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About Central Difference-Techniques in Solving SPBVPs

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Abstract

In this paper, a numerical comparison of four difference-techniques recently developed by Algelani and El-Zahar [5, 7] in solving linear SPBVPs is presented. First, the order of convergence of the difference-techniques developed in [5] is estimated. Then these four central difference-techniques are applied to SPBVPs and the numerical results are compared with other difference-techniques in the literature. The numerical results confirm the theoretical ones and show that the above techniques result in an accurate solution of SPBVPs.

Keywords: Central finite difference techniques; Order of convergence; Accuracy.

I. INTRODUCTION

Singularly perturbed boundary value problems (SPBVPs) arise very frequently in fluid mechanics, heat transfer, chemical reactions, weather prediction, nanofluid flow, optimal control theory, and other branches of Applied Mathematics. These problems depend on a small positive parameter multiply the highest derivative term in a differential equation in such a way that the solution varies rapidly in some parts and varies slowly in some other parts. Solutions of such problems display sharp boundary layers when the singular perturbation parameter is much smaller than 1. For a detailed discussion on the analytical and numerical treatment of SPBVPs we may refer the reader to the books of O'Malley [1], Doolan et al. [2], Roos et al. [3], Miller et el. [4] and references therein [5-18] . Recently El-Zahar and Algelany [5] have followed the idea in [6, 8, 9] to present three central difference-techniques for linear SPBVPs over unevenly spaced grid points and have studied uniqueness and stability conditions for each technique. Algelany and El-Zahar [7] have extended their work in [5] to present a fourth-order central difference-techniques for linear SPBVPs with variable coefficients over unevenly spaced grid points, and they have studied uniqueness and stability conditions at constant coefficients and proved that the present centered difference technique has a fourth-order of convergence at evenly spaced grid. In this paper, a numerical comparison of four difference-techniques recently developed by Algelani and El-Zahar [5, 7] in solving linear SPBVPs is presented. First, the order of convergence of the difference-techniques developed in [5] is estimated. Then these four central difference-techniques are applied to SPBVPs and the numerical results are compared with other difference-techniques in the literature. The numerical results confirm the theoretical ones and show that the above techniques result in an accurate solution of SPBVPs.

II. CENTRAL DIFFERENCE-TECHNIQUES [5, 6]

The four central difference-techniques in [5, 7] are developed for solving the linear SPBVPs defined by

$$L(y) \equiv -\varepsilon y'' + p(x)y' + q(x)y = f(x), \qquad a \le x \le b \ , \tag{1}$$

with boundary conditions

$$y(a) = \alpha \ \text{ and } \ y(b) = \beta \ ,$$

where ε is a small positive parameter $(0 < \varepsilon \ll 1)$, α and β are given constants, $p(x)$, $q(x)$ and $f(x)$ are assumed to be sufficiently continuously differentiable functions on $[a,b]$. More assumption that $q(x) > 0$ and $p(x) < P < 0$ for all $x \in [a,b]$, where P is some negative constant. Also, the interval $[a,b]$ is divided such that $x_0 = a < x_1 < x_2 < ...... < x_N = b$ with step size $h_i = x_i - x_{i-1}$, $i = 1,2,...N$. For the simplicity, the authors have used $p_i = p(x_i)$, $q_i = q(x_i)$, $f_i = f(x_i)$, $y_{i-1} = y(x_{i-1})$, $y_{i+1} = y(x_{i+1})$, and $y'_i = y'(x_i)$, etc. Algelany and El-Zahar [7] have proved that the developed central difference technique in [7] has a local truncation error at fixed step size h given by

$$\tau\Big|_{\omega\to\infty} = \left(\frac{p_{i14}}{q\left(h^2 q_i^2 + 12 p_i\, 2\right)} \right) \left(\frac{h^4}{30} \right) y^{(6)}(\zeta) \ , \tag{2}$$

where $\omega = 1/\varepsilon$ and $\zeta \in [x_{i-1}, x_i]$.

Thus, the central difference technique in [5] has a fourth-order of convergence. Using the same procedure in [7], the local truncation error τ of centered difference-techniques in [5] can be estimated and given by

For Technique **I**

$$\tau = -p_i \omega \left(\frac{h^2}{6} \right) y^{(3)}(\zeta) \,. \tag{3}$$

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For Technique **II**

$$\tau = \left(\frac{h^2\omega(p_i{}^2\omega + q_i) + 6)}{(h^2 q_i \omega + 6)}\right)\left(\frac{h^2}{12}\right) y^{(4)}(\zeta) \qquad (4)$$

For Technique **III**

$$\tau = \left(\frac{\omega p_i\,(h^2\omega(p_i{}^2\omega + 2q_i) + 12}{\left(h^4 q_i{}^2\omega^2 + 6h^2\omega(p_i{}^2\omega + 3q_i) + 72\right)}\right)\left(\frac{h^4}{20}\right) y^{(5)}(\zeta)\ . \qquad (5)$$

Thus, both of the techniques **I** and **II** have a second-order of convergence while the technique **III** has a fourth-order of convergence.

III. NUMERICAL RESULTS

In this section, a numerical comparison of the four central difference-techniques developed in [5, 7] with other techniques in literature is discussed.

Problem1.Consider the following SBPVP [6,8,9]

$$-\varepsilon y''(x) + y'(x) + y(x) = \varepsilon\pi^2 \sin(\pi x) + \pi\cos(\pi x) + \sin(\pi x)\,, \quad (6)$$

with boundary conditions $y(0) = 0$ and $y(1) = 0$. The exact solution is $y(x) = \sin(\pi x)$. Technique **I**, Technique **II**, Technique **III** in [5] and Technique **IV** in [7] are applied to the SPBVP (1) using fixed step size and the maximum absolute solution error for different values of ε and grid points N is presented in Tables1-4. Moreover, the order of convergence of each technique is estimated numerically and plotted in Figures 1-4. We will denote Technique **I**, Technique **II**, Technique **III** in [5] and Technique **IV** in [7] by **T1**, **T2, T3** and **T4** respectively.

Tables 1-4 show that **T3** and **T4** result in a more accurate solution than that obtained using **T1** and **T2** for the same values of N, ε. Figures1-4 show that each of **T1** and **T2** has at least second-order of convergence whereas each of **T3** and **T4** has at least fourth-order of convergence.

Table 1. Maximum solution error of **T1** in solving SPBVP(6) at different values of ε and N

ε	$N=10$	$N=20$	$N=100$	$N=200$
10^{-2}	1.2608E-02	3.0629E-03	1.2262E-04	3.0655E-05
10^{-3}	1.4499E-02	3.3928E-03	1.2500E-04	3.1248E-05
10^{-4}	1.4854E-02	3.6998E-03	1.2867E-04	3.1308E-05
10^{-5}	1.4891E-02	3.7376E-03	1.4659E-04	3.4369E-05
10^{-6}	1.4895E-02	3.7415E-03	1.5009E-04	3.7219E-05
10^{-7}	1.4895E-02	3.7419E-03	1.5046E-04	3.7582E-05

Table 2. Maximum solution error of **T2** in solving SPBVP(6) at different values of ε and N

ε	$N=10$	$N=20$	$N=100$	$N=200$
10^{-2}	4.1318E-03	4.1615E-04	3.9761E-06	8.8777E-07
10^{-3}	7.0612E-03	1.4499E-03	5.9568E-06	4.4168E-07
10^{-4}	7.4574E-03	1.8266E-03	3.9641E-05	3.3805E-06
10^{-5}	7.4951E-03	1.8648E-03	7.0468E-05	1.4498E-05
10^{-6}	7.4989E-03	1.8685E-03	7.4268E-05	1.8253E-05
10^{-7}	7.4993E-03	1.8689E-03	7.4642E-05	1.8629E-05

Table 3. Maximum solution error of **T3** in solving SPBVP(6) at different values of ε and N

ε	$N=10$	$N=20$	$N=100$	$N=200$
10^{-2}	5.9259E-05	3.6949E-06	6.0097E-09	3.7615E-10
10^{-3}	6.9944E-05	3.9845E-06	6.1503E-09	3.8461E-10
10^{-4}	7.3264E-05	4.5167E06	6.1953E-09	3.8611E-10
10^{-5}	7.3636E-05	4.6085E-06	7.0621E-09	4.0278E-10
10^{-6}	7.3673E-05	4.6180E-06	7.3868E-09	4.5439E-10
10^{-7}	7.3677E-05	4.6189E-06	7.4231E-09	4.6315E-10

Table 4. Maximum solution error of Technique **T4** in solving SPBVP(6) at different values of ε and N

ε	$N=10$	$N=20$	$N=100$	$N=200$
10^{-2}	1.0105E-05	2.3034E-07	1.2155E-10	7.2118E-12
10^{-3}	2.2245E-05	1.0022E-06	1.2102E-10	2.3851E-12
10^{-4}	2.4476E-05	1.4787E-06	9.2884E-10	3.5198E-10
10^{-5}	2.4685E-05	1.5320E-06	2.2244E-09	1.0096E-10
10^{-6}	2.4705E-05	1.5371E-06	2.4345E-09	1.4670E-10
10^{-7}	2.4707E-05	1.5377E-06	2.4545E-09	1.5421E-10

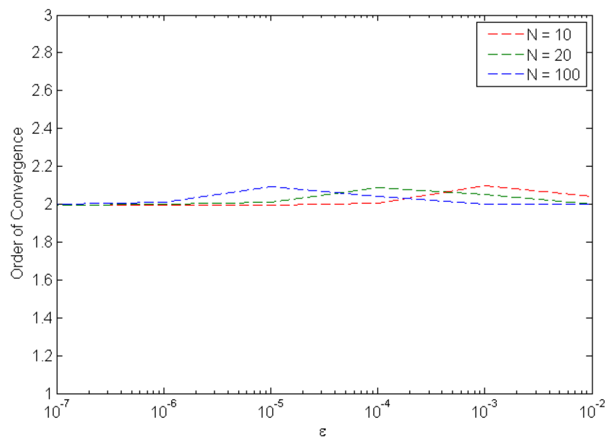

Figure 1. Computed order of convergence for **T1** in solving SPBVP(6)

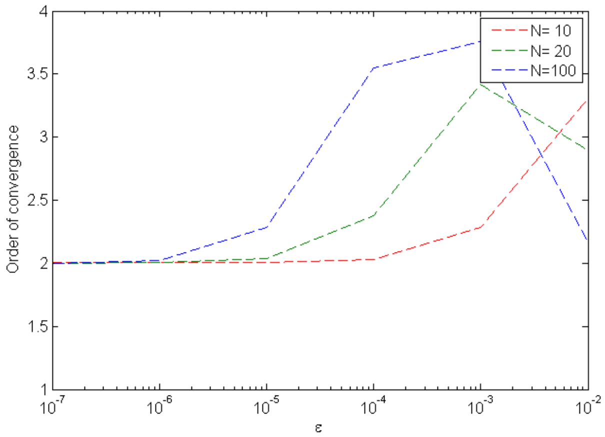

Figure 2. Computed order of convergence for **T2** in solving SPBVP(6)

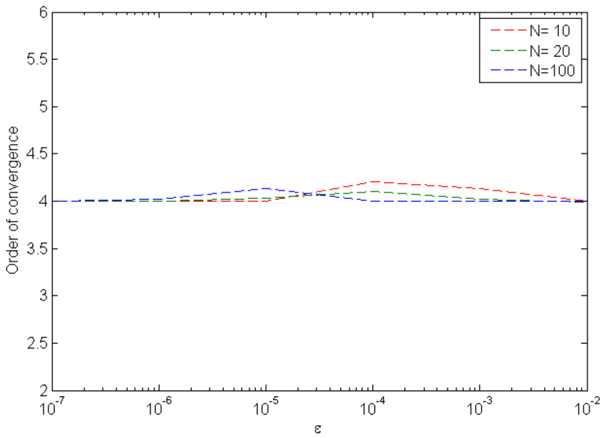

Figure 3. Computed order of convergence for **T3** in solving SPBVP(6)

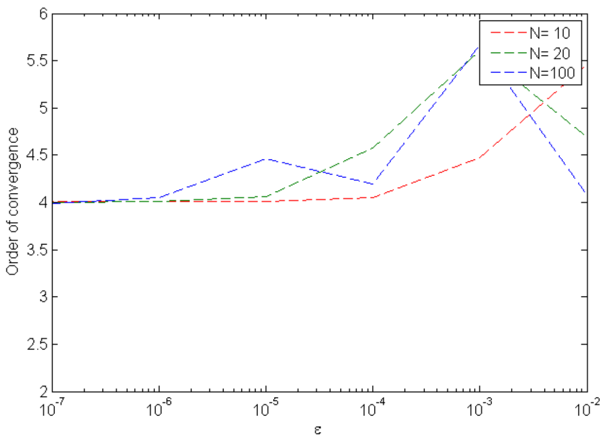

Figure 4. Computed order of convergence for **T4** in solving SPBVP(6)

Table 5. Comparison of maximum solution error for SPBVP (6) using different difference techniques

Method	N	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
UD	10	2.90E-001	3.10E-001	3.10E-001	3.10E-001
	20	1.50E-001	1.50E-001	1.60E-001	1.60E-001
IL'in	10	2.50E-001	3.00E-001	N.A	N.A
	20	9.50E-002	1.50E-001	1.60E-001	N.A
CD	10	1.60E-002	1.60E-002	1.60E-002	1.60E-002
	20	4.10E-003	4.10E-003	4.10E-003	4.10E-003
SCD2	10	6.0E-003	8.20E-003	8.30E-003	8.30E-003
	20	6.10E-004	1.90E-003	2.10E-003	2.10E-003
SCD4	10	3.60E-003	8.0E-003	8.30E-003	8.30E-003
	20	2.60E-004	1.60E-003	2.00E-003	2.10E-003
DS4	10	N.A	8.10E-005	8.10E-005	8.00E-005
	20	N.A	5.10E-006	5.10E-006	5.10E-006
F1	10	7.85E-003	8.26E-003	8.26E-003	8.26E-003
	20	5.93E-004	2.05E-003	2.05E-003	2.05E-003
F2	10	1.46E-005	2.66E-005	2.72E-005	2.72E-005
	20	3.43E-007	1.39E-006	1.68E-006	1.69E-006
T1	10	1.61E-002	1.61E-002	1.61E-002	1.61E-002
	20	4.05E-003	4.10E-003	4.10E-003	4.10E-003
T2	10	5.96E-003	8.20E-003	8.26E-003	8.26E-003
	20	6.12E-004	1.90E-003	2.05E-003	2.05E-003
T3	10	7.87E-005	8.05E-005	8.05E-005	8.05E-005
	20	4.90E-006	5.07E-006	5.09E-006	5.09E-006
T4	10	1.46E-005	2.66E-005	2.72E-005	2.72E-005
	20	3.43E-007	1.39E-006	1.68E-006	1.69E-006

In Tables 5 and 6 we compare results of the centered difference-techniques **T1**,**T2**,**T3**,**T4** in solving SPBVP1 at $q = 0$ [6,8,9] with Upwind Difference method (**UD**), the central difference method (**CD**), Il'in's scheme (**IL**), the second-order stable difference method (**SCD2**) in [9], the fourth order stable difference method

(**SCD4**) in [9], the fourth-order method (**DS4**) in [8] and the fourth-order methods, **F1**, **F2** in [6]. The abbreviation" N.A" is used to indicate 'not available in the reference'.

Table 6. Computed order of convergence in solving SPBVP (6) using different difference techniques

Method	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$
UD	1.0473	1.0473	0.9542
IL'in	1.6592	N.A	N.A
CD	1.9644	1.9644	1.9644
SCD2	3.7487	2.1271	1.9827
SCD4	4.9434	2.3750	2.0531
DS4	N.A	3.9894	3.9714
F1	3.800	2.0105	2.0105
F2	6.2771	4.2904	4.0171
T1	1.9911	1.9734	1.9734
T2	3.7440	2.1201	2.0105
T3	4.0381	3.9889	3.9833
T4	6.2771	4.2904	4.0171

Results in Tables 5 and 6 indicate that the upwind method (**UD**) is stable but not very accurate. The Il'in method does not appear to work for this problem. The central difference method (**CD**) gives reasonable results but not as good as the **SCD2** method. The **SCD4** method gives better results than the above methods. The **DS4** method gives the best results of the above methods. The **F1** method gives better results for ε equal to 10^{-1} and 10^{-2} but not as good as **DS4** method for smaller values of ε. **F2** gives better results than the above methods for all values of ε. **T1** method is equivalent to **CD** method. **T2** method is equivalent to **SCD2** method. **T3** method gives better results than the above methods for all values of ε except for **DS4** and **F2**. **T4** method has accuracy not less than all the above techniques. It is clear that the results of **CD**, **SCD2**, **F2** methods are similar to those of **T1**, **T2** and **T4** respectively when using fixed step size h.

Problem 2. Consider the following SBPVP [6,8,9]

$$\varepsilon y''(x) + y'(x) = 1 + 2x; \quad x \in [0,1], \tag{7}$$

with boundary conditions $y(0) = 0$ *and* $y(1) = 1$. The exact solution is given by

$$y(x) = x(x + 1 - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - e^{-x/\varepsilon})}{1 - e^{-1/\varepsilon}}$$

It is easily verified from Table 7 that **T3** gives fourth-order results for $\varepsilon \geq 10^{-2}$, but the obtained order is lower for $\varepsilon < 10^{-2}$ In fact, it is less than 2 for $N < 200$, then it gradually increases to 4 as N becomes larger. This is due to the existence of the boundary layer, where the solution changes rapidly over a very small interval in space. In fact the results in [6] for **F2** confirm that **F2** results are similar to **T4** results at fixed step size h. Table 8 present a comparison of **T4** with results available in [6] for **F2** and confirm that the two techniques have similar results

Table 7. Maximum solution error of Technique **T4** in solving SPBVP (7) at different values of ε and N

ε	$N = 20$	80	200	600	1000	2000
10^{-1}	**5.14e-005**	**1.99e-007**	**5.1063e-009**	**6.60e-010**	**1.17e-011**	**1.53e-012**
10^{-2}	1.37e-001	**2.49e-003**	**6.3059e-005**	**7.73e-007**	**9.97e-008**	**6.50e-009**
10^{-3}	8.13e-001	4.48e-001	1.4045e-001	**7.32e-003**	**1.05e-003**	**6.42e-005**
10^{-4}	9.39e-001	9.22e-001	8.18e-001	5.48e-001	3.67e-001	1.40e-001
10^{-5}	9.49e-001	9.83e-001	9.79e-001	9.41e-001	9.04e-001	8.18e-001
10^{-6}	9.49e-001	9.87e-001	9.93e-001	9.93e-001	9.90e-001	9.80e-001
10^{-7}	9.49e-001	9.87e-001	9.94e-001	9.98e-001	9.98e-001	9.97e-001

Table 8. Comparison of **F2** and **T4** in solving SPBVP(7), at, $\varepsilon = 10^{-2}$, $N = 2000$

Method	Maximum solution error
F2	6.3E-09
T4	6.5E-09

IV. CONCLUSIONS

In this paper, the orders of convergence of the central difference-techniques developed in [5] are estimated and shown that both of techniques **I** and **II** have a second-order of convergence while the technique **III** has a fourth-order of convergence. A numerical comparison of the four central difference-techniques developed in [5, 7] in solving linear SPBVPs is presented where two test SPBVP are solved numerically using these central difference-techniques and the results are compared with other difference-techniques in the literature. The numerical results confirm the theoretical ones and have shown that the above techniques result in an accurate solution of SPBVPs. Moreover, results showed that techniques **F2** and **T4** have similar results in solving the considered test SPBVPs.

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