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An approach to the stability of the Chen system through Hurwitz polynomials

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Abstract

In this paper a stability study of a model derived from the Lorenz system called Chen system is carried out. This stability study is carried out using the stability criteria to obtain Hurwitz polynomials which provides necessary and/or sufficient conditions to analyze the dynamics of the system by studying the location of the roots of the characteristic polynomial associated to it.

Keywords: Chen System, Lorenz System, Stability, Hurwitz Polynomials, Routh-Hurwitz Criterion.

I. INTRODUCTION

In 1963 the American mathematician and meteorologist Edward Lorenz presented a mathematical model to describe and understand atmospheric dynamics and weather predictions. Lorenz when studying weather patterns observed strange behaviors that were obtained when performing numerical simulations, noting that a small variation or perturbation in their initial values yielded completely different climatological results, this fundamental fact gave way to a growing branch of mathematics called chaos theory [1], [2], [3], [4].

This chaos theory has been a topic of great interest in recent decades due to its applicability in various areas of knowledge, among them engineering models based on electrical circuits, telecommunications, biomedical engineering, among others [5]. Generating or creating chaos in a given system of ordinary differential equations can be viable in the study of various models applied to different areas. For example, in 1976 the German biochemist Otto Eberhard Rössler constructed a chaotic system, called the Rössler system, which presents chaotic dynamics associated with the fractal properties of the attractor. Subsequently, the Chinese mathematician Chen Guanrong, who has been a pioneer in providing contributions to chaos theory and bifurcation theory, built a chaotic system in 1999 called the Chen system, which presents a variation in relation to the Lorenz system [6], [7].

Then, in 2002 the Chinese electrical engineer Jinhu Lü together with Chen presented a new chaotic system called Lü system [8], [9]. It should be noted that these chaotic systems are similar to the Lorenz system, but they are not equivalent, they have been constructed by adding a control parameter called anticontrol or chaotification [10].

The main objective of this paper is to approach the stability of the chaotic Chen system using the Routh-Hurwitz stability criteria and the Routh algorithm to obtain Hurwitz type polynomials. It is noteworthy that if the Chen system is expressed in the form $\dot{x} = \mathrm{Ax}$, then the stability analysis becomes an algebraic problem, because it is only enough to know the roots of the characteristic polynomial associated to the matrix A which correspond to the eigenvalues and observe if these have negative real part; if the above occurs, it is said that the system is asymptotically stable. If the polynomial has the above mentioned characteristic it is said to be a Hurwitz type polynomial [8], [9].

Therefore, to analyze the stability of the chaotic Chen system we will use criteria to determine when the characteristic polynomial associated to the matrix A is Hurwitz. In the literature a series of criteria are presented to obtain Hurwitz type polynomials, these criteria present some equivalences in their formulation, within them we can highlight the Routh-Hurwitz criterion, the Lienard-Chipart conditions, the Hermite-Biehler theorem, the stability test and the Routh algorithm [10], [11].

This study on the stability of the chaotic Chen system will be presented in four sections, the first section presents the Routh-Hurwitz stability criteria and the Routh algorithm; the second section presents a stability analysis of the Chen system using the Routh-Hurwitz stability criteria and the Routh algorithm to obtain Hurwitz type polynomials and the last section presents an analysis of the results through simulations performed in MATLAB for the solution of the chaotic Chen system.

II. HURWITZ POLYNOMIALS

To analyze the stability of a system of differential equations, whether linear or not, we can express it as follows:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \tag{1}$$

where A is a square matrix and $\boldsymbol{x}$ is a vector, to then establish the characteristic of the eigenvalues associated to the matrix A, which determine the asymptotic stability of the system (1) if these eigenvalues have negative real part.

The algebraic task of determining the stability of a system of differential equations expressed in the form (1) through the study of the roots of the characteristic polynomial of A associated to the linear or nonlinear system (1), therefore, the idea is to find conditions that may be necessary and/or

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sufficient for which all the roots of the characteristic polynomial have negative real part.

From the above we can mention that a polynomial with real coefficients is said to be Hurwitz if all its roots have negative real part, that is, if all its roots are in the left half-plane of the complex plane, that is, if all its roots are in the left half-plane of the complex plane, that is,

$$\mathbb{C}^- = \{ a + bi \ : a < 0 \}. \qquad (2)$$

Several criteria have been studied to obtain Hurwitz type polynomials, however, in this article we only present a brief description of the Routh-Hurwitz criterion and the Routh algorithm.

II.I Routh-Hurwitz criterion

The following is a brief deduction of the Routh-Hurwitz criterion, initially let us express the characteristic polynomial associated to the matrix A of the system (1), as follows

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n. \quad (3)$$

Using the coefficients $a_0, a_1, \ldots, a_n$ of the previous polynomial, let's construct a matrix following the next items:

- In row one are placed the coefficients of the polynomial (3) with odd location starting with a_1.
- In row two are placed the coefficients of the polynomial (3) with even location starting with a_0.
- And to complete the matrix the elements of each subsequent row are formed so that the component h_{ij} is given by:

$$h_{ij} = \begin{cases} a_{2j-i} & if \ 0 < 2j - i \leq n \\ 0 & in\ another\ case \end{cases}$$

Thus, the matrix is expressed as follows:

$$\mathcal{H} = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 \\ 0 & a_2 & a_4 & a_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix}. \qquad (4)$$

In this matrix $\mathcal{H}$ called Hurwitz matrix, the coefficients $a_1, a_2, a_3, \ldots, a_n$ are in the main diagonal and all the elements of the last column are null, except the last element which is a_n. The following is the Routh-Hurwitz theorem whose proof can be found in [11].

Routh-Hurwitz theorem

The characteristic polynomial expressed in the form (3), with its positive principal coefficient $(a_0 > 0)$, is a Hurwitz polynomial if and only if all the diagonal principal minors of the Hurwitz Matrix $\mathcal{H}$ are positive.

It is important to see that the minor principal diagonals of the matrix (4) are given by the following determinants,

$$\Delta_1 = |\, a_1 \,|, \qquad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \cdots, \ \Delta_n = a_n \cdot \Delta_{n-1}.$$

II.II Routh criterion

The following is a brief derivation of the Routh criterion, which is equivalent to the Routh-Hurwitz criterion. Initially using the coefficients of the polynomial (3), let us express the following arrangement called the Routh arrangement as follows:

$$\begin{array}{ccccc} a_0 & \boxed{a_2} & a_4 & a_6 & \cdots \\ \overline{a_1} & \boxed{a_3} & a_5 & a_7 & \cdots \\ \boxed{b_0} & b_1 & b_2 & b_3 & \cdots \\ c_0 & c_1 & c_2 & c_3 & \cdots \\ d_0 & d_1 & d_2 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \qquad (5)$$

This Routh arrangement has the following characteristics:

- In row one are the coefficients of the polynomial (3) with even location starting with a_0.
- In row two are the coefficients of the polynomial (3) with odd location starting with a_1.
- The elements of each subsequent row are formed according to the following algorithm

$$\boxed{b_0} = a_2 - \frac{a_0}{a_1}a_3, \qquad b_1 = a_4 - \frac{a_2}{a_3}a_5, \qquad \cdots$$

$$c_0 = a_3 - \frac{a_1}{b_0}b_1, \qquad c_1 = a_5 - \frac{a_3}{b_1}b_2, \qquad \cdots$$

$$d_0 = b_1 - \frac{b_0}{c_0}c_1, \qquad d_1 = b_2 - \frac{b_1}{c_1}c_2, \qquad \cdots$$

$$\vdots \qquad\qquad \vdots$$

From the above algorithm, the following theorems can be deduced, the proof of which can be found in [12].

Routh's theorem:

The number of roots of the polynomial $P(\lambda)$ in the right half-plane of the complex plane is equal to the number of sign variations of the first column in Routh's array (5).

Routh criterion:

The polynomial $P(\lambda)$ is Hurwitz polynomial if and only if when performing Routh's array (5) all values in the first column are nonzero of the same sign [13].

III. CHEN SYSTEM

Lorenz in 1963 presented the following non-linear model to describe climatological phenomena,

$$\begin{cases} \dot{x} &= \sigma\,(y-x) \\ \dot{y} &= x\,(\gamma - z) - y \\ \dot{z} &= -\beta z + xy, \end{cases} \tag{6}$$

Where the parameters σ, γ and β represent respectively the Prandtl number, the Rayleigh number and a proportionality constant. This system presents chaotic behaviors for $\sigma = 10$, $\beta = \frac{8}{3}$ and $\gamma = 28$.

In 1999 the Chinese mathematician Chen Guanrong constructed a chaotic system called Chen system, which presents a variation in relation to the Lorenz system (6). Chen added a parameter called anti-control or chaotification to the second equation of the Lorenz system presented in the following form $u = ax + by + cz$ where a, b and c are real constants to be determined. When $a = -\sigma, b = \gamma + 1$ and $c = 0$, one has Chen's chaotic system [14]-[21],

$$\begin{cases} \dot{x} &= \sigma\,(y-x) \\ \dot{y} &= x\,(\gamma - \sigma - z) + \gamma y \\ \dot{z} &= -\beta z + xy \end{cases} \tag{7}$$

In the Chen equations when taking the values of $\sigma = 35, \beta = 3$ and $\gamma = 28$ the system (7) presents a chaotic behavior whose simulation is presented in Figure 1. where a numerical approximation of a solution with two initial conditions close to the origin is given, with it we observe that a small perturbation in the initial conditions produces large changes in their respective trajectories [22].

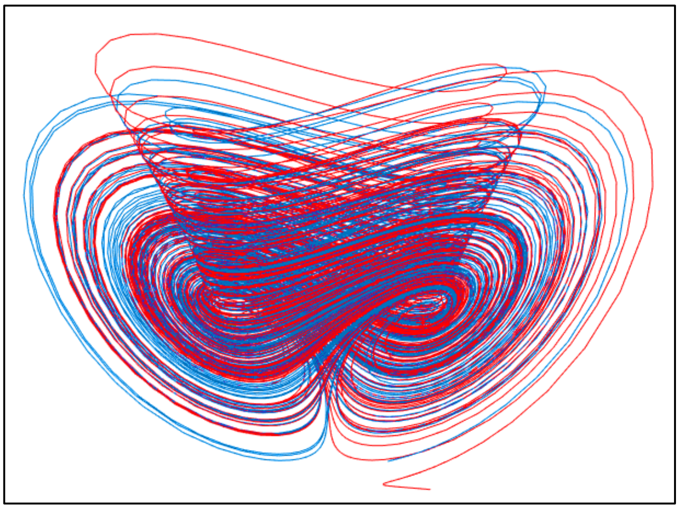

Figure. 1. Chen's chaotic attractor.

Next, we will analyze the dynamics of the Chen system using the stability criteria studied in the previous section to obtain Hurwitz polynomials.

III.I Stability analysis of the Chen system

To analyze the stability of the Chen system (7), we first establish its equilibrium points, that is:

$$\begin{aligned} \sigma\,(y-x) &= 0 \\ x\,(\gamma - \sigma - z) + \gamma y &= 0 \\ -\beta z + xy &= 0 \end{aligned} \tag{8}$$

The equilibrium points are at the origin $\mathcal{O} = (0,0,0)$ for all values of σ, γ and β, and at the points $\mathcal{C}_{1,2} = (\pm\sqrt{\beta(2\gamma-\sigma)}, \pm\sqrt{\beta(2\gamma-\sigma)}, 2\gamma-\sigma)$ for $\beta(2\gamma - \sigma) \geq 0$.

To analyze the stability at the equilibrium points O and $\mathcal{C}_{1,2}$ we first obtain the Jacobian matrix J associated to the system (7), that is:

$$\mathcal{J}(x,y,z) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \gamma - \sigma - z & \gamma & -x \\ y & x & -\beta \end{bmatrix}.$$

III.I.I Stability at the origin

Stability at $\mathcal{O} = (0,0,0)$ is obtained by linearizing the flux at $\mathcal{O}$, that is:

$$\mathcal{J}(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \gamma - \sigma & \gamma & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

The characteristic polynomial is given by:

$$\mathcal{P}(\lambda) = \lambda^3 + (\beta + \sigma - \gamma)\lambda^2 + [\beta(\sigma - \gamma) - \sigma(2\gamma - \sigma)]\lambda + \beta\,\sigma\,(\sigma - 2\gamma) \tag{9}$$

o analyze the nature of the eigenvalues that will determine the asymptotic stability in $\mathcal{O}$, we will use the Routh-Hurwitz criterion.

Thus, the Hurwitz matrix associated with the polynomial $\mathcal{P}(\lambda)$ (9) for $a_0 = 1$, $a_1 = \beta + \sigma - \gamma$, $a_2 = \beta(\sigma - \gamma) - \sigma(2\gamma - \sigma)$ and $a_3 = \beta\,\sigma\,(\sigma - 2\gamma)$ is given by:

$$\mathcal{H} = \begin{bmatrix} \beta + \sigma - \gamma & \beta\,\sigma\,(\sigma - 2\gamma) \\ 1 & \beta(\sigma - \gamma) - \sigma(2\gamma - \sigma) \end{bmatrix}.$$

Now, applying the Routh-Hurwitz theorem, we have that the polynomial $\mathcal{P}(\lambda)$ (9) is Hurwitz, if it is satisfied that:

$$\Delta_1 = |\,\beta + \sigma - \gamma\,| > 0$$

and

$$\Delta_2 = \begin{vmatrix} \beta + \sigma - \gamma & \beta\,\sigma\,(\sigma - 2\gamma) \\ 1 & \beta(\sigma - \gamma) - \sigma(2\gamma - \sigma) \end{vmatrix} > 0.$$

We can see that that Δ_1 is positive when $\beta + \sigma - \gamma > 0$, that is, if $\gamma < \beta + \sigma$, we now need Δ_2 to be positive, i.e.:

$$(\beta + \sigma - \gamma)[\,\beta(\sigma - \gamma) - \sigma(2\gamma - \sigma)] - \beta\,\sigma\,(\sigma - 2\gamma) > 0 \tag{10}$$

which is held for,

$$(\sigma - \gamma)[\beta^2 + \beta(\sigma - \gamma) + \sigma(\sigma - 2\gamma)] > 0.$$

Therefore $\sigma - \gamma > 0$ y $\sigma - 2\gamma > 0$, i.e., if $\gamma < \sigma$ y $\gamma < \frac{\sigma}{2}$.

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Therefore, the polynomial $\mathcal{P}(\lambda)$ (9) is Hurwitz if and only if $\gamma < \beta + \sigma$, $\gamma < \sigma$, $\gamma < \frac{\sigma}{2}$ and indeed the system is asymptotically stable in $\mathcal{O}$ for the above conditions.

III.I.II Estabilidad en los puntos de equilibrio $\mathcal{C}_{1,2}$

For $\beta(2\gamma - \sigma) \geq 0$ we have the points of equilibrium,

$\mathcal{C}_1 = (\sqrt{\beta(2\gamma-\sigma)}, \sqrt{\beta(2\gamma-\sigma)}, 2\gamma - \sigma)$ and

$\mathcal{C}_2 = (-\sqrt{\beta(2\gamma-\sigma)}, -\sqrt{\beta(2\gamma-\sigma)}, 2\gamma - \sigma)$

Linearizing the flow in $\mathcal{C}_1$ or $\mathcal{C}_2$ we have that,

$$\mathcal{J}(\mathcal{C}_{1,2}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \gamma - \sigma - (2\gamma - \sigma) & \gamma & -\sqrt{\beta(2\gamma-\sigma)} \\ \sqrt{\beta(2\gamma-\sigma)} & \sqrt{\beta(2\gamma-\sigma)} & -\beta \end{bmatrix}$$

whose characteristic polynomial $\mathcal{P}(\lambda)$ is given by:

$$\mathcal{P}(\lambda) = \lambda^3 + (\beta + \sigma - \gamma)\lambda^2 + \beta\gamma\lambda + 2\beta\,\sigma\,(2\gamma - \sigma) \quad (11)$$

As in the previous case, to analyze the nature of the eigenvalues of the polynomial (11) that will determine the stability in $\mathcal{C}_1$ and $\mathcal{C}_2$ we make use of Routh's criterion for $a_0 = 1$, $a_1 = \beta + \sigma - \gamma$, $a_2 = \beta\gamma$ and $a_3 = 2\beta\,\sigma\,(2\gamma - \sigma)$, thus,

$$\boxed{b_0} = \beta\gamma - \frac{1}{\beta + \sigma - \gamma} \cdot 2\beta\,\sigma\,(2\gamma - \sigma)$$

and Routh's arrangement (5) is given by

$$\begin{array}{cc} 1 & \beta\gamma \\ \beta + \sigma - \gamma & 2\beta\,\sigma\,(2\gamma - \sigma) \\ \beta\gamma - \dfrac{2\beta\,\sigma\,(2\gamma - \sigma)}{\beta + \sigma - \gamma} & \end{array} \quad (12)$$

Then, by Routh's theorem we have to analyze the signs of the first column of (12), as a_0 is positive then we are left to analyze the sign of a_1 and b_0. a_1 is positive if $\beta + \sigma - \gamma > 0$ or what is the same, if $\gamma < \beta + \sigma$, let us note that if b_0 is positive then the polynomial (11) is Hurwitz and indeed we would have stability of asymptotic type at the equilibrium points $\mathcal{C}_1$ and $\mathcal{C}_2$, therefore, let us see for which values, b_0 is positive, that is:

$$\boxed{b_0} = \beta\gamma - \frac{2\beta\,\sigma\,(2\gamma - \sigma)}{\beta + \sigma - \gamma} > 0$$

which is satisfied when

$$\begin{aligned} & \beta\gamma(\beta + \sigma - \gamma) > 2\beta\,\sigma\,(2\gamma - \sigma) \\ \Leftrightarrow \quad & \beta + \sigma - \gamma > \frac{2\sigma}{\gamma}(2\gamma - \sigma) \\ \Leftrightarrow \quad & \beta > \gamma - 2\frac{\sigma^2}{\gamma} + 3\sigma \end{aligned}$$

With the above we conclude that b_0 is positive when it is satisfied that $\gamma < \beta + \sigma$ and $\beta > \gamma - 2\frac{\sigma^2}{\gamma} + 3\sigma$ and indeed by Routh's theorem as in the first column of (12) the values of a_0, a_1 and b_0 are positive for the above restriction one has that the polynomial (11) is Hurwitz and therefore and therefore we have that $\mathcal{C}_1$ and $\mathcal{C}_2$ are asymptotically stable if the above conditions are satisfied.

IV. RESULT AND DISCUSSION

In this section we will analyze the stability results obtained previously by making use of the Routh-Hurwitz criteria and the Routh algorithm, we also perform the simulation in MATLAB for the solution of the chaotic Chen system.

In the following simulations we use the values of $\sigma = 35$, $\beta = 3$ and γ we consider it variable.

IV.I Stability at $\mathcal{O}$

If $\gamma < \beta + \sigma$, $\gamma < \sigma$, $\gamma < \frac{\sigma}{2}$, the Chen system is asymptotically stable at the origin $\mathcal{O}$, moreover, the origin is globally stable, i.e., all trajectories tend to the origin, as seen in Figures 2 and 3.

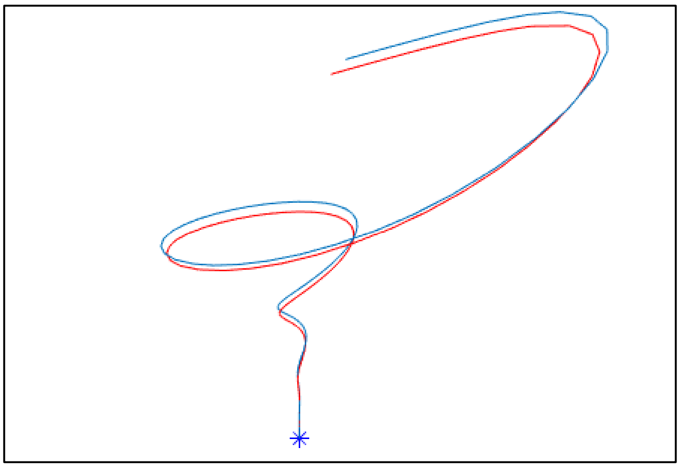

Figure. 2. $\gamma = 5$.

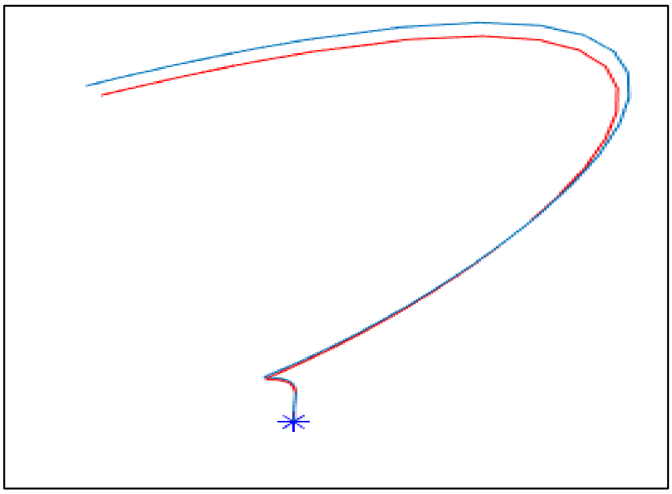

Figure. 3. $\gamma = 15$.

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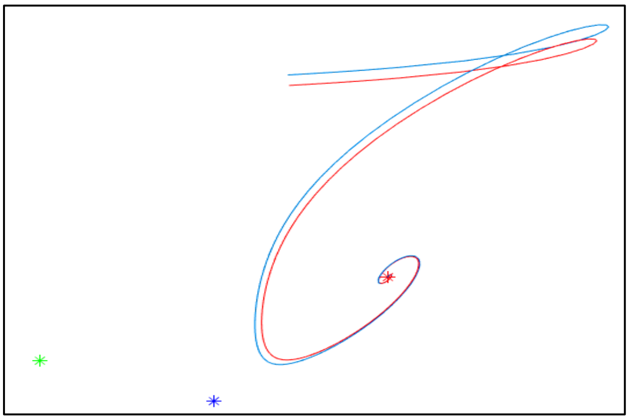
IV.I Stability at $\mathcal{C}_{1,2}$

If $\gamma < \beta + \sigma$ and $\beta > \gamma - 2\frac{\sigma^2}{\gamma} + 3\sigma$, plus the condition $\beta(2\gamma - \sigma) \geq 0$, the equilibrium points $\mathcal{C}_1$ and $\mathcal{C}_2$ are asymptotically stable.

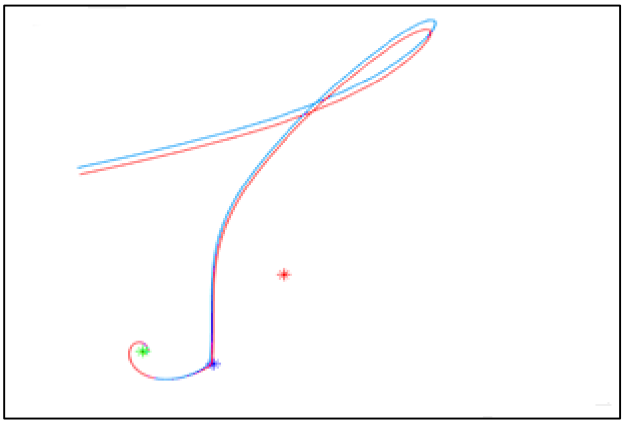
It should be noted that * in green color represents the equilibrium point $\mathcal{C}_1$ and * in red color corresponds to the equilibrium point $\mathcal{C}_2$. The stability of the Chen system at the equilibrium points $\mathcal{C}_1$ and $\mathcal{C}_2$ varies according to the value assigned in the initial conditions. For example, in Figure 4. (a) and Figure 4. (b) the stability between $\mathcal{C}_1$ and $\mathcal{C}_2$ varies, in Figure 4. (a) we have asymptotic stability for the initial condition $(1,3,3)$ and in Figure 4. (b) we have asymptotic stability for the initial condition $(-1,3,3)$, this is for $\gamma = 18$.

The same occurs when $\gamma = 19$ for the initial conditions $(1,3,3)$ and $(-1,3,3)$, as seen in Figures 4. (c) and 4. (d).

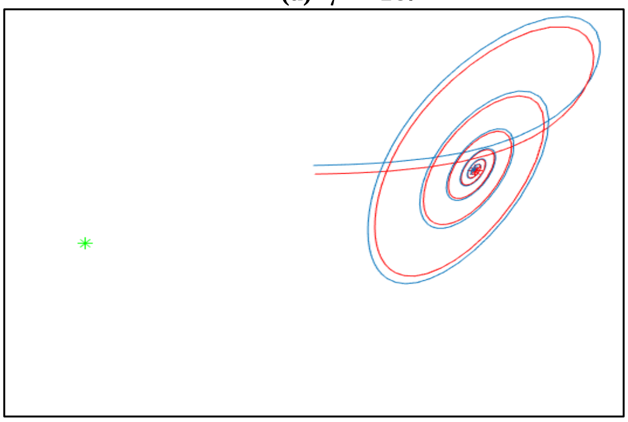
The stability of the Chen system is weak in the sense that a sensitive variation in the initial conditions can cause a change of stability between the values of $\mathcal{C}_1$ and $\mathcal{C}_2$, as observed in Figures 4. (a), (b), (c) and (d).



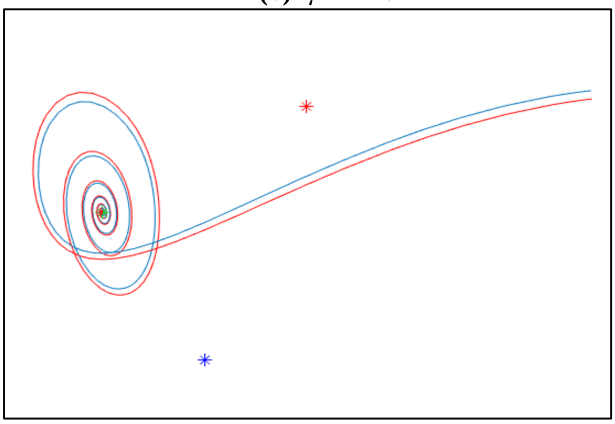
(a) $\gamma = 18$.



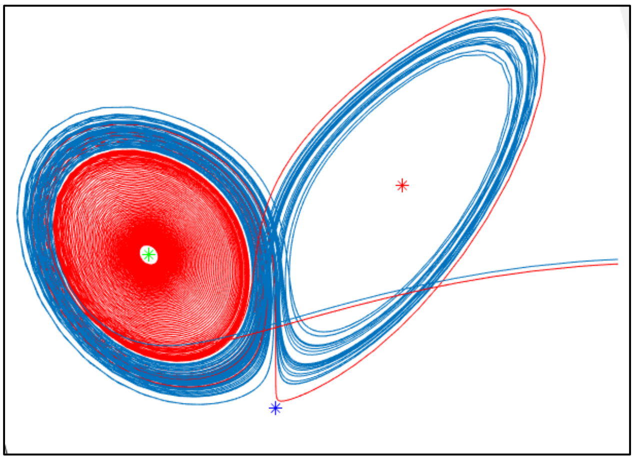
(b) $\gamma = 18$.



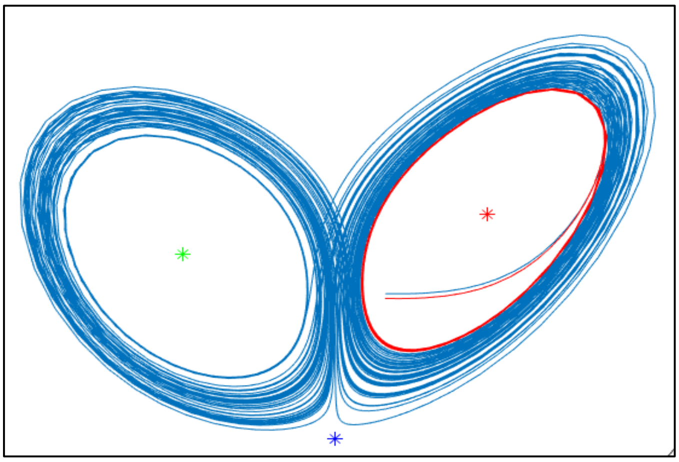
(c) $\gamma = 19$



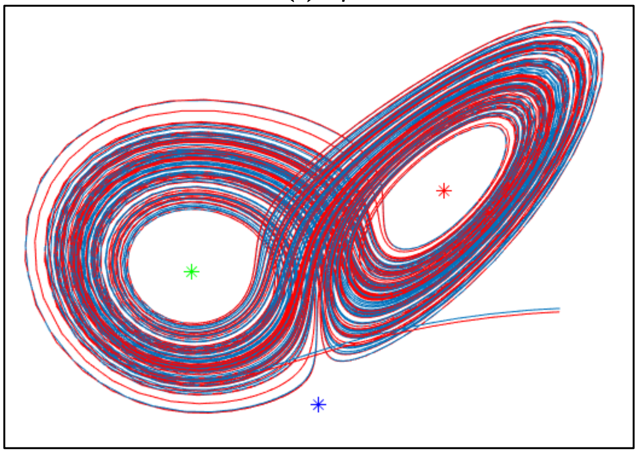
(d) $\gamma = 19$



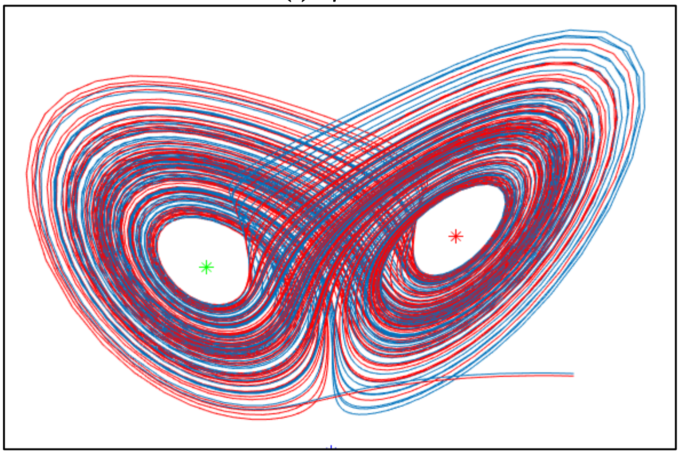
(e) $\gamma = 20$



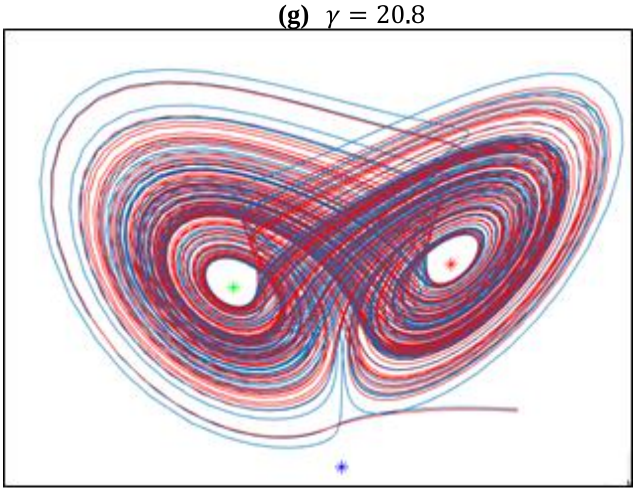
(f) $\gamma = 20.3$



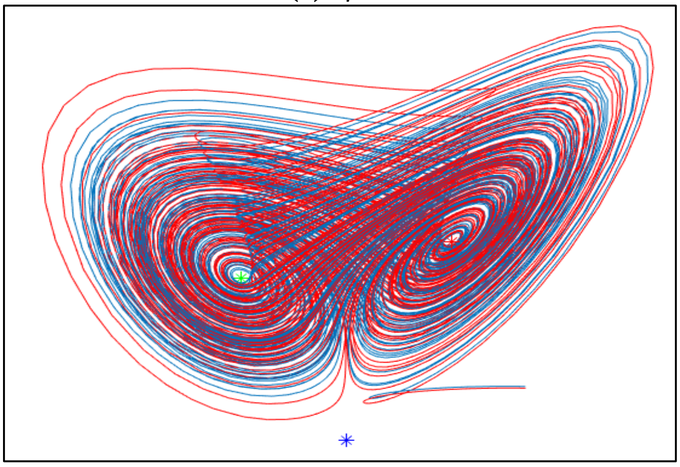
(g) $\gamma = 20.8$



(h) $\gamma = 21$



(i) $\gamma = 22$



(j) $\gamma = 23$

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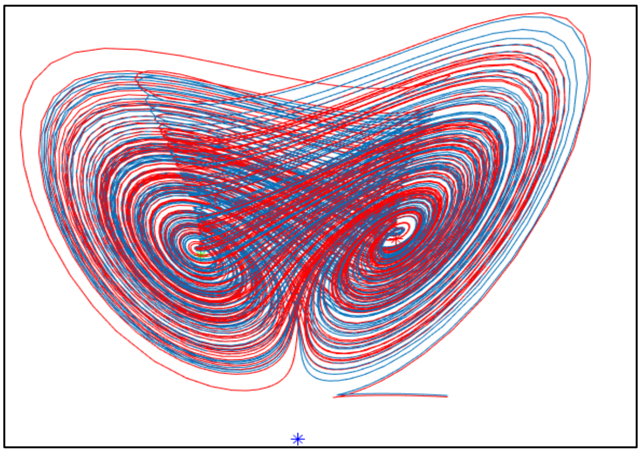
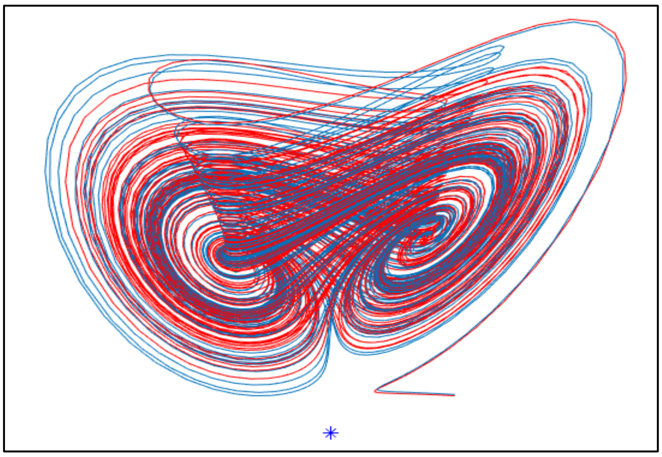
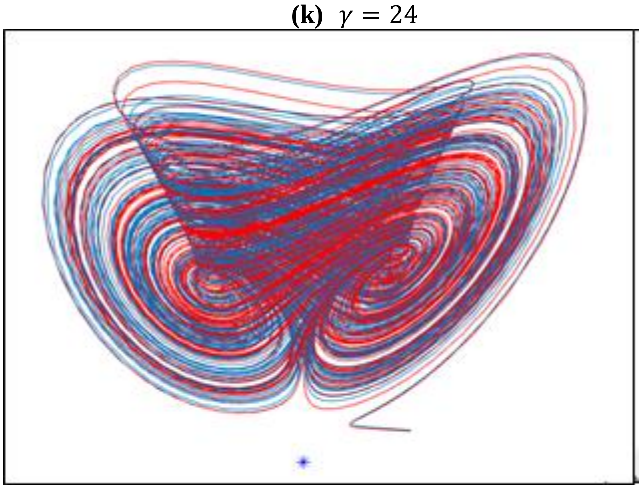


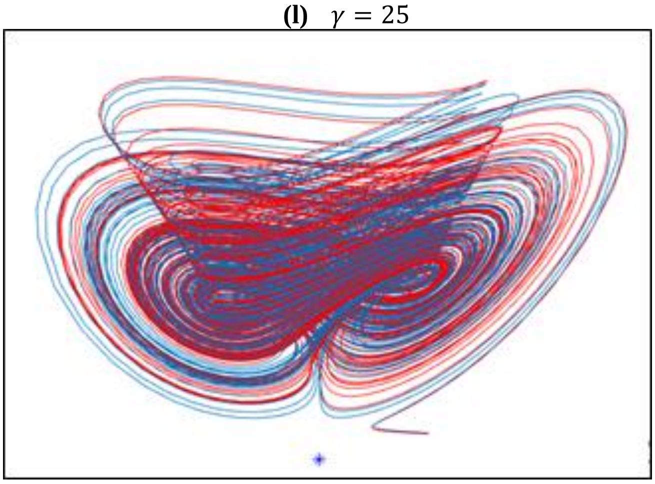
(k) $\gamma = 24$



(l) $\gamma = 25$



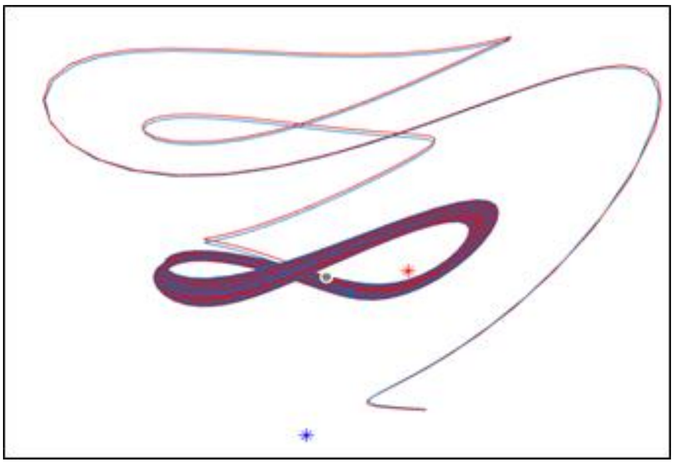
(m) $\gamma = 26$



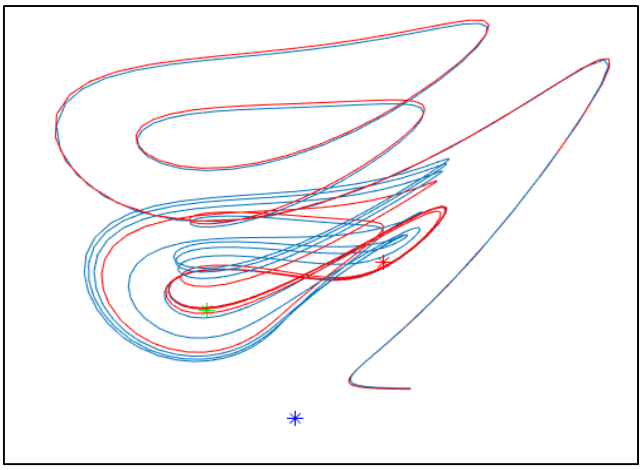
(n) $\gamma = 28$

Figure. 4. Trajectories for stability in $\mathcal{C}_1$ and $\mathcal{C}_2$.

If $\gamma > 20$ the Chen system starts to lose stability for the conditions obtained from $\gamma < \beta + \sigma$ and $\beta > \gamma - 2\frac{\sigma^2}{\gamma} + 3\sigma$, plus the condition $\beta(2\gamma - \sigma) \geq 0$, as observed in Figures 4. (e-n).



(a) $\gamma = 29$



(b) $\gamma = 30$

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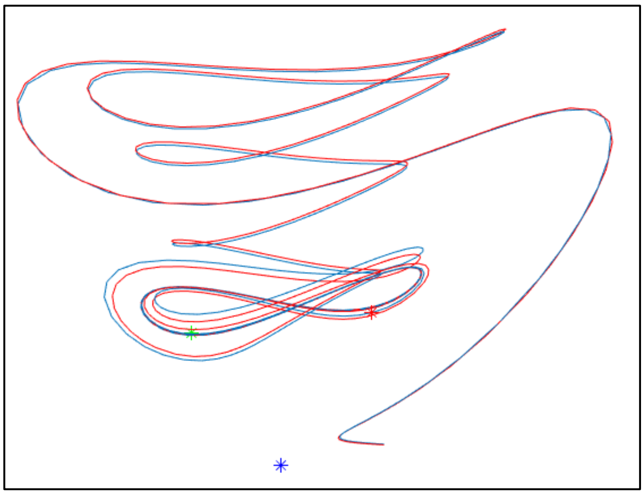
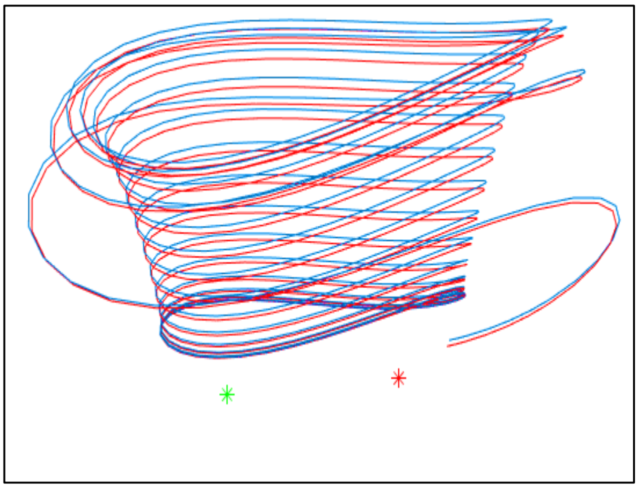


(c) $\gamma = 31$



(d) $\gamma = 33$

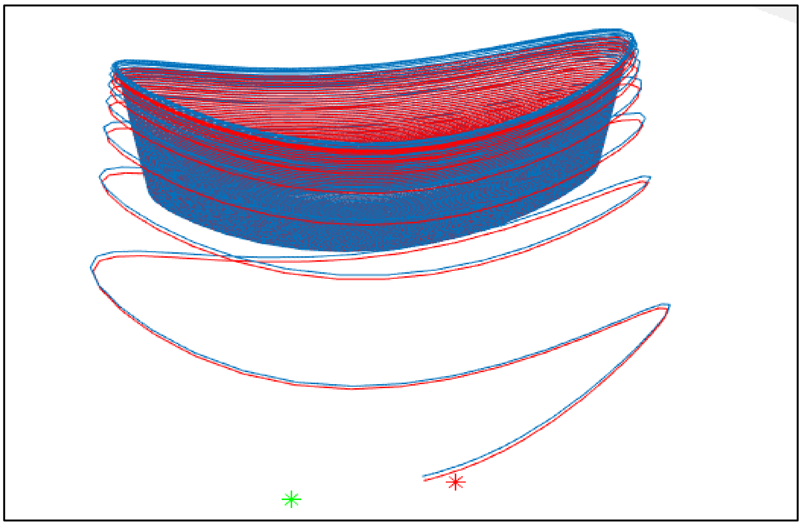


Figure. 5. Trayectorias para $\gamma > 28$.

Figures 5 (a-d) show the effect of the chaos of the Chen system, which generates chaos processes for small sensitive variations in its initial conditions.

V. CONCLUSION

By using the Routh-Hurwitz and Routh stability criteria to obtain Hurwitz polynomials, it was possible to obtain necessary and sufficient conditions to establish the stability of the Chen system. This Chen system, in spite of having been obtained through the Lorenz system by means of the process called anticontrol that turns it into a chaotic system, generates stability regions in its equilibrium points for diverse values of the studied parameters.

The Chen system is of vital importance in various applications to engineering in its branches of electrical circuits, telecommunications, biomedical engineering, among others, where stability regions and chaos play a vital role. A continuation of this work is to analyze bifurcation and chaos processes obtained for different values of the studied parameters in the Chen system..

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