devoted to the study of the solvability and the proof of the existence theorem for the solution of the boundary value problem A for a system of nonlinear partial differential equations of the second order under other boundary conditions describing the equilibrium state of elastic shallow isotropic homogeneous shells with hinged edges in the framework of shift model by S.P. Tymoshenko. The reliability of the research results is based on a rigorous mathematical study of the boundary value problem solvability, and the proof of the existence theorem. Shell structures (various building structures, domes, etc.) require more and more reliable, accurate design data and often pose completely new challenges. This work makes a significant contribution to the study of this class of problems solvability within the framework of more general models, as well as during the solution of specific applied problems for a wider class of elastic structures.

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About Author

Lilia Sergeevna Kharasova was born on February 16, 1982 in the city of Naberezhnye Chelny, RT. After graduating from the Novosibirsk State Pedagogical Institute with honors in 2004, she remained to work as an Assistant at the Department of Mathematical Analysis. Since 2013, she is the Senior Lecturer at the Department of Mathematics of the Civil Engineering Department of the Naberezhnye Chelny Institute, K(P)FU. The field of her work is physics and mathematics.