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Stability Analysis of the Lorenz System using Hurwitz Polynomials

Fabián Toledo Sánchez[1], Pedro Pablo Cárdenas Alzate[2] and José Rodrigo González Granada[3]

[1,2,3] *Department of Mathematics and GEDNOL, Universidad Tecnológica de Pereira, Pereira, Colombia.*

Abstract

In the qualitative study of differential equations, we can analyze whether or not small variations or perturbations in the initial conditions produce small changes in the future, this intuitive idea was formalized by Lyapunov in his work "The General Problem of Motion Stability". The stability analysis of a system of differential equations can be performed using the methods proposed by Lyapunov or using criteria to obtain Hurwitz type polynomials, which provide conditions to analyze the dynamics of the system by studying the location of the roots of the characteristic polynomial associated to the system. In this paper we present a stability analysis of the Lorenz system using stability criteria to obtain Hurwitz type polynomials.

Keywords: Stability of differential equations, Hurwitz polynomials, Hurwitz criterion, Lorenz system.

I. INTRODUCTION

The stability analysis of systems of ordinary differential equations is the object of study in various mathematical models applied to a varied branch of the exact sciences and control theory, where the aim is to provide necessary and sufficient conditions so that in a given problem with established initial conditions, solutions close to this initial value remain close throughout the future or in a given case tend to the equilibrium solution. This notion of closeness and proximity for small variations in the initial conditions was initially worked on by outstanding mathematicians such as Lagrange and Dirichlet, but it was not until 1892 when the Russian mathematician and physicist Aleksandr Lyapunov laid the foundations for this concept of proximity in his doctoral thesis entitled "The General Problem of Motion Stability" [1].

The study of the dynamics of these sensitive variations in the initial conditions is the fundamental work in the qualitative analysis of ordinary differential equations. The importance of such sensitive variation is observed in the Lorenz system, this nonlinear model that described the movement of air masses in the atmosphere gave way to a branch of mathematics called chaos theory, because Lorenz, studying the weather patterns, began to observe unusual behaviors to those that had been established that should happen. Lorenz using a numerical simulation used 0.506 as an initial data as a less accurate approximation of the data 0.506127, obtaining as a result, according to two tabulations, two completely different climate scenarios [2], [3]. This small variation or perturbation in their initial values yielded completely different climatological results. From the moment Lorenz formulated the problem with this question, various characteristics of the system began to be studied, such as regions of stability, attractor basins, bifurcations, chaos, geometric aspects, among others [4], [5].

Among these fundamental studies is the stability analysis; this stability analysis can be performed using the results of Lypaunov, who presented two methods to determine stability that solve particular problems according to the structure of the differential equations or using Hurwitz type polynomials.

If the nonlinear ordinary differential equation is expressed in the form $\dot{x} = \mathrm{A}x$, then the problem of analyzing the stability becomes a problem of algebraic type, since it is only enough to know the roots of the characteristic polynomial associated to the matrix A which correspond to the eigenvalues and to observe if these have negative real part; if the above occurs it is said that the system $\dot{x} = \mathrm{A}x$ is asymptotically stable. If this polynomial has the above mentioned characteristic it is said to be a Hurwitz polynomial [6], [7].

Therefore, a relevant fact to analyze the stability of the system $\dot{x} = \mathrm{A}x$ corresponds to establish criteria to determine when the characteristic polynomial associated to the matrix A is Hurwitz, i.e., if all its roots have negative real part. In [8] a series of criteria are presented to obtain Hurwitz type polynomials, these criteria present some equivalences in their formulation, among them the Routh-Hurwitz criterion, the Lienard-Chipart conditions, the Hermite-Biehler theorem, the stability test and the Routh algorithm [9].

Thus, the objective of this paper is to analyze the stability of the Lorenz system using the Routh-Hurwitz stability criteria and the Routh algorithm. For this purpose, four sections are presented: in the first one, the Routh-Hurwitz stability criteria and the Routh algorithm are deduced; in the second section, a stability analysis of the Lorenz system is performed using the criteria presented to obtain Hurwitz type polynomials and finally, the results are analyzed by means of simulations performed in MATLAB using the *ode45* function for the solution of the system of nonlinear differential equations. The simulations validate the results obtained on the stability of the Lorenz system.

II. HURWITZ POLYNOMIALS

Let

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \tag{1}$$

be a system of linear or nonlinear ordinary differential equations, where A is a square matrix and $\boldsymbol{x}$ is a vector. The stability of the system (1) at its equilibrium point can be determined by performing a study of the eigenvalues of the associated matrix A. This algebraic study establishes that if the roots of the characteristic polynomial associated to the matrix A have negative real part, then we conclude that the system (1) is asymptotically stable.

Now, the problem of determining the stability in a system of differential equations from the study of the roots of the characteristic polynomial of A associated to the linear or nonlinear system, translates into the task of finding necessary and sufficient conditions for which all the roots of this polynomial are located in the left half of the complex plane, i.e., if they have negative real part.

This way of obtaining polynomials of this type with negative real part was initially proposed by the physicist James Maxwell in 1868, who presented a solution to this problem for polynomials of degree 3, then in 1877, the Canadian mathematician Edward Routh presented an algorithm that solved the problem in a more general way providing explicit conditions for polynomials up to degree 5 and later it was the German mathematician Adolf Hurwitz in 1895 who presented a properly analytical solution to the problem [6].

Definition 1: A polynomial with real coefficients is said to be Hurwitz if all its roots have a negative real part, that is, if all its roots lie in $\mathbb{C}^-$, the left half-plane of the complex plane,

$$\mathbb{C}^- = \{ a + bi \; : a < 0 \} \tag{2}$$

II.I Criteria for obtaining Hurwitz polynomials

Let

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-2}\lambda^2 + a_{n-1}\lambda + a_n \tag{3}$$

be the characteristic polynomial associated to the matrix A of the system (1).

In the search for algorithms to determine the location of the roots of the polynomial (3) the following question was asked: how to determine if the polynomial (3) is Hurwitz?

This question was studied by several mathematicians and physicists, among the first who proposed to solve this type of problem was the Austrian engineer A. Stodola who at the end of the 19th century was interested in the problem of finding conditions under which all the roots of a polynomial had a negative real part; but in 1895 it was Hurwitz who presented a solution to the previous question based on the work of Hermite.

There are several criteria to obtain Hurwitz type polynomials. In this paper we only briefly present the Routh-Hurwitz criterion and the Routh algorithm.

II.I.I Routh-Hurwitz criterion

To present the Routh-Hurwitz criterion we first construct the following matrix from the coefficients $a_0, a_1, \ldots, a_n$ of the polynomial (3):

$$\mathcal{H} = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_2 & a_3 & \cdots & 0 \\ 0 & a_2 & a_4 & a_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{bmatrix} \tag{4}$$

This matrix is constructed as follows:

- In the first row are the coefficients of the polynomial (3) with odd location starting with a_1.
- In the second row are the coefficients of the polynomial (3) with even location starting with a_0.
- The elements of each subsequent row are formed so that the component h_{ij} is given by:

$$h_{ij} = \begin{cases} a_{2j-i} & si \;\; 0 < 2j - i \leq 0 \\ 0 & in \; other \; case \end{cases}$$

As a result of the construction, the coefficients a_1, a_2, a_3, ... , a_n are on the main diagonal of the matrix, and all the elements of the last column are null, except the last element which is a_n. The matrix $\mathcal{H}$ is called Hurwitz Matrix.

Routh-Hurwitz theorem. The polynomial (3), with its positive leading coefficient ($a_0 > 0$), is a Hurwitz polynomial if and only if all the diagonal principal minors of the Hurwitz matrix $\mathcal{H}$ are positive [10].

The principal diagonal minors of the matrix (4) are given by the following determinants,

$$\Delta_1 = |\; a_1 \;|, \qquad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \qquad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix},$$

$$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}, \cdots, \;\; \Delta_n = a_n \cdot \Delta_{n-1}.$$

II.I.II Routh criterion

For the coefficients of the polynomial (3) the Routh arrangement is constructed as follows,

$$\begin{array}{ccccc} a_0 & \boxed{a_2} & a_4 & a_6 & \cdots \\ \overline{a_1} & \boxed{a_3} & a_5 & a_7 & \cdots \\ \boxed{b_0} & b_1 & b_2 & b_3 & \cdots \\ c_0 & c_1 & c_2 & c_3 & \cdots \\ d_0 & d_1 & d_2 & d_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{5}$$

- In the first row are the coefficients of the polynomial (3) with even location starting with a_0.
- In the second row are the coefficients of the polynomial (3) with odd location starting with a_1.
- The elements of each subsequent row are formed according to the following algorithm

$$\boxed{b_0} = a_2 - \frac{a_0}{a_1} a_3 \,, \qquad b_1 = a_4 - \frac{a_2}{a_3} a_5 \,, \qquad \cdots$$

$$c_0 = a_3 - \frac{a_1}{b_0} b_1 \,, \qquad c_1 = a_5 - \frac{a_3}{b_1} b_2 \,, \qquad \cdots$$

$$d_0 = b_1 - \frac{b_0}{c_0} c_1 \,, \qquad d_1 = b_2 - \frac{b_1}{c_1} c_2 \,, \qquad \cdots$$

$$\vdots \qquad\qquad\qquad \vdots$$

Routh's theorem:

The number of roots of the polynomial $P(\lambda)$ in the right half-plane of the complex plane is equal to the number of sign variations of the first column in Routh's array (5).

Routh criterion:

The polynomial $P(\lambda)$ is Hurwitz polynomial if and only if when performing Routh's array (5) all values in the first column are nonzero of the same sign [11].

III. LORENZ SYSTEM

The American mathematician and meteorologist Edward Lorenz according to his studies on atmospheric phenomena built a mathematical model that would give rise to an important branch of dynamical systems called chaos theory whose foundations appear in his article *Deterministic Non-periodic Flow* [12] published in 1963 describing the so-called butterfly effect. This nonlinear model is given by:

$$\begin{cases} \dot{z} &= \sigma\,(y-x) \\ \dot{y} &= x\,(\gamma - z) - y \\ \dot{z} &= -\beta z + xy \end{cases} \qquad (6)$$

where σ, γ and β are positive parameters and $\sigma :=$ Prandtl number, $\gamma :=$ Rayleigh number and β is a proportionality constant. In Lorenz equations usually the values of $\sigma = 10$, $\beta = \frac{8}{3}$ are taken and γ can take any positive value. When $\gamma = 28$ the system (6) presents a chaotic behavior whose simulation is presented in Figure 1, where a numerical approximation of a solution with two initial conditions close to the origin is given, with this we observe that a small perturbation in the initial conditions produces large changes in their respective trajectories [13].

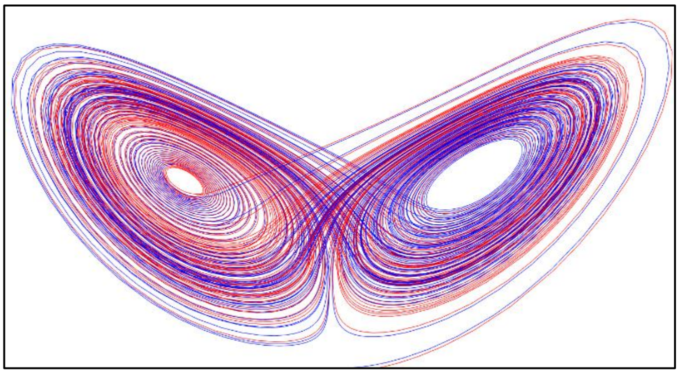


Figure. 1. Lorenz attractor.

III.I Stability analysis of the Lorenz system

To perform a stability analysis of the Lorenz system (6) we start by establishing its equilibrium points, i.e., the $\tilde{x}$ such that $f(\tilde{x}) = 0$, that is:

$$\begin{aligned} \sigma\,(y-x) &= 0 \\ x\,(\gamma - z) - y &= 0 \\ -\beta z + xy &= 0 \end{aligned} \qquad (7)$$

From (7) we have that the equilibrium points are given at the origin $\mathcal{O} = (0,0,0)$ for all values of σ, γ and β, and at the points $\mathcal{D}_{1,2} = (\pm\sqrt{\beta(\gamma-1)}, \pm\sqrt{\beta(\gamma-1)}, \gamma - 1)$ for $\gamma > 1$.

III.I.I Stability of the origin

Stability in $\mathcal{O} = (0,0,0)$ is obtained by linearizing the flux in $\mathcal{O}$, i.e:

$$\mathcal{J}(0,0,0) = \begin{bmatrix} -\sigma & \sigma & 0 \\ \gamma & -1 & 0 \\ 0 & 0 & -\beta \end{bmatrix}$$

Where $\mathcal{J}$ is the Jacobian associated to the system (6). The characteristic polynomial is given by:

$$\mathcal{P}(\lambda) = \lambda^3 + (\beta + \sigma + 1)\lambda^2 + [\beta(\sigma+1) + \sigma(1-\gamma)]\lambda + \beta\,\sigma\,(1-\gamma) \qquad (8)$$

To analyze the nature of the eigenvalues that will determine the stability in $\mathcal{O}$, we use the Hurwitz criterion.

Thus, the Hurwitz matrix associated with the polynomial $\mathcal{P}(\lambda)$ (8) for $a_0 = 1$, $a_1 = \beta + \sigma + 1$, $a_2 = \beta(\sigma+1) + \sigma(1-\gamma)$ and $a_3 = \beta\,\sigma\,(1-\gamma)$, is given by:

$$\mathcal{H} = \begin{bmatrix} \beta + \sigma + 1 & \beta\,\sigma\,(1-\gamma) \\ 1 & \beta(\sigma+1) + \sigma(1-\gamma) \end{bmatrix}$$

Applying Hurwitz's theorem, we have that the polynomial $\mathcal{P}(\lambda)$ (8) is Hurwitz, if it is satisfied that:

$$\Delta_1 = |\,\beta + \sigma + 1\,| > 0$$

$$\Delta_2 = \begin{vmatrix} \beta + \sigma + 1 & \beta\,\sigma\,(1-\gamma) \\ 1 & \beta(\sigma+1) + \sigma(1-\gamma) \end{vmatrix} > 0$$

Δ_1 is positive because the parameters β and σ are positive, we only need Δ_2 to be positive, that is:

$$(\beta + \sigma + 1)[\,\beta(\sigma+1) + \sigma(1-\gamma)] - \beta\,\sigma\,(1-\gamma) > 0 \qquad (9)$$

This is true for

$$(\sigma + 1)[\beta^2 + \beta(\sigma+1) + \sigma(1-\gamma)] > 0$$

and for all $\gamma < 1$.

Therefore, the polynomial $\mathcal{P}(\lambda)$ (8) is Hurwitz if and only if $\gamma < 1$, and indeed the system is asymptotically stable in $\mathcal{O}$ if $\gamma < 1$.

III.I.II Equilibrium point stability $\mathcal{D}_{1,2}$

For $\gamma > 1$ we have the equilibrium points,

$\mathcal{D}_1 = (\sqrt{\beta(\gamma-1)}, \sqrt{\beta(\gamma-1)}, \gamma - 1)$ and

$\mathcal{D}_2 = (-\sqrt{\beta(\gamma-1)}, -\sqrt{\beta(\gamma-1)}, \gamma - 1)$

Linearizing the flow in $\mathcal{D}_1$ or $\mathcal{D}_2$ we have that,

$$\mathcal{J}(\mathcal{D}_{1,2}) = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\gamma-1)} \\ \sqrt{\beta(\gamma-1)} & \sqrt{\beta(\gamma-1)} & -\beta \end{bmatrix}$$

whose characteristic polynomial $\mathcal{P}(\lambda)$ is given by:

$$\mathcal{P}(\lambda) = \lambda^3 + (\beta+\sigma+1)\lambda^2 + \beta(\sigma+\gamma)\lambda + 2\beta\,\sigma\,(\gamma-1) \quad (10)$$

As in the previous case, to analyze the nature of the eigenvalues of the polynomial (10) that will determine the stability in $\mathcal{D}_1$ and $\mathcal{D}_2$ we make use of Routh's algorithm for para $a_0 = 1$, $a_1 = \beta+\sigma+1$, $a_2 = \beta(\sigma+\gamma)$ and $a_3 = 2\beta\,\sigma\,(\gamma-1)$, thus,

$$\boxed{b_0} = \beta(\sigma+\gamma) - \frac{1}{\beta+\sigma+1}\cdot 2\beta\,\sigma\,(\gamma-1)$$

and Routh's arrangement (5) is given by

$$\begin{matrix} 1 & \beta(\sigma+\gamma) \\ \beta+\sigma+1 & 2\beta\,\sigma\,(\gamma-1) \\ \beta(\sigma+\gamma) - \dfrac{2\beta\,\sigma\,(\gamma-1)}{\beta+\sigma+1} & \end{matrix} \quad (11)$$

Then, by Routh's theorem we must analyze the signs of the first column of (11). Since a_0 and a_1 are positive then it only remains to analyze the sign of b_0 as a function of the variable parameter γ. Let us note that if b_0 is positive then the polynomial (10) is Hurwitz and indeed we have stability at the equilibrium points $\mathcal{D}_1$ and $\mathcal{D}_2$, therefore, let us see for which value of γ, b_0 is positive, that is:

$$\boxed{b_0} = \beta(\sigma+\gamma) - \frac{2\beta\,\sigma\,(\gamma-1)}{\beta+\sigma+1} > 0$$

This is true if

$$\gamma < \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$$

Now, if $\tilde{\gamma} = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$ [7], then we conclude that b_0 is positive when it is satisfied that $\gamma < \tilde{\gamma}$ and indeed by Routh's theorem as in the first column of (11) the values of a_0, a_1 and b_0 are positive for the above constraint it follows that the polynomial (10) is Hurwitz and therefore we have that $\mathcal{D}_1$ and $\mathcal{D}_2$ are asymptotically stable when $1 < \gamma < \tilde{\gamma}$.

IV. RESULT AND DISCUSSION

In this section we will analyze the stability results obtained previously together with their respective simulation performed in MATLAB using the *ode45* function for the solution of the system of nonlinear differential equations.

IV.I Stability at $\mathcal{O}$

- If $0 < \gamma < 1$, the origin $\mathcal{O}$ is asymptotically stable, moreover, the origin is globally stable, i.e., all trajectories tend to the origin, as seen in Figure 2.

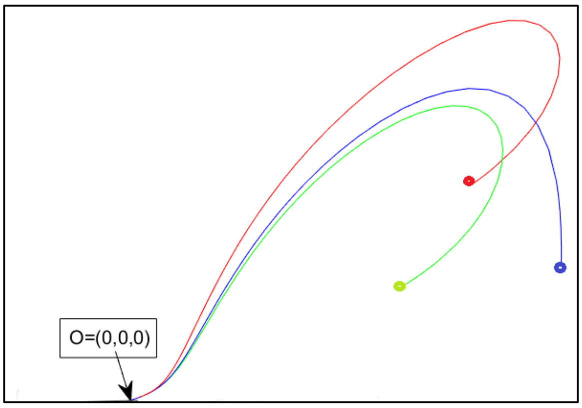


Figure. 2. $\gamma = 0.5$

- If $\gamma = 1$, the equilibrium point $\mathcal{O}$ presents a bifurcation called the hairpin.
- If $\gamma > 1$, the origin is unstable. Figure 3 shows that stability is lost at the origin $\mathcal{O}$.

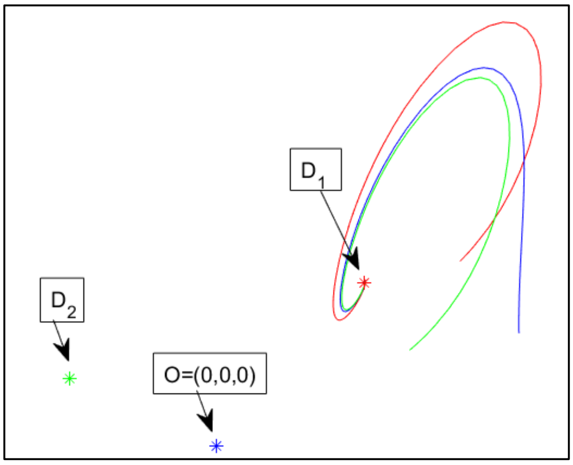
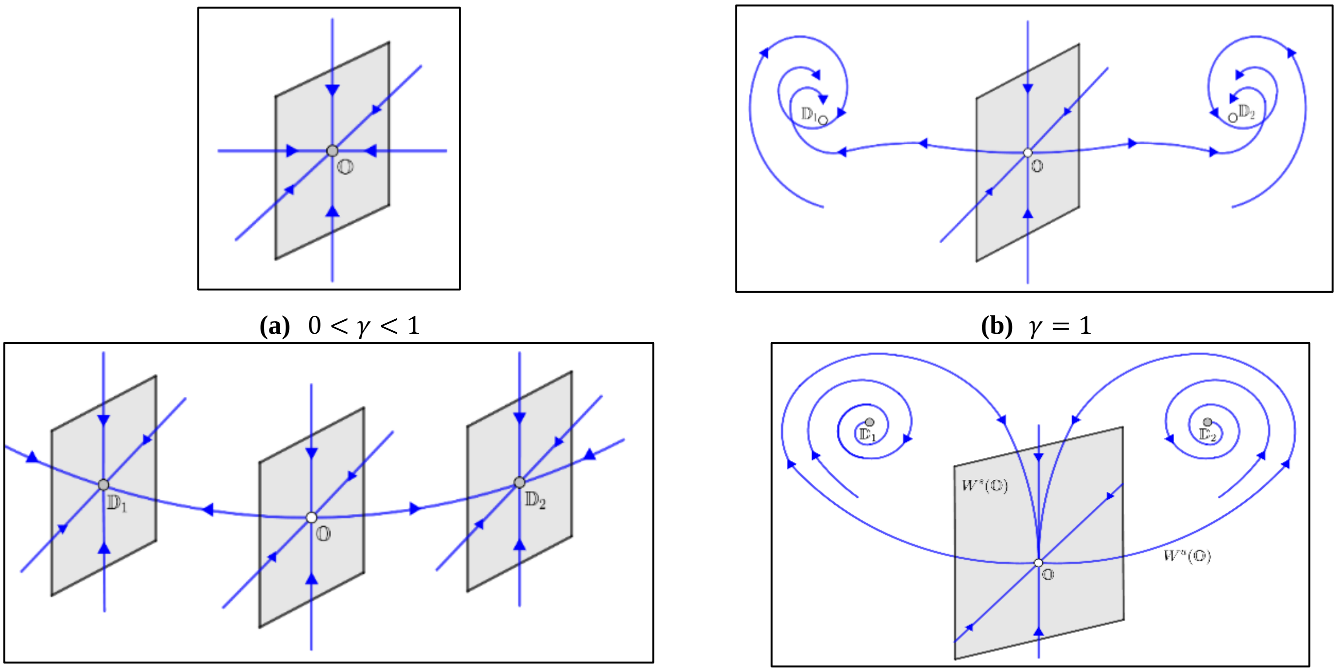


Figure. 3. $\gamma = 1.8$

Furthermore, according to the characteristic of the eigenvalues of the polynomial (8), i.e.

$$\lambda_1 = -\beta, \qquad \lambda_{2,3} = \frac{-(\sigma+1) \pm \sqrt{(\sigma+1)^2 - 4\sigma\,(1-\gamma)}}{2}$$

where λ_1 and λ_3 are negative and λ_2 is positive, the origin presents an unstable variety $W^u(0)$ one-dimensional and a stable variety $W^s(0)$ two-dimensional (see Figure 4 (a-c)). If γ grows to about 13.926 the spirals touch the stable variety of the origin and thus this point becomes homoclinic, i.e., there is an orbit that starts and ends at the origin, as seen in figure 4(d).



(a) $0 < \gamma < 1$ **(b)** $\gamma = 1$

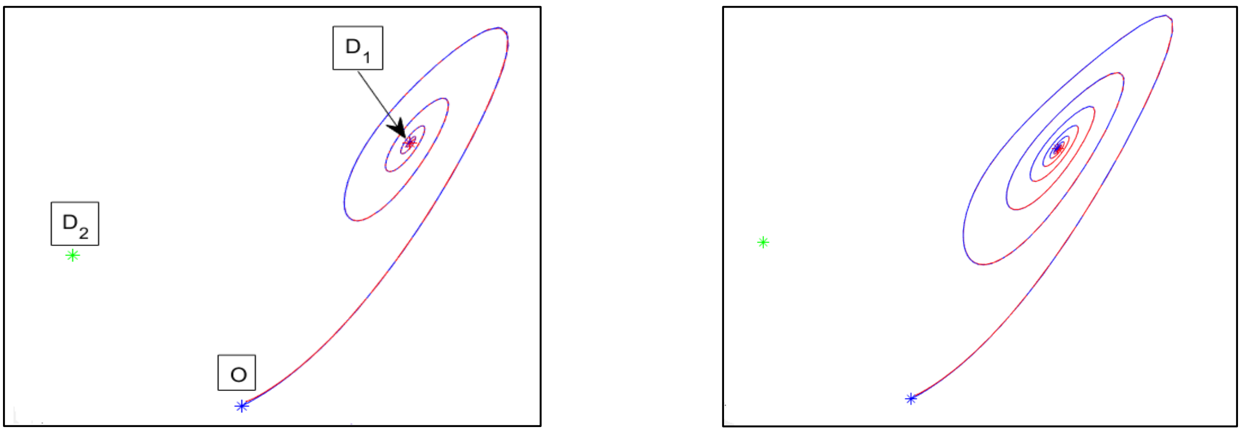
(c) $\gamma > 1$ **(d)** Homoclinic point.

Figure. 4. Flow at the origin for γ close to 1.

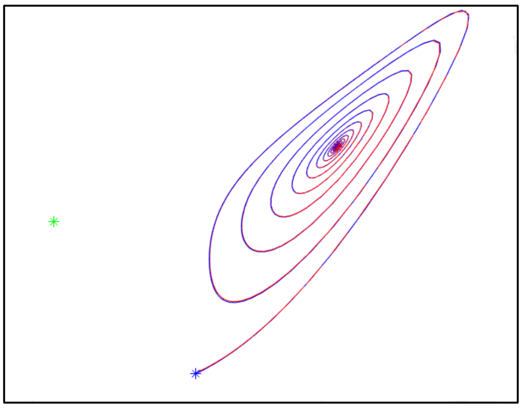
IV.I Stability at $\mathcal{D}_{1,2}$

If $1 < \gamma < \tilde{\gamma}$, the equilibrium points $\mathcal{D}_1$ and $\mathcal{D}_2$ are asymptotically stable, if $\sigma = 10$ and $\beta = \frac{8}{3}$ then $\tilde{\gamma} \approx 24.74$ and indeed $1 < \gamma < 24.74$. It is important to note that according to numerical analysis for initial conditions near the origin, if $1 < \gamma < 13.926$ then $\mathcal{D}_1$ is stable (as seen in Figure 5. (a-c)) and if $13.926 < \gamma < \tilde{\gamma} \approx 24.74$ then $\mathcal{D}_2$ is stable (as shown in Figure 5. (d-i)) i.e., there comes a time when the trajectories vary their stability course between $\mathcal{D}_1$ and $\mathcal{D}_2$.

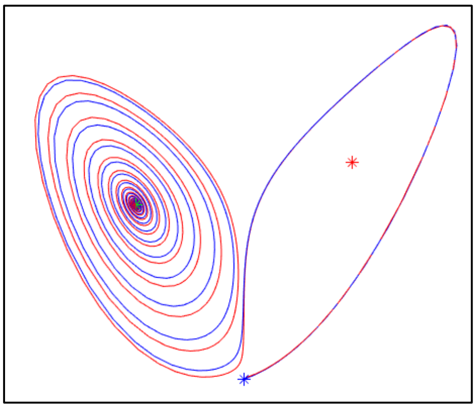
In Figure 5. two trajectories with initial conditions near the origin $\mathcal{O}$ were taken. In Figure 5. (a-c) the trajectories tend to the equilibrium point $\mathcal{D}_1$ and just at about 13.926 the trajectories change their course towards the equilibrium point $\mathcal{D}_2$, as observed in the transition from Figure 5. (c) to Figure 5. (d).



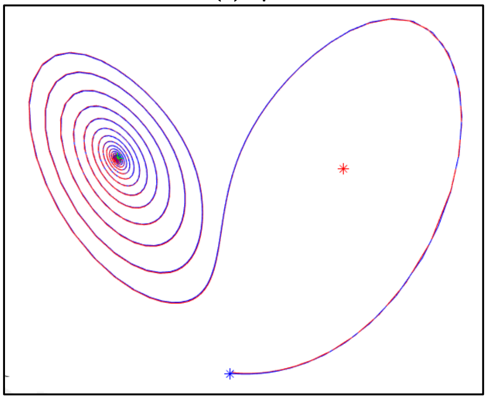
(a) $\gamma = 7$ **(b)** $\gamma = 10$



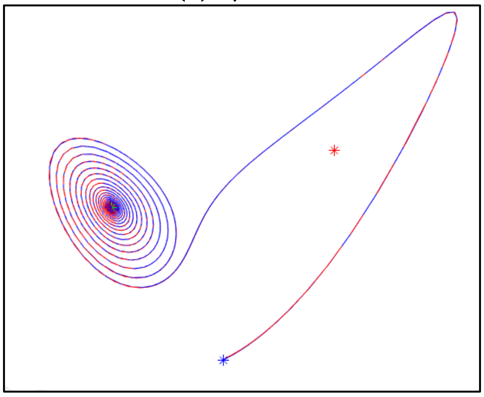
(c) $\gamma = 13$



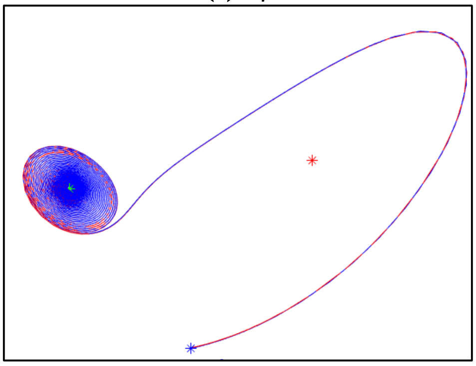
(d) $\gamma = 13.95$



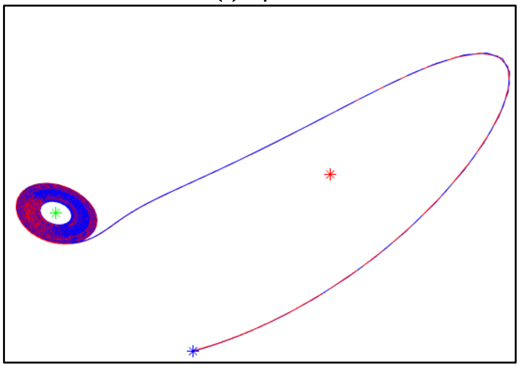
(e) $\gamma = 15$



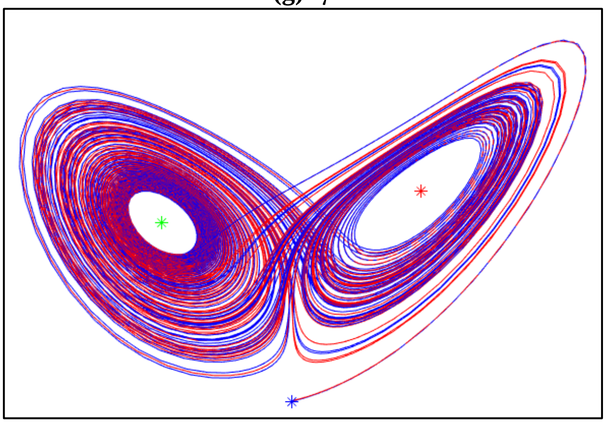
(f) $\gamma = 18$



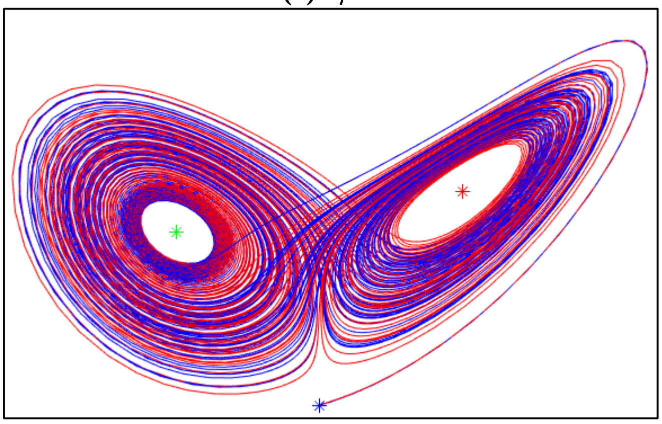
(g) $\gamma = 22$



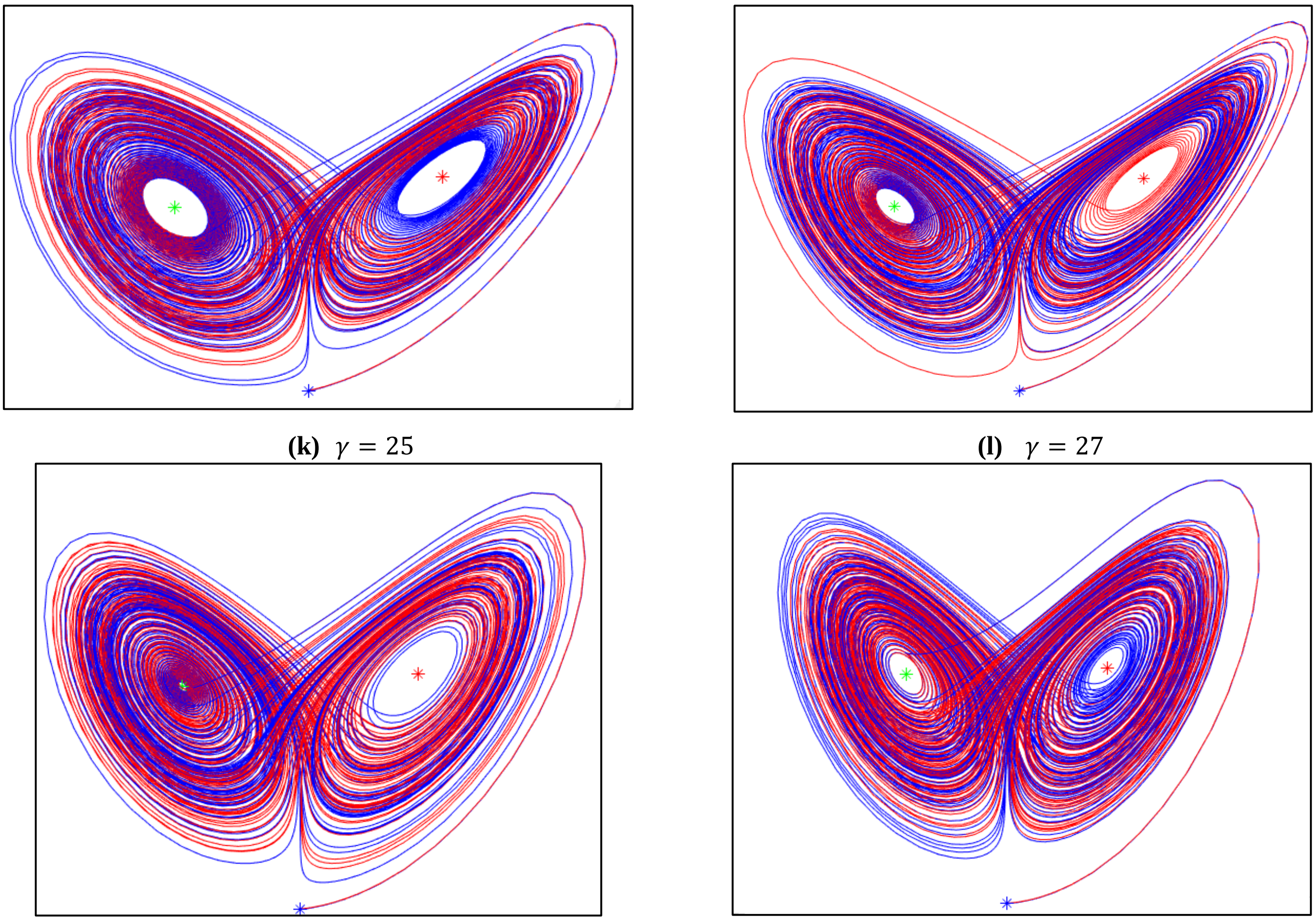
(h) $\gamma = 24$



(i) $\gamma = 24.5$



(j) $\gamma = 24.74$



(k) $\gamma = 25$ **(l)** $\gamma = 27$

(m) $\gamma = 30$ **(n)** $\gamma = 35$

Figure. 5. Trajectories for $1 < \gamma < \tilde{\gamma}$, $\gamma = \tilde{\gamma}$ and $\gamma > \tilde{\gamma}$.

For values of γ greater than 13.926 the trajectories tend to the equilibrium point $\mathcal{D}_2$ as seen in Figure 5. (d-i).

If $\gamma = \tilde{\gamma}$, one has a Hopf-type bifurcation where closed periodic orbits are formed near the equilibrium points $\mathcal{D}_1$ and $\mathcal{D}_2$.

As γ approaches the value $\tilde{\gamma} = 24.74$, the trajectories lose stability, i.e., the behavior of the trajectories as time elapses cannot be predicted, as seen in the transition in Figure 5. (i-k).

If $\gamma > \tilde{\gamma}$, the equilibrium points $\mathcal{D}_1$ and $\mathcal{D}_2$ are unstable.

For values of γ higher than 24.74 the trajectories are unstable, i.e., in spite of a small variation in the initial conditions, these trajectories have different behaviors in the future, as shown in Figure 5(j-n).

V. CONCLUSION

In the stability analysis of systems of differential equations, whether linear or nonlinear, the use of stability criteria to obtain Hurwitz type polynomials provides necessary and sufficient conditions for the study of the dynamics of mathematical models. These algebraic stability criteria made it possible to determine values for which the Lorenz system was asymptotically stable, as could be visualized by the simulations performed in MATLAB, for the values of γ yielded by the Routh-Hurwitz and Routh criterion.

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REFERENCES

[1] A. . Lyapunov, “The General Problem of the Stability of motion,” vol. 6, no. 11, pp. 951–952, 1992.

[2] P. Calderon and C. Victor, “Descripción del modelo de Lorenz con apliaciones,” Universidad Eafit, 2007.

[3] R. Buitrago, “El sistema y el atractor geométrico de Lorenz,” Universidad Nacional de Colombia, 2010.

[4] M.A. Hammami and N.H. Rettab, “On the region of attraction of dynamical systems: Application to Lorenz equations,” 2020.

[5] W. Shenpeng, C. Yan, and S. Guan, “Hopf bifurcation

analysis for a generalized Lorenz system with single delay," *Proc. - 11th Int. Conf. Progn. Syst. Heal. Manag. PHM-Jinan 2020*, pp. 486–489, 2020.

[6] B. Aguirre, C. Loredo, E. Díaz, and E. Campos, "Stability Systems Via Hurwitz Polynomials," vol. 24, no. 1, pp. 61–77, 2017.

[7] F. Toledo Sánchez, "Análsis de estabilidad de sistemas de ecuaciones difenciales utilizando polinomios de Hurwitz," Universidad Tecnologica de Pereira, 2020.

[8] C. A. Loredo, "Criterios para determinar si un polinomio es polinomio Hurwitz," Universidad Autónoma Metropolitana Unidad Iztapalapa, 2005.

[9] C. Lodero, "Factorización de Hadamard para polinomios Hurwitz," Universidad Autonoma Metropolitana, 2012.

[10] Z. Zahreddine, "On some properties of hurwitz polynomials with application to stability theory," vol. 25, no. 1, pp. 19–28, 1999.

[11] B. N. Datta, "An elementary proof of the stability criterion of Liénard and Chipart," *Linear Algebra Appl.*, vol. 22, no. C, pp. 89–96, 1978.

[12] E. N. Lorenz, "Deterministic Nonperiodic Flow," *J. Atmos. Sci.*, vol. 20, pp. 130–141, 1963.

[13] P. P. Cardenas Alzate, G. C. Velez, and F. Mesa, "Chaos control for the Lorenz system," *Adv. Stud. Theor. Phys.*, vol. 12, no. 4, pp. 181–188, 2018.