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An Approach to the Notions of the Topology of Metric Spaces through Animations in Geogebra

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Abstract

In the modern world we live in a scenario of constant changes and transformations caused by information and communication technologies (ICT), which affect in a certain sense in a new way of teaching and learning. Therefore, we must seek new ways to motivate and achieve new mechanisms to reach knowledge and therein lies the importance of the proper use of new technologies. In this article we make use of the mathematical software GeoGebra as a dynamic means of fundamental elements of the topology of metric spaces, which can be seen in some abstract sense for its understanding, but through the dynamic exploration offered by the software, facilitates the development of multiple simultaneous representations of the concept to visualize its properties or characteristics, this software is an excellent way to experiment, explore, discover, see and manipulate mathematical objects carried out within a learning process. It is noteworthy that this work is developed within the framework of the master's degree in mathematics education under the project called the topology of metric spaces animated with GeoGebra.

Keywords: Metric space, Topology of metric spaces, GeoGebra, Tic in mathematics, Mathematics Education.

I. INTRODUCTION

Topology is immersed in almost all fundamental areas of mathematics, including algebra, geometry, analysis and differential equations. Its methods and results facilitate the treatment of a variety of problems applied to the aforementioned areas. Its beginnings can be placed in the 18th century, because until that time mathematical problems were linked to the idea of measurement, magnitude or distance, and at that time problems began to be posed in which these aspects ceased to be important, they are problems that do not depend on distance or size but on place. In fact, the first mathematicians who approached them gave the study of these problems the name of *Geometria Situs* ó *Analysis Situs* [1], [2], [3].

Leibniz was the first who seems to refer to this type of problems with the previous name of Geometria Situs as mentioned by Euler in a publication in 1736 where through the problem of the Konigsberg bridges he solves problems of this type, that is why it can be said that topology arises as an "ally" of geometry [4], [5]. These considerations date as the origin of topology, but it was Listing who was the first to use the word topology. Listing's topological ideas were mainly due to Gauss, although Gauss himself chose not to publish any work in topology.

Listing wrote a paper in 1847 entitled *Vorstudien zur Topologie* although he had already been using the word for ten years in his correspondence with other mathematicians. In 1861 Listing published a paper in which he described the Möbius band (4 years before Möbius) and studied components of surfaces and connexity [6], [7].

Many mathematicians of the time were interested in the development of such geometry and made great developments until in 1906 the French mathematician Maurice Fréchet was interested in having a general definition of limit and continuity so that it could be applied in various contexts, this first step to achieve this was through the concept of metric space, in which to calculate the distance between two objects certain properties had to be met to make it a useful and applicable operation to calculate trajectories, determine geometric locations and for more elaborate measurements. If it were the case that one could define the distance between any pair of elements of a set that met certain established conditions, then the set would be a metric space or it was being given a metric structure.

It is curious that, although the problem of determining the distance between two objects is very old, it was only at the beginning of the 20th century that its definition could be formalized or axiomatized. One of these fundamental metric conditions called triangular inequality was introduced by this mathematician in his 1904 article *Généralisation d'un théoreme de Weierstrass* and was later developed by him in his 1906 thesis *Sur quelques points du Calcul fonctionnel*. From his work the triangular inequality was recognized as a central notion in the task of calculating distances in any set [4].

After 1920, metric topology is the object of exhaustive research that achieves its full development and reveals its extraordinary unifying power of a whole variety of theories, until then dispersed and apparently independent. At present, metric topology constitutes a branch of general topology and metric spaces a particular case of topological spaces. It is noteworthy that all works on general topology devote some space to the treatment of metric spaces, either as a particular case of topological spaces or as a natural way of introducing them. However, the theory of metric spaces is the indispensable foundation for a rigorous study of mathematical analysis hand in hand with geometric intuition. All this inclines one to think that the theory of metric spaces would deserve an independent study and not as a part of general topology.

Now that the fundamental importance of the study of the topology of metric spaces has been highlighted, it is necessary to carry out an adequate treatment of its concepts in such a way that it can be accessible to the mathematical academic community of any educational level, that is why through the use of GeoGebra we want to present theoretically and analytically the processes that are enlivened in the interpretation of concepts, properties and characteristics of the topology of metric spaces through visualization and representation with animated constructions that allow to rescue the new possibilities of treatment of the mathematical concept that generates clearer and more precise processes.

In the development of this article, the use of the technological resource GeoGebra, which enables a better use of creativity, sensitivity, experience, maturity and mathematical knowledge, facilitating the construction of interactive material to induce discovery and help to visualize in many ways the results of analysis and deepening of concepts. The use of the software provides ample possibilities to visualize, explore, analyze and conjecture results. The characteristics and properties of the software allow the development of dynamic and interactive geometric constructions, which strengthen in some way the teaching and learning of mathematical conceptualizations.

This study of the notions of the topology of metric spaces animated through GeoGebra will be presented in four sections, the first section presents the importance of the use of ICT in teaching and GeoGebra as a means of visualization; the second section presents the dynamic development of metric spaces along with a range of examples animated by the software; in the third section an introduction to some notions of the topology of metric spaces animated by the software is presented and in the last section a discussion of its results is made through the presentation of an interactive GeoGebra book that dynamically contains the constructions made.

II. USING GEOGEBRA IN MATHEMATICS

The use of ICTs in mathematics education as a tool to facilitate pedagogical work fosters creative capacity, creativity, innovation and accelerates the process towards change, thus presenting a transformation in teaching environments that favor didactics and playfulness for motivation and the acquisition of different knowledge. The educational use of ICT encourages the development of attitudes favorable to learning science and technology through the use of interactive programs and the search for scientific information.

The implementation of ICT in mathematics is an aid in pedagogical training, i.e., they serve as a complement or facilitator in education and the resources offered in the preparation of educational material should be used to enhance the cognitive abilities of each individual. ICT in the area of mathematics allows visualization, understood as the ability to represent, transform, generate, communicate, document and reflect on the visual information generated through the use of technology, the latter being essential for today's life.

Several researchers have been given the task of reflecting on the use of ICT and especially new technologies, which is why currently some studies [10], [11,][13], [14] have shown that the use of technological resources in a teaching environment allows the creation of learning environments in which mathematical knowledge can be produced in an alternative way, where aspects of the concepts not always explicit in the traditional model of presentation are highlighted. The use of new technologies allows working in a dynamic way with mathematical concepts and their properties, which is why the importance of computational tools for mathematics education is associated with their ability to offer alternative means of mathematical expression and their capacity to offer innovative ways of manipulating mathematical objects.

Currently there are many computational means for teaching mathematical entities, one of them of great acceptance by the educational community is the mathematical software GeoGebra, which is an interactive mathematical software with dynamic components for teaching geometry, algebra, calculus, among others. It is developed by Markus Hohenwarter together with an international team of developers. With this software, interactive graphics are generated and related to algebra, obtaining dynamic spreadsheets. It covers all educational levels, from the most basic school level to the most advanced university level, and allows the development of free learning materials.

The mathematical assistant GeoGebra, integrates the work in the areas of geometry, algebra and mathematical analysis in a dynamic environment enhancing, among others, the development of variational thinking. In this sense, by recreating dynamic environments, the software allows users to visualize and represent variation relationships through the use of sliders. Based on the above, this software can be assumed as a didactic tool, since it is a physical or symbolic element that, within a learning environment, provides tools for the presentation of a particular subject, and at the same time provides the user with a form of representation, visualization and organization of the concepts worked on in the study of certain mathematical objects [12-15].

The use of this dynamic geometry program allows approaching geometry and other aspects of mathematics through experimentation and manipulation of different elements, facilitating the realization of constructions to deduce results and properties from direct observation. The use and applicability of the software has been in the focus of several researchers in the field, as Godoy states that "GeoGebra is an educational software that allows experimental and discovery learning, where the designer creates rich environments in situations that the user can explore, i.e., they can build their elements and draw conclusions according to certain properties" [8]. The student must arrive at knowledge from experiences by creating their own models of thought, their own interpretations of the problem, so it provides an adequate means for our goal.

According to Espina, "the software allows to perform dynamic constructions in an easy and intuitive way" [11], in this sense, students can work this application in an interactive and simple way, affirming that it is not a complicated process and that, in addition, extensive sections are not required for its explanation and operation. These appreciations guide us to the use of dynamic geometry software as a means of visualization to verify concepts, characteristics and properties of these mathematical objects through dynamic constructions.

The virtues of GeoGebra are strengthened by the visualization processes that it provides through its dynamic character, which is why we present the views of researchers on these processes. Arcavi defines visualization as "the ability, process and product of the creation, interpretation, use and reflection on figures, images, diagrams, in our mind, on paper or with technological tools for the purpose of representing and communicating information, thinking and developing ideas and advancing understanding" [9].

It is for this reason that visualization placed at the service of the interpretation of concepts or properties of a mathematical object can also go beyond its procedural role and inspire a general and creative solution. Moreover, representations of visual forms can be legitimate elements in mathematical demonstrations.

The main characteristic attributed by Arcavi to visualization is that it offers a method of seeing the invisible, hence many people believe that visualization is an innate ability and a matter that should remain on the margin of educational activity [9]. However, in our case, it takes on a fundamental role in the understanding of the concepts, characteristics and properties of the topology of metric spaces, given the processes of manipulating, experimenting and generating visual conjectures, through the use of GeoGebra software.

The visualization made possible with dynamic geometry software allows the user to not only see but also explore mathematical and conceptual relationships that can be difficult to "understand" without the use of technological resources, which is a major reason why it is necessary to incorporate resources such as GeoGebra in a teaching environment.

In our proposal, visualization would be associated with the geometric figures presented for the understanding of concepts, properties and characteristics of the topology of metric spaces. In this sense, with the application of GeoGebra software we intend that, from the elements designed in this tool, visualization processes are achieved for the understanding and construction of knowledge to address the thematic presented.

III. METRIC SPACES USING GEOGEBRA ANIMATIONS.

In this chapter we present the definition of metric space providing examples that appear naturally in many applications, each example of metric space is accompanied by a graphical representation made from GeoGebra software as a means of visualization of properties and characteristics of each metric space.

It is noteworthy that, as mentioned above, some fundamental concepts such as the passage to the limit or the continuity of functions in Euclidean spaces are defined exclusively in terms of distance. The sets endowed with a distance are called metric spaces, whose formal definition was presented by the French mathematician Maurice Fréchet, which plays a preponderant role in modern mathematics.

III.I Metric Spaces

Let X be a nonempty set. A metric or distance on X is a function $d: X \times X \longrightarrow \mathbb{R}$ that satisfies the following properties:

i. $d(x,y) = 0$ if and only if $x = y$.
ii. $d(x,y) = d(y,x)$ for all $x, y \in X$.
iii. $d(x,z) = d(x,y) + d(y,z)$ for all $x, y, z \in X$.

The inequality of property (iii.) is called a triangular inequality. A metric space is the set X with a given metric d, i.e., (X, d).

A very important observation is that the distance between two points is never negative, that is, $d(x,y) \geq 0$ for all $x, y \in X$. This holds because from properties (i.), (iii.) and (ii.) respectively it follows that:

$$0 = d(x,x) \leq d(x,y) + d(y,x) = 2d(x,y)$$

Therefore, $d(x,y) \geq 0$ for all $x, y \in X$.

III.I.I Examples of metric spaces

A variety of examples of particular metric spaces are presented below, in each of which the definition of metric is highlighted.

Discrete metric.

Let X be a nonempty set and $d: X \times X \longrightarrow \mathbb{R}$ the function defined by:

$$d(x,y) = \begin{cases} 0, & if\ x = y \\ 1, & if\ x \neq y \end{cases} \qquad (1)$$

The proof that (X, d) is a metric space is reduced to a simple check. The metric space (X, d) is called discrete, although it is of no further interest, given its obvious triviality, it tells us that any nonempty set can be provided with a metric. On the other hand, discrete spaces are often used as counterexamples.

Usual metric in $\mathbb{R}$.

If $X = \mathbb{R}$ and the function $d: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $d(x,y) = |x - y|$, for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, d)$ is a metric space.

The metric conditions are immediately deduced from the known properties of the absolute value. We will call this metric the usual or Euclidean metric of $\mathbb{R}$.

Euclidian metric in $\mathbb{R}^2$.

If $X = \mathbb{R}^2$ and $d_2: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$is defined by $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$:

$$d_2(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \qquad (2)$$

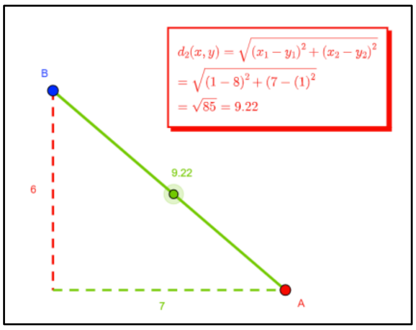

Figure 1. Euclidean metric in $\mathbb{R}^2$.

Figure 1. shows the usual way of measuring in $\mathbb{R}^2$ and intuitively verifies the axioms corresponding to the definition of metric. This metric is called the usual or Euclidean metric in $\mathbb{R}^2$.

Other ways to measure in $\mathbb{R}^2$are presented below:

Taxicab or Manhattan metric in $\mathbb{R}^2$.

If $X = \mathbb{R}^2$ and $d_1: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$is defined by $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$:

$$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2| \quad (3)$$

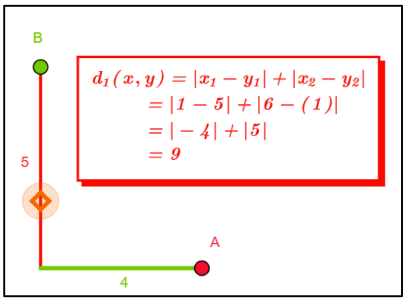

Figure 2. Taxicab metric in $\mathbb{R}^2$.

Figure 2. shows the distance between two points whose measurement is determined by a generalization of the usual metric on $\mathbb{R}$, in which it mentions that to measure the distance between point B and A we first find the horizontal distance and add the vertical distance.

The name comes from the fact that the distance can be interpreted as the length of a cab ride, which in a grid city like " *Manhattan* " in New York, goes from one point to another with a single turn of the steering wheel.

Maximum or chess metric in $\mathbb{R}^2$.

If $X = \mathbb{R}^2$ and $d_\infty: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$is defined by $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$:

$$d_\infty(x, y) = máx\{|x_1 - y_1|, |x_2 - y_2|\} \quad (4)$$

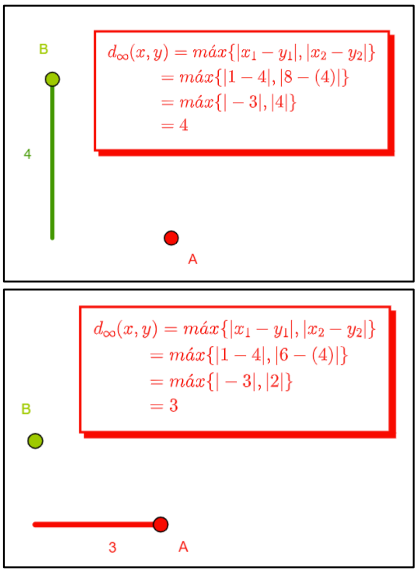

Figure 3. Maximum metric in $\mathbb{R}^2$.

In the maximum metric the distance between two points is determined by the maximum horizontal or vertical distance between two points. The name is due to the fact that this metric can be interpreted in the following way: if we think of a chessboard and on it only one piece, the king, it can reach in a single move the eight squares that surround it. Well, the distance between two squares is the minimum number of moves that the king must make to go from one square to the other.

Lift metric in $\mathbb{R}^2$.

If $X = \mathbb{R}^2$ and $d: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$is defined by $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$:

$$d(x, y) = \begin{cases} |x_2 - y_2|, & if\ x_1 = y_1 \\ |x_2| + |x_1 - y_1| + |y_2|, & if\ x_1 \neq y_1 \end{cases} \quad (5)$$

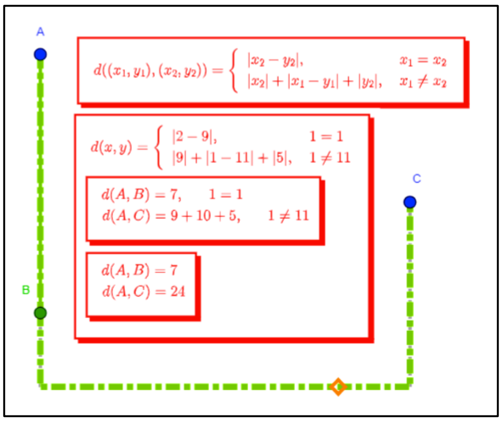

Figure 4. Lift metric in $\mathbb{R}^2$.

In the metric of the maximum in $\mathbb{R}^2$ (4), the metric of the elevator is defined, this can be interpreted as follows: If we think of the plane as the union of all vertical straight lines and in turn, we think of these as if they were buildings, then the distance between two points that are on the same vertical straight line, that is, the distance between A and B (see figure 4.) is just the absolute value of the difference of the vertical

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coordinates, this can be interpreted as the path of an elevator going from one floor to another in the same building, that is, from point A which is on floor 9 to point B which is on floor 2.

If the points are on different verticals, then the distance is the sum $|x_2| + |x_1 - y_1| + |y_2|$, which can be interpreted as the route that consists of going down the elevator of the first building to the first floor (abscissa line), go down the street to the second building and go up the elevator of the second building to the floor that indicates the second coordinate of the second point, that is, if we wanted to go from point A to point C which is on another vertical line we would have to perform the movements mentioned above.

Messenger metric in $\mathbb{R}^2$.

If $X = \mathbb{R}^2$ and $d: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is defned by $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$:

$$d(x,y) = \begin{cases} \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, & if\ x_1y_1 = x_2y_2 \\ \sqrt{{x_1}^2 + {x_2}^2} + \sqrt{{y_1}^2 + {y_2}^2}, & if\ x_1y_1 \neq x_2y_2 \end{cases} \quad (6)$$

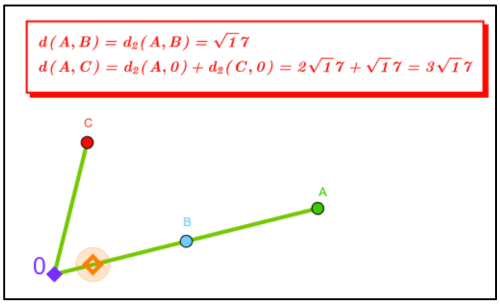


Figure 5. Messenger metric in $\mathbb{R}^2$.

In the Euclidean space $\mathbb{R}^2$ the messenger metric is defined, which is considered as the distance between two different points of the plane which is equivalent to the sum of the Euclidean distances of both points to the origin, that is, to go from point A to point C (see figure 5.) first we find the Euclidean distance from A to the origin which corresponds to 2 times the distance between A and B and we add the Euclidean distance between the origin and the end point C.

This metric can be interpreted as follows: if one were to measure the path taken by a letter leaving from a first point A, arriving at a second point B (which is right in the middle of point A to the origin) and then passing through the post office, located at the origin, and from there going to the final point C, where the addressee of the letter is [16].

Supremum metric in $\mathbb{R}^2$.

Let $\mathcal{C}[a\,,b]$ be the set of continuous real functions on the closed interval $[a\,,b]$. If $X = \mathcal{C}[a\,,b]$ and $d: \mathcal{C}[a\,,b] \times \mathcal{C}[a\,,b] \longrightarrow \mathbb{R}$ is defined for $f\,,g \in \mathcal{C}[a\,,b]$ by:

$$d(f,g) = sup\{|f(x) - g(x)|: x \in [a,b]\} \quad (7)$$

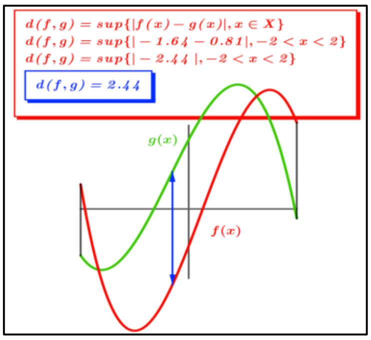


Figure 6. Supremum metric in $\mathbb{R}^2$.

The metric of the supremum corresponds to the largest vertical separation between the graphs of the functions, in Figure 6. the largest vertical distance between the functions $f(x)$ and $g(x)$ for the interval -2 to 2 is shown in blue.

Demonstrations that the above examples do indeed correspond to metrics can be found in [1].

IV. TOPOLOGY OF METRIC SPACES THROUGH ANIMATIONS IN GEOGEBRA

In this section the notion of topology associated to a metric space is presented by introducing the open balls and from there the open sets, closed sets and their properties are studied. Graphical representations are presented using GeoGebra software of established definitions and properties of the topology of metric spaces.

In metric spaces, there are certain subsets with very remarkable properties and which prove to be the indispensable tool for a rigorous study of the analysis. These are the open sets. Given a metric space $d(x,y)$ there are relevant subsets of it capable of describing the neighbors of a point by controlling the distance (degree of closeness) and which would also be responsible for defining the topology inherent to the metric.

IV.I Topology induced by a metric

Let X be a nonempty set. A topology on X is a family τ of subsets of X such that:

i. $\emptyset, X \in \tau$.

ii. If $(A_i)_{i\in I}$ is a collection of elements of τ, then $\bigcup_{i\in I} A_i \in \tau$. That is, the arbitrary union of elements of τ is an element of de τ.

iii. If $A_1, A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$. That is, the intersection of a finite number of elements of τ is an element of τ.

iv. If τ is a topology in X, then the pair (X, τ) is called a topological space and the elements of τ are called open in X.

A *basis* for a topology τ in X is a family $\boldsymbol{\mathcal{A}}$ of elements of τ such that every element of τ (i.e., every open set of X) can be expressed as a union of elements of $\boldsymbol{\mathcal{A}}$. For $\boldsymbol{\mathcal{A}}$ to be a basis for a topology in X it is necessary and sufficient that it satisfies the following conditions:

i. X is the union of elements of $\boldsymbol{\mathcal{A}}$.

ii. Given A_1, A_2 elements of $\boldsymbol{\mathcal{A}}$ and $x \in A_1 \cap A_2$, then there exists $A \in \boldsymbol{\mathcal{A}}$ such that $x \in A \subset A_1 \cap A_2$.

IV.I.I Open balls in metric spaces

Let (X, d) be a metric space, x_0 an element of X and $r > 0$. The set,

$$\mathcal{B}(x_0, r) = \{x \in X: d(x, x_0) < r\} \qquad (8)$$

is called an *open ball* in X of center x_0 and radius r. The set,

$$\overline{\mathcal{B}}(x_0, r) = \{x \in X: d(x, x_0) < r\} \qquad (9)$$

is called a *closed ball* in X of center x_0 and radius r.

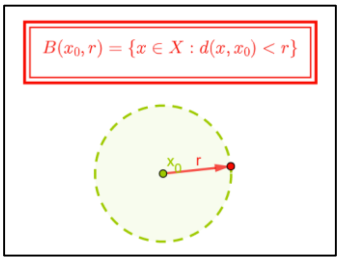


Figure 7. Open ball in $\mathbb{R}^2$.

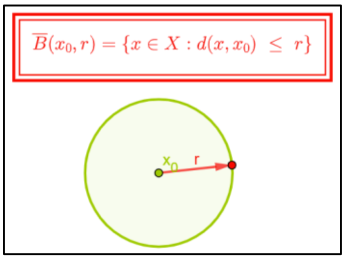


Figure 8. Closed ball in $\mathbb{R}^2$.

Let (X, d) be a metric space and $\boldsymbol{\mathcal{A}}_d = \{\mathcal{B}(x, r): r > 0, x \in X\}$ the family of all open balls in X. Then, $\boldsymbol{\mathcal{A}}_d$ is a basis for a topology τ_d on X, called the *topology induced by the metric d*.

It should be noted that a metric space is always considered to be endowed with the topology induced by its metric.

Open ball with discrete metric.

Let (X, d_D) be the metric space with the discrete metric defined by (1). The open ball $\mathcal{B}(x_0, r)$ in this metric corresponds to the set:

$$\mathcal{B}(x_0, r) = \begin{cases} \{x_0\}, & if\ r \leq 1 \\ X, & if\ r > 1 \end{cases} \qquad (10)$$

which corresponds to the set X if the radius is greater than 1 and corresponds to its center when its radius is less than or equal to 1.

Open ball with the usual metric in $\mathbb{R}$.

In the metric space $(\mathbb{R}, d)$ where $d(x, y) = |x - y|$, as defined in (2), for all $x, y \in \mathbb{R}$, given $x_0 \in \mathbb{R}$ and $r > 0$. The open ball of center x_0 and radius r, is the open interval:

$$\mathcal{B}(x_0, r) = (x_0 - r, x_0 + r) = \{x \in \mathbb{R}: x_0 - r < x < x_0 - r\}$$

and the closed ball of center x_0 and radius r, is the closed interval:

$$\overline{\mathcal{B}}(x_0, r) = [x_0 - r, x_0 + r] = \{x \in \mathbb{R}: x_0 - r \leq x \leq x_0 - r\}$$

Open ball with Euclidean metric in $\mathbb{R}^2$ ***and*** $\mathbb{R}^3$.

In the metric space $(\mathbb{R}^2, d_2)$ we have that:

$$\mathcal{B}((x_0, y_0), r) = \{(x, y) \in \mathbb{R}^2: (x - x_0)^2 + (y - y_0)^2 < r^2\}$$

It corresponds to the circle of radius r centered at a point (x_0, y_0) (see figure 9).

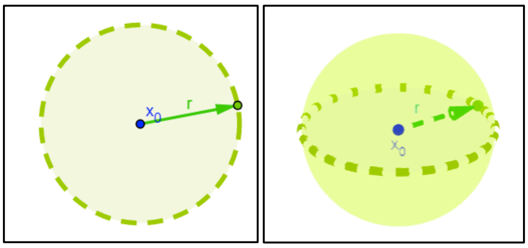


Figure 9. Open ball in $\mathbb{R}^2$ *and* $\mathbb{R}^3$.

In the metric space $(\mathbb{R}^3, d_2)$ we have that:

$$\mathcal{B}((x_0, y_0, z_0), r) =$$
$$\{(x, y, z) \in \mathbb{R}^3: (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2\}$$

It corresponds to the interior of the sphere of radius r centered at a point (x_0, y_0, z_0) (see figure 9, right).

It should be noted that the geometric appearance of a ball in $\mathbb{R}^2$ or $\mathbb{R}^3$ is not necessarily spherical, it depends on the metric under consideration.

Open ball with the metric of the maximum in $\mathbb{R}^2$ ***and*** $\mathbb{R}^3$.

In the metric space $(\mathbb{R}^2, d_\infty)$, the ball with center at (0,0) and radius r, is given by:

$$\mathcal{B}((0,0), r) = \{(x, y) \in \mathbb{R}^2: máx\{|x|, |y|\} < r\} \qquad (11)$$

The open ball $\mathcal{B}((0,0), r)$ corresponds to the interior of the square with center (0,0) and sides parallel to the coordinate axes and length $2r$ (see figure 10).

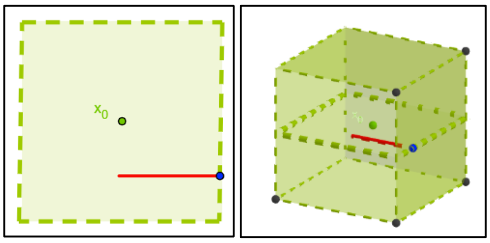


Figure 10. Open ball in $\mathbb{R}^2$ *and* $\mathbb{R}^3$in the maximum metric.

The points of the plane that verify the condition $máx\{|x|,|y|\} < r$ is $|x| < r$ and $|y| < r$, these coordinates x and y are in the interval $(-r,r)$ so the ball will be:

$$\mathcal{B}((0,0),r) = (-r,r) \times (-r,r)$$

Similarly the open ball with center at (x_0,y_0) and radius r with this metric, is given by:

$$\mathcal{B}((x_0,y_0),r) = (x_0 - r, x_0 + r) \times (y_0 - r, y_0 + r)$$

In the metric space $(\mathbb{R}^3, d_\infty)$ the ball corresponds to a cube as presented in Figure 10, right.

Open ball with taxicab metric in $\mathbb{R}^2$ and $\mathbb{R}^3$.

In the metric space $(\mathbb{R}^2, d_1)$, the ball with center at $(0,0)$ and radius r, is given by:

$$\mathcal{B}((0,0),r) = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < r\} \qquad (12)$$

The open ball $\mathcal{B}((0,0),r)$ corresponds to the interior of the rhombus centered at the origin $(0,0)$ and with vertices at the points $(0,r), (0,-r), (r,0), (-r,0)$ (see figure 11).

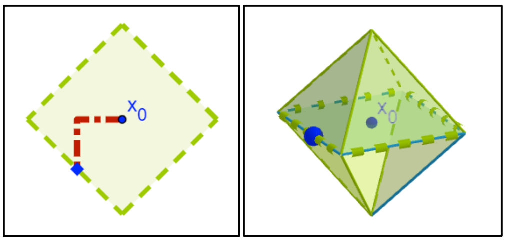


Figure 11. Open ball in $\mathbb{R}^2$ and $\mathbb{R}^3$ in the taxicab metric.

This open ball corresponds to the points of the plane that verify $|x| + |y| < r$. If we assume that $x, y \geq 0$ it must be fulfilled that $x + y < r$, that is, these are the points of the plane whose coordinates are non-negative and verify $y < r - x$; in short, the points of the first quadrant that are below the straight line $y = r - x$. Reasoning in the same way about the possible signs of the coordinates we obtain the open ball with this metric.

In the metric space $(\mathbb{R}^3, d_1)$ the ball corresponds to an octahedron as shown in figure 11, right.

Open ball with metric d_p in $\mathbb{R}^n$.

We can generalize the above examples and define a metric d_p in $\mathbb{R}^n$ for every real number $p \geq 1$, thereby having an infinite collection of metrics. If $X = \mathbb{R}^n$ and $d_p : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is defined for $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ by:

$$d_p(x,y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p} \qquad (13)$$

For the particular case of $\mathbb{R}^2$, Figure 12 presents the open balls corresponding to the previous generalization for $p = 1,2,7$ and $p = 20$.

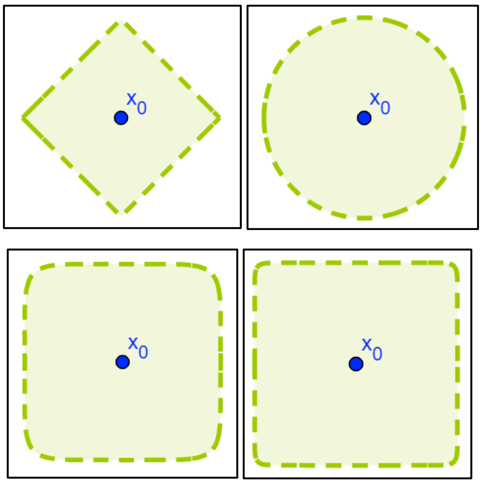


Figure 12. Open balls with metrics d_1, d_2, d_7 and d_{20}, respectively.

The metric d_p with condition $p \geq 1$ should not go unnoticed, since in the case $p < 1$ we do not obtain a metric and in its corresponding balls it is not verified that the distance from a point to the center is less than the radius, as seen in Figure 13.

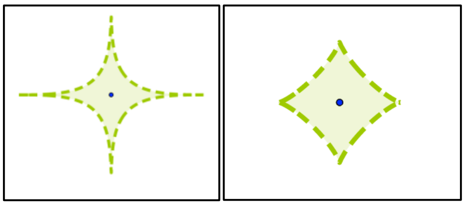


Figure 13. Open balls with metric $d_{0.5}, d_{0.8}$.

Open ball with lift metric in $\mathbb{R}^2$.

In the metric space $(\mathbb{R}^2, d)$, the open ball with center at point $A = (x_0, y_0)$ and radius r corresponding to the metric d of the elevator defined as in (5), corresponds to an extension of the ball with the metric of the maximum with center at a point $A = (x_0, y_0)$.

The open ball corresponding to the elevator metric has the following specifications according to the place in the plane where we consider the center:

1. If $y_0 = 0$.
   Let us see which points $(x,y) \in \mathbb{R}^2$ belong to the $\mathcal{B}((x_0,y_0),r)$. Indeed, if $x = x_0$, then $d((x_0,0),(x_0,y)) < r$, that is $|y| < r$. Now if $x \neq x_0$, then $((x_0,0),(x,y)) < r$, that is $|y| + |x - x_0| < r$. Thus for this case the open ball $\mathcal{B}((x_0,0),r)$ corresponds to a rhombus, without the "edge" with vertices at $(x_0 - r, 0), (x_0 + r, 0)(x_0, r), (x_0, -r)$, i.e. a rhombus on the abscissa axis, as seen in Figure 14.

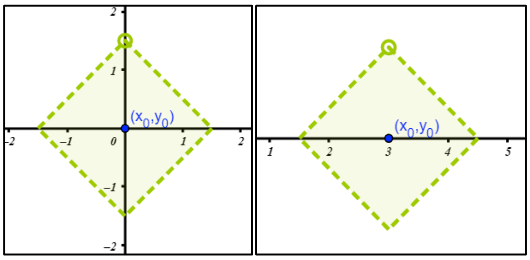


Figure 14. Open balls with elevator metric. Case 1.

2. If $y_0 > 0$.
   Let us see which points $(x, y) \in \mathbb{R}^2$ belong to the $\mathcal{B}\big((x_0, y_0), r\big)$. Indeed, if $x = x_0$, then $d((x_0, y_0), (x_0, y)) < r$, that is $|y - y_0| < r$. Now if $x \neq x_0$, then $d((x_0, y_0), (x, y)) < r$, that is $|y_0| + |y| + |x - x_0| < r$, or what is the same, $|y| + |x - x_0| < r - |y_0|$. We must analyze the case $r \leq |y_0|$ and $r > |y_0|$, for the first case there are no points (x, y) with $x \neq x_0$ belonging to $\mathcal{B}((x_0, y_0), r)$, and therefore, for this case the open ball corresponds to a line segment, as seen in Figure 15.

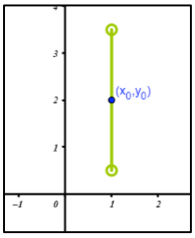


Figure 15. Open ball with lift metric. Case 2a.

For the second case, $r > |y_0|$,, the ball $\mathcal{B}((x_0, y_0), r)$ consists of the line segment $\{(x_0, y): y_0 - r < y < y_0 + r\}$ and the interior of the rhombus with vertices at $(x_0 + y_0 - r, 0)$, $(x_0 - y_0 + r, 0), (x_0, r - y_0), (x_0, y_0 - r)$ as seen in Figure 16.

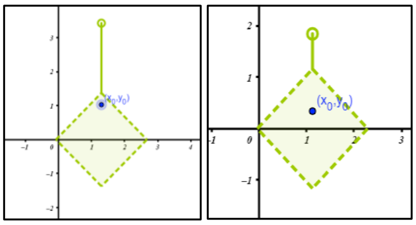


Figure 16. Open ball with lift metric. Case 2b.

3. If $y_0 < 0$.
   This case is analyzed similarly to case 2 and its open balls $\mathcal{B}((x_0, y_0), r)$ are presented in Figure 17.

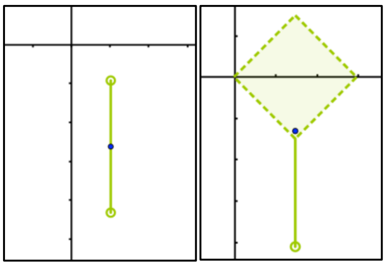


Figure 17. Open ball with lift metric. Case 2c.

Open ball with messenger's metric in $\mathbb{R}^2$.

In the metric space $(\mathbb{R}^2, d)$, the open ball with center at point $A = (x_0, y_0)$ and radius r corresponding to the messenger's metric d (6), corresponds to an extension of the ball with Euclidean metric with center at a point $A = (x_0, y_0)$.

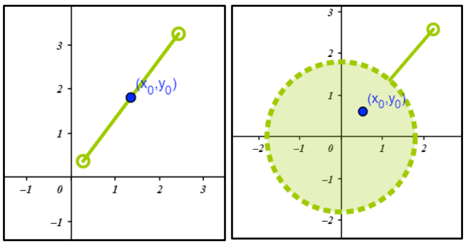


Figure 18. Open balls with messenger metric.

The balls for the messenger metric centered at a point and with positive radius consist of the center joined with the Euclidean ball centered at the origin and of radius r minus the distance from the center to the origin. If the radius is less than or equal to that distance, then the ball shrinks to the center [16].

Open ball with the supremum metric in $\mathbb{R}^2$.

In the metric space $(\mathbb{R}^2, d)$,, given $r > 0$ and a function $f_0 \in \mathcal{C}[a, b]$, then the open ball $\mathcal{B}(f_0, r)$ consists of all continuous functions $f(x)$ whose graphs lie in the area bounded by $f_0 - r$ and $f_0 + r$, as shown in Figure 19. The ball $\mathcal{B}(f_0, r)$ is the set,

$$\mathcal{B}(f_0, r) = \{\text{f} \in \mathcal{C}[a, b]: sup\{|f(x) - f_0(x)| < r, x \in [a, b]\}\}$$

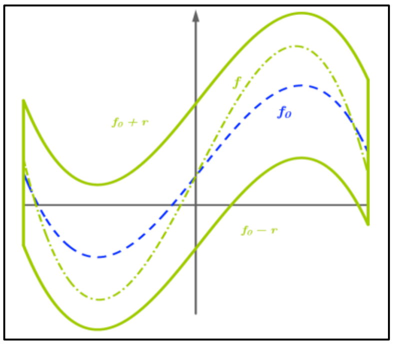


Figure 19. Open ball with the messenger metric.

IV. RESULT AND DISCUSSION

This section presents the fundamental results of the work carried out, in which the applets built in GeoGebra are presented, which have been shared in the software's own network called *GeoGebraTube*, which allows globalizing the knowledge, not only allowing their download, but also facilitating their modification to adapt them to particular needs [17].

With each of the applets a virtual book has been created in the GeoGebra page distributed by chapters, in which we visualize the characteristics and properties of the metric spaces. Each chapter of the virtual book presents detailed information on each mathematical object of the topology of metric spaces and its specification to perform the animation using the software.

With this interactive book we can experience all the concepts, characteristics and properties of the topology of metric spaces by manipulating its applets, since it is an excellent resource to visualize and understand the different concepts. The GeoGebra book called *topology of metric spaces* is composed of 4 chapters, see figure 20, each of them enriched with representative illustrations that help to visualize the fundamental characteristics of each subject. It should be noted that in this article we present only the introductory part of the topology of metric spaces, that is, only chapters one and two.

In the first chapter, the definition of distance or metric is presented, addressing examples that appear naturally in many applications, see figure 21. This chapter contains 14 applets in which a brief explanation of the concept is given and the use of animations through sliders is mentioned in detail, as shown in figure 22, to verify properties or characteristics of the metric in question. In chapter 2, as shown in figure 24, the notion of topology associated to a metric space is presented by introducing the open balls and from there, open sets, closed sets and their properties and characteristics are studied.

In all applets, sliders have been created, as shown in figure 23, to animate the image or figure, so that the user can visualize and verify the properties of the concepts presented. It is worth mentioning that when the automatic animation is activated, a button appears in the lower left corner of the graphical view. This button allows the user to stop and restart the progress (see figure 23, bottom left). It should be noted that a slider is a controller that allows you to move or, as its name suggests, slide a point over a certain figure and display an animation. These controllers or action objects add interactivity and control possibilities over the objects.

In each of the applet's the constructive process is taken into account which allows us to analyze the situation in successive steps starting from the simplest ones. The sliders presented in each applet allow the ease with which we can drag the objects, forcing them to acquire many different positions which allow the observation of the characteristics and properties inherent to the topology of metric spaces, with this we seek to establish conjectures by varying the parameters of the slider obtaining several representations in a direct way. This dynamism of the applet's constructions with GeoGebra allows us to make inferences through visualization processes that we experience through the use of sliders.

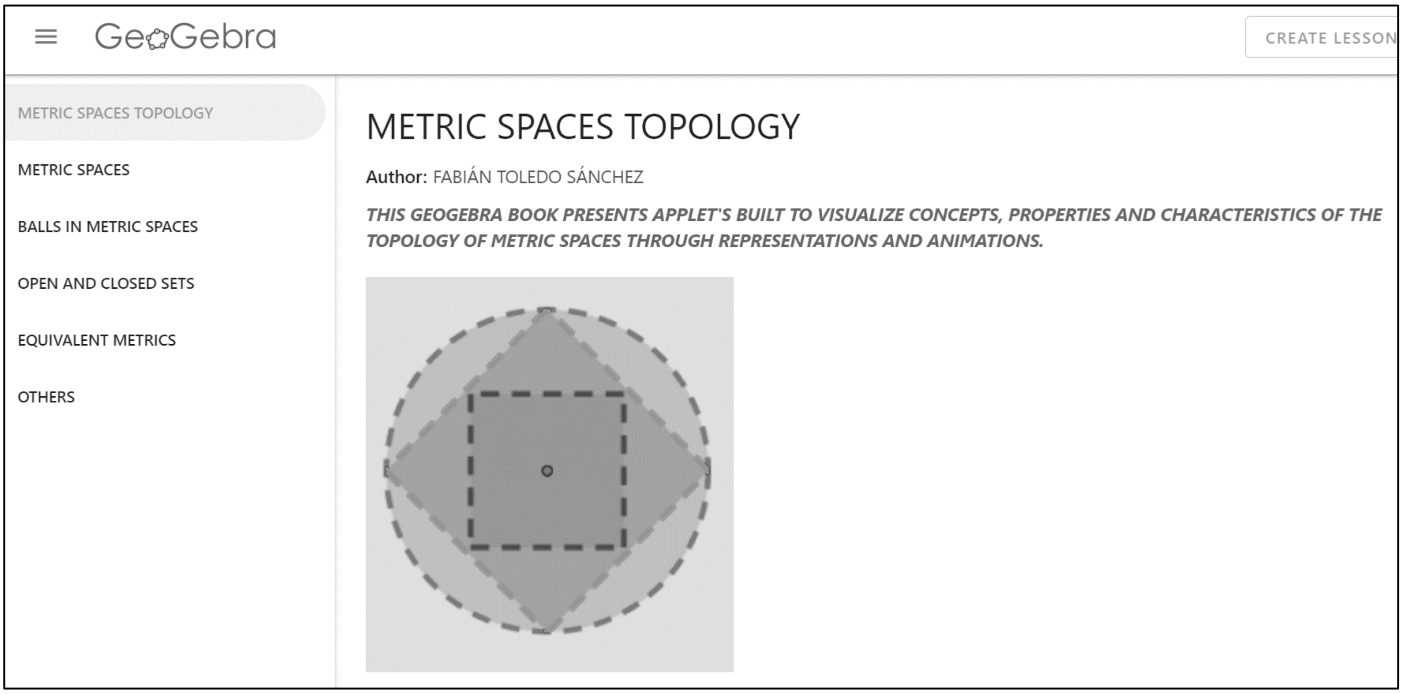


Figure 20. GeoGebra book on the Web.

International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 14, Number 12 (2021), pp. 1179-1190

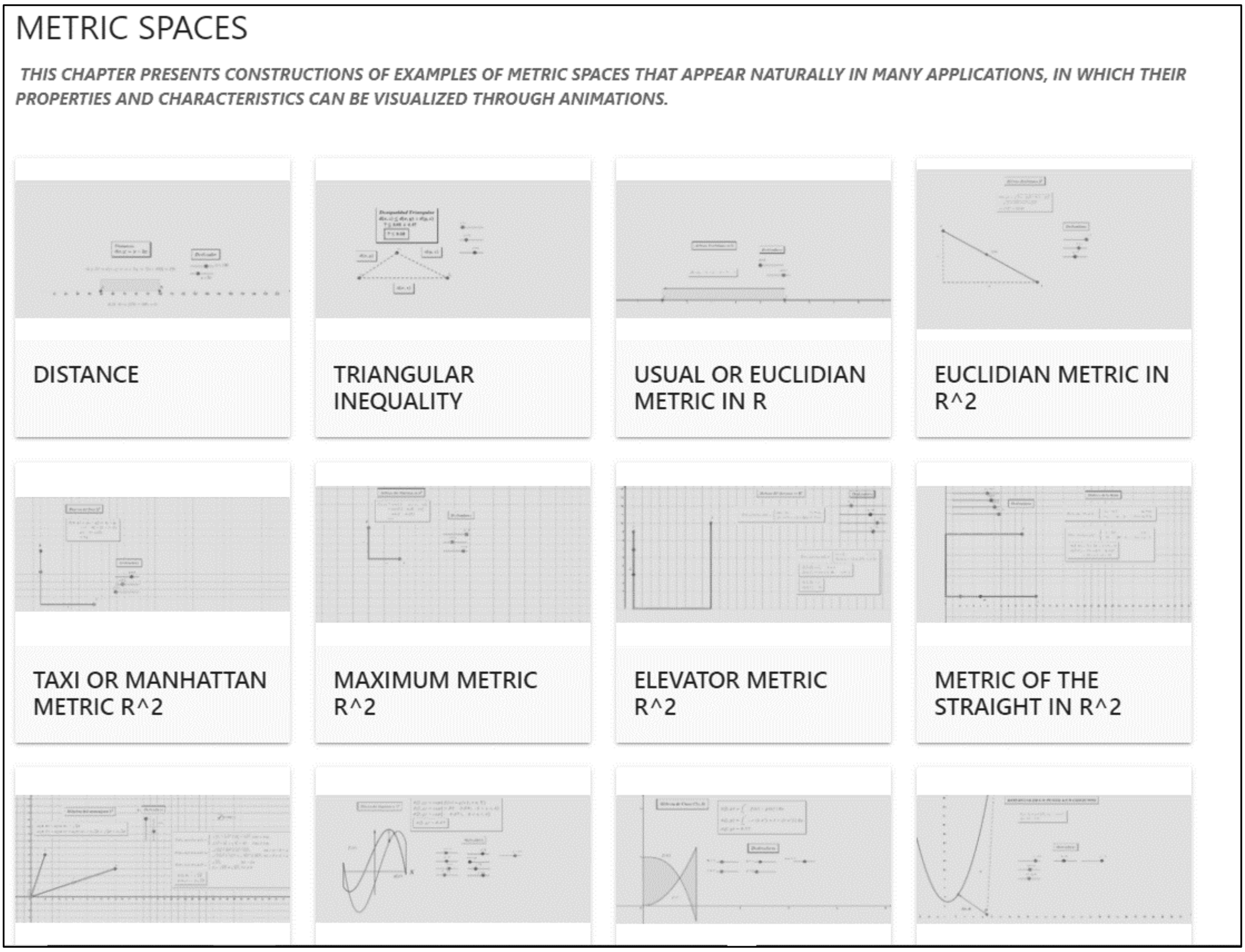
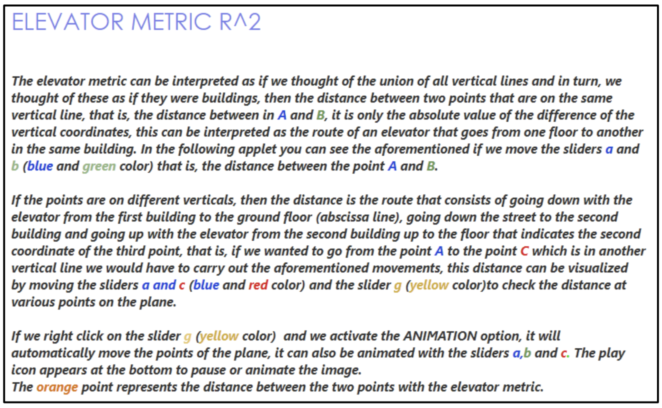


Figure. 21. First chapter of the GeoGebra book: Metric spaces.



ELEVATOR METRIC R^2

The elevator metric can be interpreted as if we thought of the union of all vertical lines and in turn, we thought of these as if they were buildings, then the distance between two points that are on the same vertical line, that is, the distance between in A and B, it is only the absolute value of the difference of the vertical coordinates, this can be interpreted as the route of an elevator that goes from one floor to another in the same building. In the following applet you can see the aforementioned if we move the sliders a and b (blue and green color) that is, the distance between the point A and B.

If the points are on different verticals, then the distance is the route that consists of going down with the elevator from the first building to the ground floor (abscissa line), going down the street to the second building and going up with the elevator from the second building up to the floor that indicates the second coordinate of the third point, that is, if we wanted to go from the point A to the point C which is in another vertical line we would have to carry out the aforementioned movements, this distance can be visualized by moving the sliders a and c (blue and red color) and the slider g (yellow color)to check the distance at various points on the plane.

If we right click on the slider g (yellow color) and we activate the ANIMATION option, it will automatically move the points of the plane, it can also be animated with the sliders a,b and c. The play icon appears at the bottom to pause or animate the image.
The orange point represents the distance between the two points with the elevator metric.

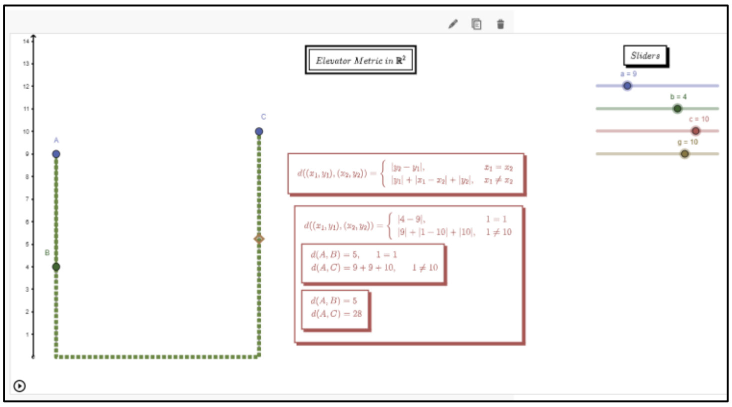


Figure 22. Description of each Applet's.

Figure 23. Interactive interface with its sliders.

International Journal of Engineering Research and Technology. ISSN 0974-3154, Volume 14, Number 12 (2021), pp. 1179-1190

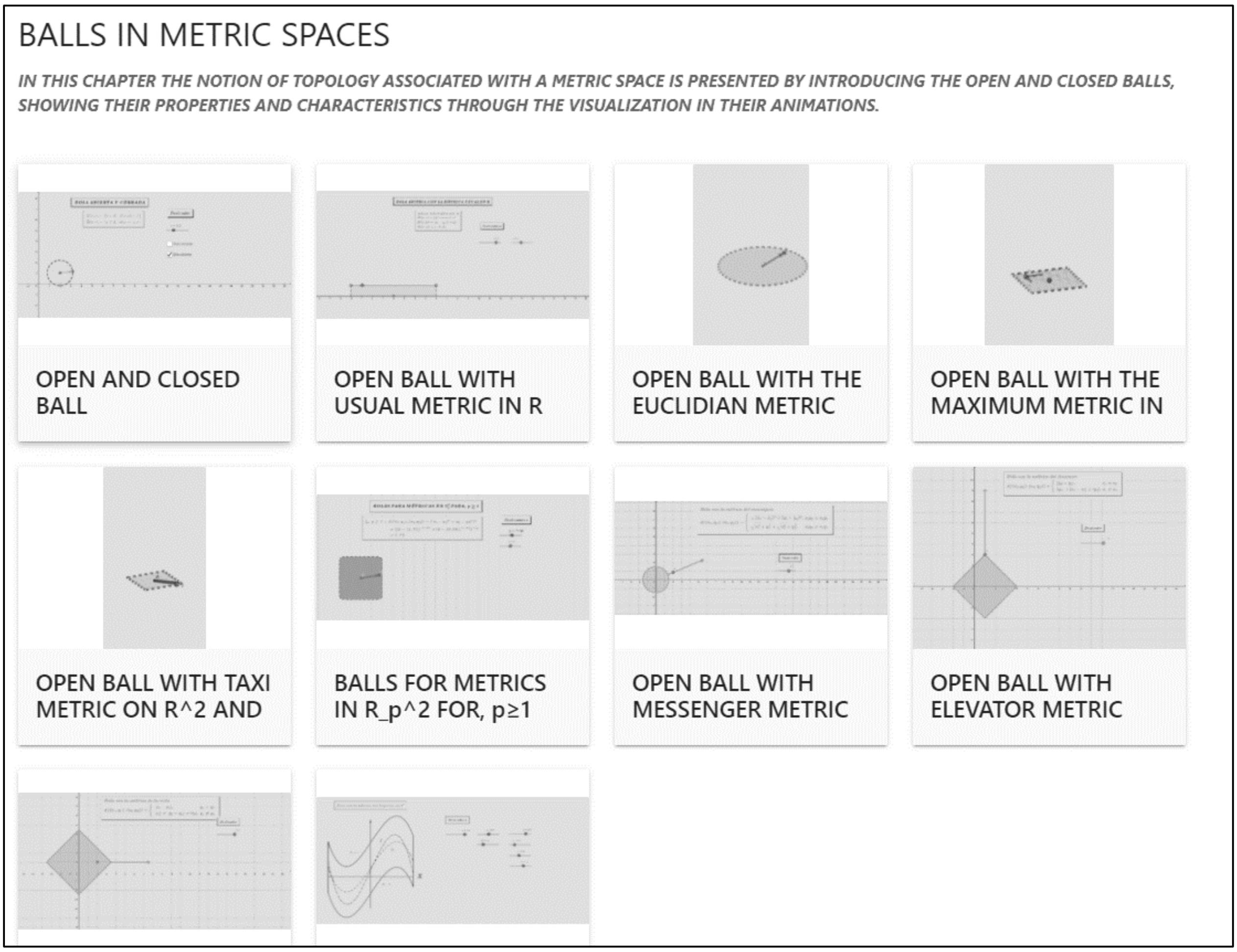


Figure 24. Second chapter of the GeoGebra book: Open balls in metric spaces.

V. CONCLUSION

As a result of the interaction with GeoGebra software through the construction of its applets for the visualization of concepts, properties and characteristics of the topology of metric spaces, it is observed that the geometric character of the topology of metric spaces facilitated the construction of dynamic applets in GeoGebra for the visualization of concepts, properties and characteristics.

The constructions in GeoGebra are a means that make possible the understanding of some concepts, properties and characteristics immersed in the topology of metric spaces, since they propitiate processes of visualization, experimentation, generation and validation of visual conjectures which contribute to the production of knowledge.

learn mathematics in an easier way, but rather we consider that, through the dynamic process mediated by thought processes, the construction of mathematical knowledge is different and seems to harmonize with the elements of a part of our society where the use of new technologies has become so powerful and incorporated into everyday life that they are now an inherent part of the culture.

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REFERENCES

[1] Toledo Sánchez, Fabián, “La topología de espacios métricos animada con Geogebra,” Tesis de Maestria en enseñanza de las Matemáticas, Universidad Tecnologica de Pereira, 2017.

[2] Adams, Colin and Franzosa, Robert, Introduction to topology pure and applied, New Delhi: Pearson prentice hall, first impression, 2009.

[3] Bombal, L. Rodríguez y G. Vera, Problemas de análisis Matemático: espacios métricos y normados, AC Madrid, 1982.

[4] Munkres, James, Topología, Madrid: prentice hall, inc., 2da edición, 2002.

[5] Iribarren, Ignacio, Topología de Espacios Métricos, Limusa, 1973.

[6] J.R. Giles, Introduction to the Analysis of Metric Spaces, Cambridge University Press, 1987.

[7] Lages Lima, Elon, Espacos métricos, Projeto Euclides, 1977.

[8] Godoy Villanueva, Carlos M, Papel y Lápiz y Programas de Geometría Dinámica en el aprendizaje de conceptos geométricos y su aplicación a resolución de problemas, Master de Recerca en Didáctica de les Matematiques i de les Ciencies Experimentals, Universitat Autónoma de Barcelona, 2011.

[9] Arcavi, A, The role of visual representations in the learning of mathematics, Educational Studies in Mathematics, 52(3), pp 215-241, 2003.

[10] Ávila, P, Razonamiento covariacional a través de software dinámico. El caso de la variación lineal y cuadrática, Trabajo de Maestría, Universidad Nacional de Colombia, Medellín, 2012.

[11] Espina P, GeoGebra, Revista de didáctica de las matemáticas, 69, pp.19-24, 2006.

[12] Moritz Lucius, Summermann Daniel, Benjamin Rott, Mathematics in the Digital Age: The Case of Simulation-Based Proofs, International Journal of Research in Undergraduate Mathematics Education (2021) 7:438–465, 2021.

[13] Benning I., Enacting core practices of effective mathematics pedagogy with GeoGebra, Mathematics Teacher Education and Development, Volume 23, Issue 2, Pages 102 – 12719, July 2021.

[14] Hernández A. Hernández A, Perdomo Díaz J, Camacho Machín M, Mathematical understanding in problem solving with GeoGebra: a case study in initial teacher education, International Journal of Mathematical Education in Science and Technology, Volume 51, Issue 2, Pages 208 - 22317 February 2020.

[15] Pirklová P, Pirklová P, Bímová D, Bímová D, Metric problems in monge projection in Geogebra, 18th Conference on Applied Mathematics, APLIMAT 2019, Volume 2, Pages 949 - 9582019 18th Conference on Applied Mathematics, APLIMAT 2019.

[16] Vanegas, Johan. y Zapata Zulma, "Algunos lugares geométricos desde la métrica del mensajero," Universidad Pedagógica Nacional, 2020.

[17] GeoGebra, Libro de GeoGebra: Topología de espacios métricos, Fabian Toledo Sanchez, [en línea], citado el 02 de Noviembre del 2021.