International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 13, Number 12 (2020), pp. 4679-4685

Hyers-Ulam-Rassias stability of nonlinear fractional differential equation with three point integral boundary conditions

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Abstract

In this paper, we prove the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of a nonlinear fractional differential equation with three point integral boundary conditions.

MSC: 34A08; 34K10, 34K20

Keywords: fractional differential equation, Hyers-Ulam-Rassias stability, boundary condition

1. Introduction

The study of differential equation with fractional order has great attention in recent years. This concept is not new and is very much as old as classical differential equations but no one has the answer at that time. Nowadays fractional calculus is the area which deals about it and we notice that fractional derivative means, the derivative of arbitrary order. Fractional differential equation is considered to be an alternate model for nonlinear differential equations. Therefore, it get more attention to study. There are many authors discussed the existence results of fractional differential equations using various fixed point theorems. For example, one can refer the monographs of Kilbas et al. [12], Miller and Ross [16], Podulbny [17], Diethelm et al. [5, 6], Benchora [3] and so on. Obviously, the differential equations of fractional order has been proved to be a valuable tool in the modeling of many phenomena in various fields of science and engineering. Indeed, one can find many applications in electromagnetic, control, electrochemistry etc. (see [7]- [9]).

On the otherhand, the stability concept is widely studied on functional equations. But the analysis of stability concepts of fractional differential equations has been very slow and there are only countable numbers of works. In 2009, Li [14], first proposed the Mittag-Leffler stability and in 2010 [15], the fractional Lyapunov's second method. In the next year, Li and Zhang [13] have been given a brief overview on the stability of the fractional differential equations. However, there are only few works available on the local stability and Mittag-Leffler stability for fractional differential equations and very rare works on the Ulam stability of fractional differential equations.

In 2011, Wang [21] carried out a pioneering work on the Hyers-Ulam stability and data dependence for fractional differential equations with Caputo derivative. Wang [22] proved the Hyers-Ulam stability of fractional differential equation of order $0 < \alpha < 1$ via a generalized fixed point approach, by adopting some part idea of Wang et al. [21], Cadariu and Radu [4] and Jung [11] in the next year. Particularly, there are very rare works on the Hyers-Ulam stability of fractional differential equations with boundary conditions. Recently, Rabha [10], have given Ulam stabilities with boundary conditions in the interval$(0, 1)$. For more information on functional equations and their stability problems, see [18]- [20].

In this paper, the Hyers-Ulam stability of the following fractional boundary value problem is proven.

$$^{C}D^{\alpha}x(t) = f\big(t, x(t)\big), \quad 0 < t < 1, \quad 1 < \alpha < 2 \qquad (1.1)$$

$$x(0) = 0, \quad x(1) = a\int_0^{\mu} x(s)ds, \quad 0 < \mu < 1 \qquad (1.2)$$

Where $^{C}D^{\alpha}$ denotes the caputo fractional derivative of order α, $f : [0,1] \times X \to X$ is continuous, and $a \in R$ is such that $a \neq \frac{2}{\mu^2}$.

This paper is organized as follows: In Section 2, basic definitions and notations are given. In Section 3, the Generalised Hyers-Ulam stability of the above fractional boundary value problem is proved. In section 4, the Hyer-Ulam stability of given boundary value problem is proved.

2. Preliminaries

Throughout this paper, we assume that Y is a normed space and $I = [0, T]$ is a given interval.

Definition 2.1 [3]

Given an interval [a, b] of R. The fractional order integral of a function □ $\in L^1([a, b], R)$ of order $\alpha \in R_+$ is defined by

$$I^{\alpha}_{a+}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)ds$$

Where Γ is the gamma function.

Definition 2.2 [3]

For a function h given on the interval [a, b], the Caputo fractional order derivative of h, is defined by

$$({}^{C}D^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{n}(s)ds$$

Where $n = [\alpha] + 1$.

Lemma 2.4 [1]

A unique solution of the boundary value problem (1.1) is given by

$$x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s)\big)ds-\frac{2t}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f\big(s,x(s)\big)ds$$
$$+\frac{2at}{(2-a\mu^2)\Gamma(\alpha)}\int_0^{\mu}\left(\int_0^s (s-m)^{\alpha-1} f\big(m,x(m)\big)dm\right)ds$$

Definition 2.5 [22]

A function $d: X \times X \to [0,\infty]$ is called a generalized metric on X if and only if d satisfies

(A1) $d(x,y)=0$ if and only if $x=y$;

(A2) $d(x,y)= d(y,x)$ for all $x,y\in X$;

(A3) $d(x,z) \le d(x,y)+d(y,z)$ for all $x,y,z \in X$;

Theorem 2.1 [4]

Let (X, d) be a generalized complete metric space. Assume that $\Lambda: X\to X$ is a strictly contractive operator with the Lipschitz constant L<1. If there exists a nonnegative integer k such that $d\big(\Lambda^{k+1},\Lambda^{k}x\big)<\infty$ for some $x\in X$, then for following are true:

(a) The sequence $\{\Lambda^n x\}$ converges to a fixed point x^* of Λ.

(b) x^* is the unique fixed point of Λ in $x^* = \{y\in X \mid d\big(\Lambda^k x,\ y\big)<\infty\}$;

(c) If $y\in X^*$ then $d(y,x^*)=\frac{1}{1-L}d(\Lambda y,y)$.

3. Hyers-Ulam-Rassias stability

In this section, we first prove the generalized Hyers-Ulam stability of the fractional differential equation (1.1) with given integral boundary condition (1.2) using Theorem (2.1).

Theorem 3.1

Let $I=[0,T]$ be a closed interval. Assume that $f: I \times R \to R$ is a continuous function which satisfies the standard Lipschitz condition

$$|f(t,y)-f(t,z)|<L|y-z| \tag{3.1}$$

for all $t\in I$ and $y,z\in R$. If a continuously differential function $x: I\to R$ satisfies

$$|\,{}^{c}D^{\alpha}x(t)-f(t,x(t))|<\varphi(t) \tag{3.2}$$

for all $t\in I$, where $\varphi: I \to (0,\infty)$ is a continuous function with

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)ds\right|<K_1\varphi(t) \tag{3.3}$$

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^{\mu}\varphi(s)ds\right|< K_2\varphi(t) \tag{3.4}$$

for all $t\in I$, then there exists a unique continuous function $x_0: I \to R$ such that

$$x_0(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,x(s)\big)ds-\frac{2t}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f\big(s,x(s)\big)ds$$
$$+\frac{2at}{(2-a\mu^2)\Gamma(\alpha)}\int_0^{\mu}\Big(\int_0^s (s-m)^{\alpha-1} f\big(m,x(m)\big)dm\Big)\,ds \tag{3.5}$$

and

$$|x(t) - x_0(t)| \le \frac{K_1}{1 - \left(LK_1 + \frac{2tLK_1}{(2 - a\mu^2)} + \frac{2atLK_1K_2}{(2 - a\mu^2)}\right)} \varphi(t) \tag{3.6}$$

for all $t \in I$.

Proof

Let us define a set $X$of all continuous functions F: $I \ \rightarrow R$ by

$$X = \{F: I \ \rightarrow R \ |F \ is \ continuous\} \tag{3.7}$$

Introduce a generalized complete metric on X as follows

$$d(F,G) = \inf\{C \in [0,\infty] | \ |F(t) - G(t)| < C\varphi(t) \ for \ all \ t \in I\} \tag{3.8}$$

Define an operator $\Lambda: X \ \rightarrow X$ by

$$(\Lambda F)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\big(s, x(s)\big) ds - \frac{2t}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f\big(s, x(s)\big) ds$$
$$+ \frac{2at}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^\mu \left(\int_0^s (s-m)^{\alpha-1} f\big(m, x(m)\big) dm \right) ds \tag{3.9}$$

for all $F \in X$.

Since F and f are continuous functions, it is easy to see that Λ is well defined.

To achieve our aim, we need to prove that Λ is strictly contractive X.

For any $F, G \ \in X$, let $C_{FG} \in [0,\infty]$ be an arbitrary constant with $d(F,G) < C_{FG}$

That is by (3.7), we have

$$|F(t) - G(t)| < C_{FG}\varphi(t) \tag{3.10}$$

for all $t \in I$.

It then follows from (3.1), (3.3), (3.7), (3.9) and (3.10) that

$|(\Lambda F)t - (\Lambda G)t|$

$$\le \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f\big(s, F(s)\big) - f(s, G(s))\right| ds + \frac{2t}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \left|f\big(s, F(s)\big) - f(s, G(s))\right| ds$$
$$+ \frac{2at}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^\mu \left(\int_0^s (s-m)^{\alpha-1} \left|f\big(m, F(m)\big) - f(m, G(m))\right| dm \right) ds$$

$$\le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |F(s) - G(s)| ds + \frac{2tL}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} |F(s) - G(s)| ds$$
$$+ \frac{2atL}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^\mu \left(\int_0^s (s-m)^{\alpha-1} |F(m) - G(m)| dm \right) ds$$

$$\le \frac{LC_{FG}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds + \frac{2tLC_{FG}}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \varphi(s) ds + \frac{2atLC_{FG}}{(2 - a\mu^2)\Gamma(\alpha)} \int_0^\mu \left(\int_0^s (s-m)^{\alpha-1} \varphi(m) dm \right) ds$$

$$\le LK_1C_{FG}\varphi(t) + \frac{2tLK_1C_{FG}\varphi(t)}{(2 - a\mu^2)} + \frac{2atLK_1K_2C_{FG}\varphi(t)}{(2 - a\mu^2)}$$

$$\le \left(LK_1 + \frac{2tLK_1}{(2 - a\mu^2)} + \frac{2atLK_1K_2}{(2 - a\mu^2)}\right) C_{FG}\varphi(t)$$

for all $t \in I$.

That is,

$$d(\Lambda F,\Lambda G)\le \left(LK_1+\frac{2tLK_1}{(2-a\mu^2)}+\frac{2atLK_1K_2}{(2-a\mu^2)}\right)C_{FG}$$

Hence we can conclude that

$$d(\Lambda F,\Lambda G)\le \left(LK_1+\frac{2tLK_1}{(2-a\mu^2)}+\frac{2atLK_1K_2}{(2-a\mu^2)}\right)d(F,G)$$

for all F, $G \in X$, where we note that 0 < $\left(LK_1+\frac{2tLK_1}{(2-a\mu^2)}+\frac{2atLK_1K_2}{(2-a\mu^2)}\right)$< 1.

It follows from (3.7) and (3.9) that for an arbitrary $g_0\in X$, there exists a constant $0<C<\infty$ with

$$\begin{aligned}|(\Lambda g_0)(t)-(g_0)(t)|&\\ \le \Bigg|\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds&-\frac{2t}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s))ds\\ &+\frac{2at}{(2-a\mu^2)\Gamma(\alpha)}\int_0^\mu\left(\int_0^s(s-m)^{\alpha-1}f(m,x(m))dm\right)-g_0(t)\Bigg|\\ &\le C\varphi(t)\end{aligned}$$

for all$t\in I$, since $f(t,g_0(t))$ and $g_0(t)$ are bounded on I and $\min_{t\in I}\varphi(t)>0$

Thus (3.8) implies that $d(\Lambda g_0,g_0)<\infty$

Therefore, according to Theorem (2.1), there exists a continuous function $x_0: I\to R$ such that $\Lambda^n g_0\to x_0$ in (X,d) and $\Lambda x_0=x_0$, that is, x_0 satisfies (3.5) for every $t\in I$.

We will now verify that $\{g\in X\ /\ d(g_0,g)<\infty\}=X$.

For any $g\in X$, since g and g_0 are bounded on I and $\min_{t\in I}\varphi(t)>0$, there exists a constant

$0<C_g<\infty$ such that $|g_0(t)-g(t)|\le C_g\varphi(t)$.

Hence, we have $d(g_0,g)<\infty$ for all $g\in X$, that is $\{g\in X/d(g_0,g)<\infty\}=X$.

Hence in view of Theorem (2.1), we conclude that x_0 is the unique continuous function with the property (3.5). On the other hand, it follows from (3.2) that

$$-\varphi(t)\le {}^CD^\alpha x(t)-f(t,x(t))\le\varphi(t)$$

for all $t\in I$.

If we integrate each term in the above inequality and substitute the boundary conditions, then we obtain

$$\left|x(t)-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s))ds+\frac{2t}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s))ds-\frac{2at}{(2-a\mu^2)\Gamma(\alpha)}\int_0^\mu\left(\int_0^s(s-m)^{\alpha-1}f(m,x(m))dm\right)ds\right|\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\varphi(s)ds$$

for all $t\in I$.

Thus by (3.3) and (3.9) we get

$$|x(t)-(\Lambda x)(t)|\le K_1\varphi(t)$$

for each $t\in I$, which implies that

$$d(x,\Lambda x)\le K_1\varphi(t) \tag{3.11}$$

Finally, Theorem (2.1) and (3.11) imply that

$$\begin{aligned}d(x,x_0)&\le\frac{1}{1-\left(LK_1+\frac{2tLK_1}{(2-a\mu^2)}+\frac{2atLK_1K_2}{(2-a\mu^2)}\right)}d(x,\Lambda x)\\ &\le\frac{K_1}{1-\left(LK_1+\frac{2tLK_1}{(2-a\mu^2)}+\frac{2atLK_1K_2}{(2-a\mu^2)}\right)}\varphi(t)\end{aligned}$$

which implies the validity of (3.6) for each $t\in I$

4. Hyers-Ulam stability

In this section we will prove the Hyers-Ulam stability of (1.1) with boundary conditions (1.2).

Theorem 4.1

Let $I=[0,T]$ be a closed interval. Assume that $f: I \times R \to R$ is a continuous function which satisfies a Lipschitz condition (3.1) for all $t \in I$ and $y, z \in R$, where L is a constant. If a continuously differentiable function $x: I \to R$ satisfying the differential inequality

$$|\,{}^cD^\alpha x(t) - f(t, x(t))| < \varepsilon \tag{4.1}$$

for all $t \in I$and for some $\varepsilon \geq 0$, then there exists a unique continuous function $x_0: I \to R$ satisfying (3.5) and

$$|x(t) - x_0(t)| \leq \frac{\varepsilon}{1 - L\left(1 + \frac{2}{(2-a\mu^2)} + \frac{2a\mu^{\alpha+1}}{(2-a\mu^2)(\alpha+1)}\right)} \tag{4.2}$$

for all $t \in I$.

Proof

Let us define a set $X$of all continuous functions $F: I \to R$ by

$$X = \{F: I \to R \,|F\, is\ continuous\}$$

Introduce a generalized complete metric on X as follows

$$d(F,G) = \inf\{C \in [0,\infty] |\, |F(t) - G(t)| < C\ for\ all\ t \in I\} \tag{4.3}$$

Define an operator $\Lambda: X \to X$ by (3.9)

We now assert that Λ is strictly contractive on X.

For all F, $G \in X$, let $C_{FG} \in [0,\infty]$ be an arbitrary constant with $d(F,G) \leq C_{FG}$ that is, let us assume that

$$|F(t) - G(t)| \leq C_{FG}$$

for any $t \in I$. It then follows from (3.1), (3.9) and (4.3) that

$$|(\Lambda F)t - (\Lambda G)t|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,F(s)) - f(s,G(s))\right|ds + \frac{2t}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\left|f(s,F(s)) - f(s,G(s))\right|ds$$

$$+\frac{2at}{(2-a\mu^2)\Gamma(\alpha)}\int_0^\mu\left(\int_0^s (s-m)^{\alpha-1}\left|f(m,F(m)) - f(m,G(m))\right|dm\right)ds$$

$$\leq \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|F(s) - G(s)|ds + \frac{2tL}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|F(s) - G(s)|ds$$

$$+\frac{2atL}{(2-a\mu^2)\Gamma(\alpha)}\int_0^\mu\left(\int_0^s (s-m)^{\alpha-1}|F(m) - G(m)|dm\right)$$

$$\leq \frac{LC_{FG}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\,ds + \frac{2tLC_{FG}}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\,ds$$

$$+\frac{2atLC_{FG}}{(2-a\mu^2)\Gamma(\alpha)}\int_0^\mu\left(\int_0^s (s-m)^{\alpha-1}\,dm\right)ds$$

$$\leq \frac{LC_{FG}}{\Gamma(\alpha+1)} + \frac{2LC_{FG}}{(2-a\mu^2)\Gamma(\alpha+1)} + \frac{2a\mu^{\alpha+1}LC_{FG}}{(2-a\mu^2)\Gamma(\alpha+2)}$$

$$\leq \frac{LC_{FG}}{\Gamma(\alpha+1)}\left(1 + \frac{2}{(2-a\mu^2)} + \frac{2a\mu^{\alpha+1}}{(2-a\mu^2)(\alpha+1)}\right)$$

for all $t \in I$. That is

$$d(\Lambda F, \Lambda G) \leq \frac{LC_{FG}}{\Gamma(\alpha+1)}\left(1 + \frac{2}{(2-a\mu^2)} + \frac{2a\mu^{\alpha+1}}{(2-a\mu^2)(\alpha+1)}\right)$$

Thus it follows that

$$d(\Lambda F,\Lambda G) \leq \frac{L}{\Gamma(\alpha+1)}\left(1+\frac{2}{(2-a\mu^2)}+\frac{2a\mu^{\alpha+1}}{(2-a\mu^2)(\alpha+1)}\right)d(f,g)$$

for all $f,g \in X$, and we note that $0<\frac{LC_{FG}}{\Gamma(\alpha+1)}\left(1+\frac{2}{(2-a\mu^2)}+\frac{2a\mu^{\alpha+1}}{(2-a\mu^2)(\alpha+1)}\right)<1$

Analogously to the proof of Theorem (2.1), we can show that each $g_0 \in X$ satisfies the property $d(\Lambda g_0, g_0)<\infty$.

Therefore, Theorem (2.1) implies that there exists a continuous function $x_0: I \to R$ such that $\Lambda^n g_0 \to x_0$ in (X,d) as $n\to\infty$, and such that $x_0=\Lambda x_0$, that is, x_0 satisfies the equation (3.4) for all $t\in I$.

If $g\in X$, then g_0 and g are continuous functions defined on a compact interval I. Hence, there exists a constant $C>0$ with $|g_0(t)-g(t)|<C$ for all $t\in I$. This implies that $d(g_0,g)<\infty$ for every g$\in X$. Therefore, according to Theorem (2.1), x_0 is a unique continuous function with property (3.4). Furtheremore, it follows from (4.1) that

$$-\varepsilon \leq {}^{C}D^{\alpha}x(t)-f(t,x(t)) \leq \varepsilon$$

for all $t\in I$. If we integrate each term of the above inequality and applying the boundary conditions, we have

$$\left|y(t)-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\big(s,x(s)\big)ds-\frac{2t}{(2-a\mu^2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}f\big(s,x(s)\big)ds+\frac{2at}{(2-a\mu^2)\Gamma(\alpha)}\int_0^{\mu}\Big(\int_0^s (s-m)^{\alpha-1}f\big(m,x(m)\big)dm\Big)ds\right| \leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varepsilon ds$$

Thus by and we get,

$$|x(t)-\Lambda x(t)| \leq \frac{\varepsilon}{\Gamma(\alpha+1)}$$

for all $t\in I$, that is, it holds that $d(x,\Lambda x)\leq \frac{t^{\alpha}}{\Gamma(\alpha+1)}\varepsilon$

It now follows from Theorem (2.1) that

$$d(x,x_0) \leq \frac{1}{1-\frac{L}{\Gamma(\alpha+1)}\left(1+\frac{2}{(2-a\mu^2)}+\frac{2a\mu^{\alpha+1}}{(2-a\mu^2)(\alpha+1)}\right)}d(y,\Lambda y)$$

$$\leq \frac{\varepsilon}{1-L\left(1+\frac{2}{(2-a\mu^2)}+\frac{2a\mu^{\alpha+1}}{(2-a\mu^2)(\alpha+1)}\right)}$$

which implies the validity of (4.2) for each $t\in I$.

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