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Stochastic Volatility Estimation to Determine Price Process with Compensated Poisson Jumb Using Fourier Transformation

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Abstract

This paper deal with Stochastics volatility estimation using Fourier Transformation to determine price process through compensated Poission jumb. Here we adopt theoretical basis of Fourier transformations by examine Bohr's convolution which revealed similar relation with the Fourier transformation of the price determination with a compensated Poisson jump and the volatility with estimated instantaneous volatility. The similar relation for a special case when $\beta(t)$ were included in the jump was revealed with direct variation to the volatility with all parameters constant.which is applicable. to both univariate and multivariate volatility settings. This revealed the effect of the dynamics of volatility on finances through various kinds of jumb difussion processes

Keywords: Convolution, Compensated, Estimation, Stochastic, Transformation, Volatility

INTRODUCTION

Measurement of an uncertainty of returns through volatility involves an important role in transacting cash flows, assets selling and purchases at a giving time. This is very important and necessary in market institutions because of unstable and stock prices forecast, hedging as well as portfolio management. Board managers and policy makers depend much on this to ascertain unstable nature of the market to minimize losses. Constant changes in its nature is so challenging to forecast and predictions. Black-Scholes models proposed a constant volatility with empirical findings shows that researchers need to intensify more into asset volatility modelling as Black-Scholes model, price processes does not include Poisson jump It is now of interest to verify and examine at what level the dynamics of stock price volatility affected by jumps price inclusion

Empirical evidence suggests that the volatility of many assets prices is stochastic which affects pricing and hedging of options assets thus creates the need for simple and efficient models. Volatility as a key parameter in financial economics-mathematics and the recent econometric literature has devoted much attention to its computation which is crucial in hedging strategies in the classical Black-Scholes environment or in more sophisticated stochastic volatility models.

[1] Examined and evaluate time-varying methods on the development of new tools for volatility measurement, modeling and forecasting which been motivated by the empirical observation in financial asset return and persistent fashion in asset classes and time periods. [2] Opined that volatility can be determined using both parametric and non-parametric methods. The parametric procedures rely on explicit functional form with the assumptions regarding the expected and instantaneous volatility which include discrete-time volatility model

[3] Developed and applied Fourier series analysis to compute time series volatility when the data observed are semi martingale based on Fourier coefficients computations which relies on the integration of the time series rather than on its differentiation. The method is fully model free and non-parametric which makes the method well suitable in financial transaction and in the analysis of high frequency time series. [4] revealed the potential and bias that covariance estimator has especially when the regular interval size h is small relative to its frequency. He then propose a new estimator based only on original data which requires no prior synchronization of transaction-based data and independent from the choice of h with imputation of missing values. It is also free from extraneous biases, consistency estimator as the observe time intensity which represents the liquidity of the market increases to infinity.

The finite sample property of the Fourier estimator for integrated volatility under market microstructure noise with an observed value of the contaminated process, derive an analytic expression for the bias and the mean squared error of the contaminated estimator can be practically used to design optimal MSE-based estimators, which are very robust and efficient in the presence of noise as opined by [5].

[6] Evaluate and provides a non-parametric method for computation of instantaneous multivariate volatility for continuous semi-martingales based on Fourier series analysis where the co-volatility was adjusted as a stochastic function of time by establishing a relationship between Fourier volatility processes. He then derived a non-parametric estimator from a discrete unevenly spaced and a synchronously sampled.

The forecasting performance of the Fourier estimator with the inclusion of microstructure noise and analytic comparison with simulation studies indicate that the Fourier estimator significantly outperforms realized estimators, particularly for high-frequency and when noise component is relevant as analyze and opined by [7].

1. FOURIER TRANSFORMSATION AND FORMATION OF PRICE PROCESS WITH COMPENSATED POISSON JUMB

Representation of frequency domain and mathematical operation linking to a function of time is refers to as Fourier transformation as opined by [8], A time varying data can be transformed from one domain into a different domain called the main ideal of Fourier methods and transformed to be a non-periodic functions which is absolutely converges and proposed by [9] in this case, a function can be reconstructed from its original Fourier stage using inverse Fourier transform.. In Fourier transformation the volatility is constructed as a function of it's iteration and computation of the cross-correlation between price and volatility where all the observed values are taking into consideration to avoids inconsistency in data.

1.1. Formation of Price process with compensated Poisson jump

We let $p(t)$ be the log-price of assets which is a continuous semi−martingale on a fixed time window, then

$$dp(t) = \alpha(t,B)dt + \sigma(t,B)dB(t) + dM(t) \tag{1}$$

where $M(t) = N(t) - \lambda t$ and $N(t)$ is a Poisson process with intensity λ, α is the drift, σ is the volatility, time is t and standard Brownian motion B. $B(t)$ and $M(t)$ are independent. Solving

$$p(t) = p(0) + \int_0^t \alpha(s,B)ds + \int_0^t \sigma(s,B)dB(s) + \int_0^t dM(s) \tag{2}$$

where σ is adapted to a filtration and it's bounded by $|\alpha| + |\sigma| \leq c$ for $c \in \mathbb{R}$,

Given that $p(t) = p^1(t), \cdots, p^n(t)$, satisfying

$$dp^j(t) = \sum_{i=1}^d \sigma_i^j(t) + \alpha^j(t)dt + dM^j(t), \quad j = 1, \cdots, n, \tag{3}$$

where $B = B^1, \cdots B^d$ are independent Brownian motions on a probability space, such that σ_i^j and α^j are random processes which are adapted to a filtration and satisfies the following conditions;

$$E\left[\int_0^T (\alpha^i(t))^2 dt\right] < \infty \tag{4}$$

$$E\left[\int_0^T (\sigma_i^j(t))^4\right] \infty, i = 1, \cdots d, \quad j = 1, \cdots, n. \tag{5}$$

Theorem 1

Given that *function* $\phi(v)$ *has Fourier transform:* $\mathcal{F}(\phi)(k) := \frac{1}{2\pi}\int_0^{2\pi} \phi(v)e^{-ikv}dv, \quad k \in \mathbb{Z}$

and its differential form: $\mathcal{F}(d\phi)(k) := \frac{1}{2\pi}\int_0^{2pi} e^{-ikv}d\phi(v)$

then ,$\mathcal{F}(\phi)(k) = \frac{i}{k}\left[\frac{1}{2\pi}(\phi(2\pi) - \phi(o)) - \mathcal{F}d\phi(k)\right]$

Proof:

Given that

$$\mathcal{F}(\phi)(k)\frac{1}{2\pi}\left[-\phi(v)\frac{e^{-ikv}}{ik} - \int -\frac{e^{-ikv}}{ik}d\phi(v)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi ik}[-\phi(v)e^{-ikv} + \int e^{ikv}d\phi(v)]_0^{2\pi}$$

$$= -\frac{1}{2\pi ik}[\phi(v)e^{-ikv} - \int e^{-ikv}d\phi(v)]_0^{2\pi}$$

Then,

$$\mathcal{F}(\phi)(k) = \left[-\frac{1}{2\pi ik}\phi(v)e^{-ikv}\right]_0^{2\pi} + \frac{1}{ik}\mathcal{F}d\phi(k)$$

$$= -\frac{1}{2\pi ik}\left[\phi(2\pi)e^{-ik(2\pi)} - \phi(0)\right] + \frac{1}{ik}\mathcal{F}d\phi(k)$$

but $e^{-ik(2\pi)} = \cos(2\pi k) - i\sin(2\pi k)$ and $\cos(2\pi k) = 1, \quad \sin(2\pi k) = 0$, then we have

$$\mathcal{F}(\phi)(k) = -\frac{1}{2\pi ik}[\phi(2\pi)(\cos(2\pi k) + i\sin(2\pi k)) - \phi(0)] + \frac{1}{ik}\mathcal{F}d\phi(k)$$

$$= -\frac{1}{2\pi ik}[\phi(2\pi) - \phi(0)] + \frac{1}{ik}\mathcal{F}d\phi(k)$$

that is

$$\mathcal{F}(\phi)(k) = \frac{i}{k}\left[\frac{1}{2\pi}(\phi(2\pi) - \phi(0)) - \mathcal{F}d\phi(k)\right]$$

3. THE IDENTITY RELATION FOR A COMPLEX MARTINGALE CASE

Proposition 1

Given that the identity that relates to the price process and volatility matrix with the compensated poisson jump as

$$\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(k) + \frac{1}{2\pi}\mathcal{F}(dN)(k) = \mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j \quad (6)$$

Proof.

Let (t) be Price process of the Fourier transformation

Let $\Sigma(t)$ be volatility matrix for Fourier transform

Given that α=0; and p be a semi−martingale.

$$dp^j(t) = \sum_{i=1}^{d} \sigma_i^j(t)\sigma_i^k(t)dB^i(t) + \alpha^j(t)dt + dM^j(t), \quad j = 1, \cdots, n,$$

where $\sigma(t, B(t))$ does not depend on $B(t)$.

$$\mathcal{F}(dp^j)(k) = \mathcal{F}\left(\sum_{i=1}^{d} \sigma_i^j\sigma_i^k dB^i\right)(k) + \mathcal{F}(\alpha^j dt)(k) + \mathcal{F}(dM^j)(k)$$

Given that $dp_\sigma^i(t) = \sum_{j=1}^{d} \sigma_i^k(t)\sigma_i^k(t)dB^i(t)$, then,

$$\mathcal{F}(dp^j)(k) = \mathcal{F}(dp_\sigma^j)(k) + \mathcal{F}(\alpha^j dt)(k) + \mathcal{F}(dM^j)(k)$$

$$\mathcal{F}(dp_\sigma^j)(k) = \phi_m(k), \quad \mathcal{F}(\alpha^j dt)(k) = \phi_\alpha(k) \quad and \quad \mathcal{F}(dM^j)(k) = \phi_M(k)$$

$$(\phi_\sigma + \phi_\alpha + \phi_M)_{*B}(\phi_\sigma + \phi_\alpha + \phi_M) = (\phi_{\sigma*B} + \phi_{\alpha*B} + \phi_{M*B})(\phi_\sigma + \phi_\alpha + \phi_M)$$

$$= \phi_{\sigma*B}\phi_\sigma + \phi_{\alpha*B}\phi_\alpha + \phi_{\sigma*B}\phi_M + \phi_{\alpha*_B}\phi_\sigma + \phi_{\alpha*B}\phi_\alpha$$

$$+\phi_{\alpha*B}\phi_M + \phi_{M*B}\phi_\sigma + \phi_{M*B}\phi_\alpha + \phi_{M*B}\phi_M$$

But $\alpha = 0$

Then by Bohr's convolution

$$(\phi_\sigma + \phi_\alpha + \phi_M)_{*B}(\phi_\sigma + \phi_\alpha + \phi_M) = \phi_{\sigma*B}\phi_\sigma + \phi_{M*B}\phi_M$$

Thus:

$$(\phi_{\alpha*B}\phi_\alpha)(k) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{s=-n}^{n} \phi_\alpha(s)\phi_\alpha(k-s)$$

$$(\phi_{\sigma*B}\phi_\sigma)(k) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{s=-n}^{n} \phi_\sigma(s)\phi_\sigma(k-s)$$

$$(\phi_{M*B}\phi_M)(k) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{s=-n}^{n} \phi_M(s)\phi_M(k-s)$$

$$(\phi_\sigma + \phi_\alpha + \phi_M)_{*B}(\phi_\sigma + \phi_\alpha + \phi_M) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{s=-n}^{n} \phi_\sigma(s)\phi_\sigma(k-s)$$

$$+\lim_{n\to\infty}\frac{1}{2n+1}\sum_{s=-n}^{n} \phi_M(s)\phi_M(k-s)$$

$$\mathcal{F}(dp^j)_{*B}\mathcal{F}(dp^j) = (\phi_\sigma + \phi_\alpha + \phi_M)_{*B}(\phi_\sigma + \phi_\alpha + \phi_M) = \phi_{\sigma*B}\phi_\sigma + \phi_{M*_B}\phi_M$$

$$\mathcal{F}(dX)_{*B}\mathcal{F}(dX) = \frac{1}{2\pi}\mathcal{F}([dX,dX]) = \frac{1}{4\pi^2}\int_0^t e^{-iks}d[X,X](s),$$

$$\mathcal{F}(dp^j)_{*B}\mathcal{F}(dp^j) = (\phi_{\sigma*B} * \phi_\sigma) + (\phi_{M*B} * \phi_M)$$

$$= \mathcal{F}(dp_\sigma^j)_{*B}\mathcal{F}(dp_\sigma^j) + \mathcal{F}(dM^j)_{*B}\mathcal{F}(dM^j)$$

$$= \frac{1}{2\pi}\mathcal{F}([dp_\sigma^j, dp_\sigma^j]) + \frac{1}{2\pi}\mathcal{F}([dM^j, dM^j])$$

$$= \frac{1}{4\pi^2}\int_0^t e^{-iks}d[p_\sigma^i, p_\sigma^i](s) + \frac{1}{4\pi^2}\int_0^t e^{-iks}d[M^j, M^j](s)$$

Where

$d[M^j, M^j](s) = dN^j(s)$ is:

$$\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j) = \frac{1}{4\pi^2}\int_0^t e^{-iks}d[p_\sigma^i, p_\sigma^j](s) + \frac{1}{4\pi^2}\int_0^t e^{-iks}dN^j(s)$$

Let $d[p_\sigma^i, p_\sigma^j](s) = \Sigma^{ij}(t) \Rightarrow \frac{1}{4\pi^2}\int_0^t e^{-iks}d[p_\sigma^i, p_\sigma^j](s) = \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(k)$

$$\frac{1}{4\pi^2}\int_0^t e^{-iks}dN^j(s) = \frac{1}{2}\mathcal{F}(dN^j)(k$$

$$\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij}(k) = \mathcal{F}(dp^j)_{*B}\mathcal{F}(dp^j) - \frac{1}{2\pi}\mathcal{F}(dN^j)(k)$$

Hence the volatility matrix and the compensated Poisson jump is:

$$\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(k) + \frac{1}{2\pi}\mathcal{F}(dN^j)(k) = \mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j)$$

Proposition 2

Given that

$$(\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j))(q) = \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q) + \frac{1}{2\pi}\mathcal{F}(dN^j)(q) \quad (7)$$

Proof.

We let (t) be independent of Bt

Let (t),$\Gamma rj(t)$ be complex martingales for any integers r,k,

Let p be the price process, then, i,=1,2.

Then the complex martingale for Fourier transformation yield:

$$\Gamma_k^i(t) := \frac{1}{2\pi}\int_0^t e^{-iks}dp^i(s)$$

$$\Gamma_k^j(t) := \frac{1}{2\pi}\int_0^t e^{-irs}dp^j(s)$$

$$\mathcal{F}(dp^i)(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-iks}dp^i(s)$$

which implies $\Gamma_k^j(2\pi) = \mathcal{F}(dp^i)(k)$

Apply Ito formula,

$$d(\Gamma_k^i\Gamma_r^j)(t) = \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_k^j(t)d\Gamma_k^i(t) + d\Gamma_k^i(t)d\Gamma_r^j(t)$$

$$= \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^i(t)$$

$$+ \left(d(\frac{1}{2\pi}\int_0^t e^{-iks}dp^i(s))d(\frac{1}{2\pi}\int_0^t e^{-irs}dp^j(s))\right)$$

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Let $\alpha = 0$

$$d(\Gamma_k^i\Gamma_r^j)(t) = \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_k^j(t)d\Gamma_k^i(t)$$

$$+\left(d\left(\frac{1}{2\pi}\int_0^t e^{-iks}\left(\sum_{i=0}^{d}\sigma_l^i dB^l + dM^j(s)\right)\right)d\left(\frac{1}{2\pi}\int_0^t e^{-irs}\left(\sum_{i=0}^{d}\sigma_l^j dB^l + dM^j(s)\right)\right)\right)$$

$$= \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^i(t) + \left(d\left(\tfrac{1}{2\pi}\int_0^t e^{-iks}\left(\textstyle\sum_{i=0}^{d}\sigma_i^j dB^i\right) + d\left(\tfrac{1}{2\pi}\int_0^t e^{-iks} dM^j(s)\right)\right)\right)$$

$$\times\left(d\left(\tfrac{1}{2\pi}\int_0^t e^{-irs}(\textstyle\sum_{i=0}^{d}\sigma_i^j dB^i\right) + d\left(\tfrac{1}{2\pi}\int_0^t e^{-irs} dM^j(s)\right)\right)$$

$$= \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^i(t) + d\left(\frac{1}{2\pi}\int_0^t e^{-iks}\sum_{i=0}^{d}\sigma_l^i dB^l\right)d\left(\frac{1}{2\pi}\int_0^t e^{-irs}\sum_{i=0}^{d}\sigma_l^j dB^l\right)$$

$$+d\left(\tfrac{1}{2\pi}\int_0^t e^{-iks}\textstyle\sum_{i=0}^{d}\sigma_l^i dB^l\right)d\left(\tfrac{1}{2\pi}\int_0^t e^{-irs} dM^j(s)\right)$$

$$+d\left(\tfrac{1}{2\pi}\int_0^t e^{-iks} dM^j(s)\right)d\left(\tfrac{1}{2\pi}\int_0^t e^{-iks}\textstyle\sum_{i=0}^{d}\sigma_l^j dB^l\right)$$

$$+d\left(\tfrac{1}{2\pi}\int_0^t e^{-iks} dM^j(s)\right)d\left(\tfrac{1}{2\pi}\int_0^t e^{-irs} dM^j(s)\right)$$

$$= \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^i(t) + \left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)l}\textstyle\sum_{l=0}^{d}\sigma_l^i\sigma_l^j(t)(dB^l(s))^2\right)$$

$$+\left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)l} dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^i\sigma_l^j(t)dB^l(t)\right)$$

$$+\left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)l} dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^i\sigma_l^j(t)dB^l(t)\right) + \left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)l}(dM^j(t))^2\right)$$

But $\sum_{l=0}^{d}\sigma_l^i\sigma_l^j = \sum ij$ and $(dM^j(t))^2 = dN_t^j$, which implies,

$$d\left(\Gamma_k^i\Gamma_r^j\right)(t) = \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^i(t) + \left(\tfrac{1}{2\pi}\right)^2\Sigma^{ij}e^{-i(k+r)t}dt$$

$$+\left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^i(t)dB^l(t)\right) + \left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^i(t)dB^l(t)\right)$$

$$+\left(\frac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dN_t^j\right)$$

$$\int_0^{2\pi} d\left(\Gamma_k^i\Gamma_r^j\right)(t) = \int_0^{2\pi}\left(\Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_k^i(t)\right)d\Gamma_k^i(t) + \int_0^{2\pi}\left(\tfrac{1}{2\pi}\right)^2\Sigma^{ij}e^{-i(k+r)t}dt$$

$$+\int_0^{2\pi}\left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^j(t)dB^l(t)\right)$$

$$+\int_0^{2\pi}\left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^j(t)dB^l(t)\right) + \int_0^{2\pi}\left(\tfrac{1}{2\pi}\right)^2$$

$$\int_0^{2\pi} d(\Gamma_k^i\Gamma_r^j)(t) = (\Gamma_k^i\Gamma_r^j)(2\pi) - (\Gamma_k^i\Gamma_r^j)(0) = \Gamma_k^i(2\pi)\Gamma_r^j(2\pi) - \Gamma_k^i(0)\Gamma_r^j(0),$$

and $\Gamma_k^i(0)\Gamma_r^j(0) = 0$

$$\Gamma_k^i(2\pi)\Gamma_r^j(2\pi) = \int_0^{2\pi}(\Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^i(t)) + \int_0^{2\pi}\left(\tfrac{1}{2\pi}\right)^2\Sigma^{ij}e^{-i(k+r)t}dt +$$

$$\int_0^{2\pi}\left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^i(t)dB^l(t)\right) + \int_0^{2\pi}\left(\tfrac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\textstyle\sum_{l=0}^{d}\sigma_l^i(t)dB^l(t)\right)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dN_t^j\right)$$

Let $H^{ij}(k,r) = \int_0^{2\pi} (\Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^j(t))$

Thus

$\frac{1}{2\pi}\int_0^{2\pi} \sum ij\, e^{-i(k+r)t}dt = \mathcal{F}(\sum ij)(k+r)$, which follows that,

$$\Gamma_k^i(2\pi)\Gamma_r^j(2\pi) = \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(k+r) + H^{ij}(k,r)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dM^j(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-i(k+r)t}dN_t^j\right)$$

for $n \geq 1$, then for any integer q, where $|q| \leq n$

$$g_q^{ij}(n) = \frac{1}{2n+1}\sum_{s=-n}^n \Gamma_{q+s}^i(2\pi)\Gamma_{-s}^j(2\pi)$$

$$\Gamma_{q+s}^i(2\pi)\Gamma_{-s}^j(2\pi) = \frac{1}{2\pi}\mathcal{F}(\sum ij\,(q+s-s) + H^{ij}(q+s,-s)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-i(q+s-s)t}dM^j(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-i(q+s-s)t}dM^j(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-i(q+s-s)t}dM_t^j\right)$$

$$\Gamma_{q+s}^i(2\pi)\Gamma_{-s}^j(2\pi) = \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij}(q+s-s) + H^{ij}(q+s,-s)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right) + \int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right)$$

$$+\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dN_t^j\right)$$

$$g_q^{ij}(n) = \frac{1}{2n+1}\sum_{s=-n}^n \Gamma_{q+s}^i(2\pi)\Gamma_{-s}^j(2\pi)$$

$$= \frac{1}{2n+1}\sum_{s=-n}^n\left[\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q) + H^{ij}(q+s,-s) + \int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2(e^{-iqt}dM^j(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t))\right]$$

$$+\frac{1}{2n+1}\sum_{s=-n}^n\left[\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j)(t)\sum_{l=0}^d \sigma_l^i(t)\right) + \int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2(e^{-iqt}dN_t^j)\right]$$

$$= \frac{1}{2n+1}\left[\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q)(2n+1) + \sum_{s=-n}^n H^{ij}(q+s,-s)\right]$$

$$+\frac{1}{2n+1}\left[\sum_{s=-n}^n\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j)(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right)\right]$$

$$+\frac{1}{2n+1}\sum_{s=-n}^n\left[\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j)(t)\sum_{l=0}^d \sigma_l^i(t)\right)\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dN_t^j)\right)\right]$$

$$= \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q) + \frac{1}{2n+1}\sum_{s=-n}^n\left[H^{ij}(q+s,-s) + \int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j)(t)\sum_{l=0}^d \sigma_l^i(t)\right)\right]$$

$$+\frac{1}{2n+1}\sum_{s=-n}^n\left[\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j)(t)\sum_{l=0}^d \sigma_l^i(t)dB^l(t)\right) + \int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dN_t^j)\right)\right]$$

Hence,

$$g_q^{ij}(n)\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q) + H_n^{ij}(q+s,-s) + Y_n^{ij}$$

$$H_n^{ij}(q+s,-s) = \frac{1}{2n+1}\sum_{s=-n}^{n} H^{ij}(q+s,-s$$

$$H^{ij}(k,r) = \int_0^{2\pi} \Gamma_k^i(t)d\Gamma_r^j(t) + \Gamma_r^j(t)d\Gamma_k^i(t$$

Thus

$$H_n^{ij}(q+s,-s) = \frac{1}{2n+1}\sum_{s=-n}^{n} \int_0^{2\pi} \Gamma_{q+s}^i(t)d\Gamma_{-s}^j(t) + \Gamma_{-s}^j(t)d\Gamma_{q+s}^i(t)$$

$$Y_n^{ij}(q) = \frac{1}{2n+1}\sum_{s=-n}^{n} \int_0^{2\pi} \left(\frac{1}{2\pi}\right)^2 \left(e^{-iqt}dM^j)(t)\sum_{l=0}^{d} \sigma_l^i(t)dB^l(t)\right)$$

$$+\frac{1}{2n+1}\sum_{s=-n}^{n}\left[\int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2\left(e^{-iqt}dM^j)(t)\sum_{l=0}^{d} \sigma_l^i(t)dB^l(t)\right) + \int_0^{2\pi}\left(\frac{1}{2\pi}\right)^2 (e^{-iqt}dN_t^j)\right]$$

$H_n^{ij}(q+s,-s)$ can also be reduced to Q_n

$$Q_n = \frac{1}{2n+1}\sum_{s=-n}^{n} \int_0^{2\pi} d\Gamma_{-s}^j(t_2)\int_0^{t_2} d\Gamma_{q+s}^i(t_1)$$

from Q_n,

$$\int_0^{2\pi} d\Gamma_{-s}^j(t_2)\int_0^{t_2} d\Gamma_{q+s}^i(t_1) = \left[\Gamma_{-s}^j(t_2)\right]_0^{2\pi}\left[\Gamma_{-s}^j(t_1)\right]_0^{t_2}$$

$$= (\Gamma_{-s}^j(2\pi) - \Gamma_{-s}^j(0))(\Gamma_{q+s}^i(t_2) - \Gamma_{q+s}^i(0))$$

We let $\Gamma(0) = 0$

$$\int_0^{2\pi} d\Gamma_{-s}^j(t_2)\int_0^{t_2} d\Gamma_{q+s}^i(t_1) = \Gamma_{-s}^j(2\pi)\Gamma_{q+s}^i(t_2$$

$$\int d(\Gamma_{-s}^j(2\pi)\Gamma_{q+s}^i(t_2)) = \int \Gamma_{-s}^j(2\pi)d\Gamma_{q+s}^i(t_2) + \int \Gamma_{q+s}^i(t_2)d\Gamma_{-s}^j(2\pi)$$

Since $(t_1, t_2) \in t$ and $t \in (0,2\pi)$, we have

$$\int_0^{2\pi} d(\Gamma_{-s}^j(t)\Gamma_{q+s}^j(t)) = \int_0^{2\pi} \Gamma_{-s}^j(t)d\Gamma_{q+s}^i(t) + \Gamma_{q+s}^i(t)d\Gamma_{-s}^j(t)$$

Hence by symmetry as $n \to \infty$,

Given a Dirichlet kernel $(D_n(t))$ such that:

$$D_n(t) = \frac{1}{2n+1}\sum_{s=-n}^{n} e^{ist} = \frac{1}{2n+1}\frac{\sin(n+1)t}{\sin(t/2)},$$

$$Q_n = \frac{1}{2n+1}\sum_{s=-n}^{n} \int_0^{2\pi} d\Gamma_{-s}^j(t_2)\int_0^{t_2} d\Gamma_{q+s}^i(t_1)$$

$$= \frac{1}{2n+1}\sum_{s=-n}^{n} \Gamma_{-s}^j(2\pi)\Gamma_{q+s}^i(t_2)$$

$$= \frac{1}{2n+1}\sum_{s=-n}^{n} \frac{1}{2\pi}\int_0^{2\pi} e^{ist_2}dp^j(t_2) \times \frac{1}{2\pi}\int_0^{t_2} e^{-i(q+s)t_1}dp^i(t_1)$$

$$= \frac{1}{4\pi^2}\left(\frac{1}{2n+1}\right)\sum_{s=-n}^{n} \int_0^{2\pi} e^{ist_2}dp^j(t_2)\int_0^{t_2} e^{-i(q+s)t_1}dp^i(t_1)$$

$$= \frac{1}{4\pi^2}\left(\frac{1}{2n+1}\right)\sum_{s=-n}^{n} \int_0^{2\pi} e^{ist_2}dp^j(t_2)\int_0^{t_2} e^{-iqt_1}e^{-ist_1}dp^i(t_1)$$

$$= \frac{1}{4\pi^2}\left(\frac{1}{2n+1}\right)\sum_{s=-n}^{n} \int_0^{2\pi}\int_0^{t_2} e^{-iqt_2} \times e^{-ist_1} \times e^{-ist_1}dp^i(t_1)dp^j(t_2)$$

$$Q_n = \frac{1}{4\pi^2}\left(\frac{1}{2n+1}\right)\sum_{s=-n}^{n} \int_0^{2\pi}\int_0^{t_2} e^{is(t_2-t_1)} \times e^{-iqt_1}dp^i(t_1)dp^j(t_2)$$

$$= \frac{1}{4\pi^2}\left(\frac{1}{2n+1}\right)\int_0^{2\pi}\int_0^{t_2} e^{-iqt_2} \times \sum_{s=-n}^{n} e^{is(t_2-t_1)}dp^i(t_1)dp^j(t_2)$$

$= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{t_2} e^{-iqt_2} \times \frac{1}{2n+1}\sum_{s=-n}^{n} e^{is(t_2-t_1)} dp^i(t_1)dp^j(t_2)$

$D_n(t_2-t_1) = \frac{1}{2n+1}\sum_{s=-n}^{n} e^{is(t_2-t_1)}$

$Q_n = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{t_2} e^{-iqt_1} D_n(t_2-t_1)dp^i(t_1)dp^j(t_2$

$Q_n = \frac{1}{4\pi^2}\int_0^{2\pi} dp^j(t_2)\int_0^{t_2} e^{-iqt_1} D_n(t_2-t_1)dp^i(t_1)$

But $\alpha = 0$

$$Q_n = \frac{1}{4\pi^2}\int_0^{2\pi} dp^j(t_2)\int_0^{t_2}(\cos(qt_1) - i\sin(qt_1))D_n(t_2-t_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1) + dM^i(t_1)\right)$$

$$|Q_n|^2 = \left(\frac{1}{4\pi^2}\int_0^{2\pi} dp^j(t_2)\int_0^{t_2}(\cos(qt_1) - i\sin(qt_1))D_n(t_2-t_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1) + dM^i(t_1)\right)\right)^2$$

$$|Q_n|^2 = \left(\frac{1}{4\pi^2}\right)^2\int_0^{2\pi}\left(\sum_{k=1}^{2}\sigma_k^i(t_2)dB^k(t_1) + dM^i(t_2)\right)^2$$

$$\left[(\cos(qt_1) - i\sin(qt_1))D_n(t_2-t_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1) + dM^i(t_1)\right)\right]^2$$

We let

$$\zeta(t_2) = \int_0^{t_2}(\cos(qt_1)D_n(t_2-t_1))^2\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1) + dM^i(t_1)\right)^2, dM_t.dB_t = 0$$

$= \int_0^{t_2}(\cos(qt_1)D_n(t_2-t_1))^2\left(\left(\sum_{k=1}^{2}\sigma_k^i(t_1)\right)^2 dt_1 + dN^i(t_1)\right)$

$\varpi^2(t_2) = \int_0^{t_2}(\sin(qt_1)D_n(t_2-t_1))^2\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k t_1 + dM^i(t_1)\right)^2$

$= \int_0^{t_2}(\sin(qt_1)D_n(t_2-t_1))^2\left(\left(\sum_{k=1}^{2}\sigma_k^i(t_1)\right)^2 dt_1 + dN^i(t_1)\right)^2$

$|Q_n|^2 = \left(\frac{1}{16\pi^4}\right)\int_0^{2\pi}\left(\left(\sum_{k=1}^{2}\sigma_k^i(t_2)\right)^2 dt_2 + dN^j(t_2)\right)[\varsigma^2(t_2) + \varpi^2(t_2)]$

$16\pi^4|Q_n|^2 = \sum_{k=1}^{2}\int_0^{2\pi}[\varsigma^2(t_2) + \varpi^2(t_2)]\left((\sigma_k^j(t))^2 dt_2 + dN^j(t_2)\right)$

Using Ito identity Equation:

$16\pi^4[|Q_n|^2] = \sum_{k=1}^{2} E\left[\int_0^{2\pi}[\varsigma^2(t_2) + \varpi^2(t_2)]\left((\sigma_k^j(t))^2 dt_2 + dN^j(t_2)\right)\right]$

Apply Cauchy−Schwarz inequality

$(16\pi^4[|Q_n|^2])^2 \le 4\sum_{k=1}^{2} E\left[\int_0^{2\pi}(\sigma_k^j(t_2))^4 dt_2 + (dN^j(t_2))^2\right]$

$\times \{E\left[\int_0^{2\pi}\left(\int_0^{t_2} D_n(t_2-t_1)\cos(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1) + dM^i(t_1)\right)\right)^4 dt_2\right]$

$+E\left[\int_0^{2\pi}\left(\int_0^{t_2} D_n(t_2-t_1)\sin(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1) + dM^i(t_1)\right)\right)^4 dt_2)\right]$ $E[\int_0^{2\pi}(\sigma_k^i(t))^4 dt] < \infty$

$\Rightarrow E\left[\int_0^{2\pi}(\sigma_k^j(t_2))^4 dt_2 + (dN^i(t_2))^2\right] < \infty$

Apply Burkholder−Gundy's inequality

$$E\left[\int_0^{2\pi}\left(\int_0^{t_2} D_n(t_2-t_1)\cos(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)\right)^4 dt_2\right]$$

$$\leq 4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(t_2-t_1)\cos^4(qt_1)\left(\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)\right)^4 dt_2\right]$$

$$= 4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(t_2-t_1)\cos^4(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)dt_2\right]$$

$-1\leq\cos(qt_1)\leq 1$ and $\cos^4(qt_1)$ takes the interval $0\leq\cos^4(qt_1)\leq 1$

Note that he maximum of cos is 1,then

$$E\left[\int_0^{2\pi}\left(\int_0^{t_2} D_n(t_2-t_1)\cos(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)\right)^4 dt_2\right]$$

$$\leq 4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(t_2-t_1)\left(\sum_{k=1}^{2}(\sigma_k^i(t_1))^4 d(t_1)+(dM^i(t_1))^2\right)dt_2\right]$$

$$E\left[\int_0^{2\pi}\left(\int_0^{t_2} D_n(t_2-t_1)\sin(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)\right)^4 dt_2\right]$$

$$\leq 4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(t_2-t_1)\sin^4(qt_1)\left(\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)\right)^4 dt_2\right]$$

$$= 4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(t_2-t_1)\left(\sum_{k=1}^{2}(\sigma_k^i(t_1))^4 dt_1+(dN^i(t_1))^2\right)dt_2\right]$$

$$E\left[\int_0^{2\pi}\left(\int_0^{t_2} D_n(t_2-t_1)\cos(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)\right)^4 dt_2\right]$$

$$+E\left[\int_0^{2\pi}\left(\int_0^{t_2} D_n(t_2-t_1)\sin(qt_1)\left(\sum_{k=1}^{2}\sigma_k^i(t_1)dB^k(t_1)+dM^i(t_1)\right)\right)^4 dt_2\right]$$

$$\leq 4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(t_2-t_1)\left(\sum_{k=1}^{2}(\sigma_k^i(t_1))^4 dt_1+(dN^i(t_1))^2\right)dt_2\right]$$

$$+4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(t_2-t_1)\left(\sum_{k=1}^{2}(\sigma_k^i(t_1))^4 dt_1+(dN^i(t_1))^2\right)dt_2\right]$$

We let $t_1=u, t_2-t_1=v, ,t_2=u+v,\frac{du}{dt_1}=1,\quad \frac{du}{dt_2}=1,\frac{dv}{dt_2}=1$

$$4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(v)\left(\sum_{k=1}^{2}(\sigma_k^i(u))^4 du+(dN^i(u))^2\right)d(u+v)\right]$$

$$+4E\left[\int_0^{2\pi}\int_0^{t_2} D_n^4(v)\left(\sum_{k=1}^{2}(\sigma_k^i(u))^4 du+(dN^i(u))^2\right)d(u+v)\right]$$

$$= 4E\left[\int_0^{2\pi}\left(\sum_{k=1}^{2}(\sigma^i+_k(u))^4\right)du+(dN^i(u))^2\right]\int_0^{2\pi} D_n^4(v)dv$$

$$+4E\left[\int_0^{2\pi}\left(\sum_{k=1}^{2}(\sigma^i+_k(u))^4\right)du+(dN^i(u))^2\right]\int_0^{2\pi} D_n^4(v)dv$$

$$= 8E\left[\int_0^{2\pi}\left(\sum_{k=1}^{2}(\sigma_k^i(u))^4\right)du+(dN^i(u))^2\right]\int_0^{2\pi} D_n^4(v)dv$$

As $|D_n(v)|\leq 1$, $\int_0^{2\pi} D_n^4(v)dv\leq\int_0^{2\pi} D_n^2(v)dv\leq\int_0^{2\pi}|D_n(v)|dv$

$$\int_0^{2\pi} D_n^2(v)dv=\frac{2\pi}{2n+1}$$

As $n\to\infty,\quad \int_0^{2\pi} D_n^2(v)dv=0.\quad Q_n^2=0\Rightarrow Q_n=0.$

$$if\quad Q_n=0,\quad then\quad H_n^{ij}=0$$

Evaluating $Y_n^{ij}(q)$;

$$Y_n^{ij}(q) = \frac{1}{2n+1}\sum_{s=-n}^{n} \int_0^{2\pi} \left(\frac{1}{2\pi}\right)^2 \left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^i(t) dB^l(t)\right)$$

$$+\frac{1}{2n+1}\sum_{s=-n}^{n} \left[\int_0^{2\pi} \left(\frac{1}{2\pi}\right)^2 \left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^j(t) dB^l(t)\right) + \int_0^{2\pi} \left(\frac{1}{2\pi}\right)^2 \left(e^{-iqt} dN_t^j\right)\right]$$

$$= \left(\frac{1}{4\pi^2}\right)\left(\frac{1}{2n+1}\right)\left[\int_0^{2\pi} \left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^i(t) dB^l(t)\right)(2n+1)\right]$$

$$+\left(\frac{1}{4\pi^2}\right)\left(\frac{1}{2n+1}\right)\left[\int_0^{2\pi} \left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^j(t) dB^l(t)\right)(2n+1) + \int_0^{2\pi} e^{-iqt} dN_t^j (2n+1)\right]$$

$$= \left(\frac{1}{4\pi^2}\right)\int_0^{2\pi} \left[\left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^i(t) dB^l(t)\right)\right.$$

$$\left. + \left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^i(t) dB^l(t)\right) + e^{-iqt} dN_l^j\right]$$

$$\frac{1}{4\pi^2}\int_0^{2\pi} e^{-iqt} dN^j(t) = \frac{1}{2\pi}\mathcal{F}(dN)(q)$$

$$Y_n^{ij}(q) = \frac{1}{4\pi^2}\int_0^{2\pi} \left[\left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^i(t) dB^l(t)\right) + \left(e^{-iqt} dM^j(t) \sum_{l=0}^{d} \sigma_l^j(t) dB^l(t)\right)\right]$$

$$+\frac{1}{2\pi}\mathcal{F}(dN^j)(q)$$

Given that e^{-iqt} as $\cos(qt) + i\sin(q$

$$Y_n^{ij}(q) = \frac{1}{4\pi^2}\int_0^{2\pi} \left[\left((\cos(qt) + i\sin(qt)) dM^j(t) \sum_{l=0}^{d} \sigma_l^i(t) dB^l(t)\right)\right]$$

$$+\frac{1}{4\pi^2}\int_0^{2\pi} \left[\left((\cos(qt) + i\sin(qt)) dM^j(t)\right) \sum_{l=0}^{d} \sigma_l^j(t) dB^l(t)\right)\right] + \frac{1}{2\pi}\mathcal{F}(dN^j)(q)$$

But $i = j$,

$$Y_n^{ij}(q) = \frac{1}{4\pi^2}\int_0^{2\pi} \left[(\cos(qt) + \sin(qt)) dM^j(t)\right) \left(\sum_{l=0}^{d} \sigma_l^j(t) dB^l(t) + \sum_{l=0}^{d} \sigma_l^j(t) dB^l(t)\right)\right]$$

$$+\frac{1}{2\pi}\mathcal{F}(dN^j)(q$$

$$\eta(t) = (\cos(qt) + i\sin(qt)) dM^j(t)\left(\sum_{l=0}^{d} \sigma_l^j(t) dB^l(t) + \sum_{l=0}^{d} \sigma_l^j(t) dB^l(t)\right)$$

$$\eta(t) = (\cos(qt) + i\sin(qt))\left(2\sum_{l=0}^{d} \sigma_l^j(t) dB^l(t)\right) dM^j(t)$$

$$\eta^2(t) = (\cos(qt) + i\sin(qt))^2 \left(4\sum_{l=0}^{d} \sigma_l^j(t) dB^l(t) dM^j(t)\right)^2$$

By De−moivre's formula, $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$

$$\eta^2(t) = (\cos(2qt) + i\sin(2qt))\left(4\sum_{l=0}^{d} \sigma_j^i(t))^2 dt (dN_t^j)\right)$$

$dt \cdot dN_i^j = 0 \Rightarrow \eta^2(t) = 0 \Rightarrow \eta(t) = 0.$

$$Y_n^{ij}(q) = \frac{1}{2\pi}\mathcal{F}(dN^j)(q)$$

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$$g_q^{ij}(N) = \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q) + \frac{1}{2\pi}\mathcal{F}(dN^j)(q$$

$g_q^{ij}(N) = \frac{1}{2n+1}\sum_{s=-n}^{n} \Gamma_{-s}^{i}(2\pi)\Gamma_{-s}^{j}(2\pi)$

$= \frac{1}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp^i)(-s)\mathcal{F}(dp^j)(q+s)$

$= (\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j))(q)$

$(\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j))(q) = \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q) + \frac{1}{2\pi}\mathcal{F}(dN^j)(q)$

$\mathcal{F}(dN^j)(q) = \frac{1}{2\pi}\int_0^t e^{-iks}dN^j(s)$

Theorem 2

Let p be the price process and let (p) be the volatility of the price process $p(t)$ at time t , q,n are integers.

Then,

$$Vol(p) = \lim_{n\to\infty}\sum_{|q|<n}\left(1-\frac{|q|}{n}\right)(\frac{2\pi}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp^j)(s)\mathcal{F}(dp^j)(q-s+\frac{q}{i}\mathcal{F}(N)(q) - \frac{1}{2\pi}(N(2\pi)-N(0)))\exp(iqt) \quad (8)$$

Proof

Let $\mathcal{F}(\phi)(k) = \frac{i}{k}\left[\frac{1}{2\pi}(\phi(2\pi)-\phi(0)) - \mathcal{F}d\phi(k)\right]$

$$\mathcal{F}d(\phi)(k) = -\frac{k}{i}\mathcal{F}(\phi)(k)\frac{1}{2\pi}(\phi(2\pi)-\phi(0))$$

$\mathcal{F}(dN)(q) = -\frac{q}{i}\mathcal{F}(N)(q) + \frac{1}{2\pi}(N(2\pi)-N(0))$

$(\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j))(q) = \frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q) + \frac{1}{2\pi}\mathcal{F}(dN^j)(q)$

$2\pi(\mathcal{F}(dp^i)_{*B}(\mathcal{F}(dp^j))(q) = \mathcal{F}(\Sigma^{ij})(q) + \mathcal{F}(dN^j)(q)$

$2\pi(\mathcal{F}(dp^i)_{*B}(\mathcal{F}(dp^j))(q) = \mathcal{F}(\Sigma^{ij})(q) - \frac{q}{i}\mathcal{F}(N)(q) + \frac{1}{2\pi}(N(2\pi)-N(0))$

$\mathcal{F}(\Sigma^{ij})(q) = 2\pi(\mathcal{F}(dp^i)_{*B}(\mathcal{F}(dp^j))(q) + \frac{q}{i}\mathcal{F}(N)(q) - \frac{1}{2\pi}(N(2\pi)-N(0))$

$(\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j))(q) = \lim_{n\to\infty}\frac{1}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp^j)(s)\mathcal{F}(dp^j)(q-s)$

Then

$\mathcal{F}(\Sigma^{ij})(q) = \lim_{n\to\infty}\frac{2\pi}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp^j)(s)\mathcal{F}(dp^j)(q-s)$

$+\frac{q}{i}\mathcal{F}(N)(q) - \frac{1}{2\pi}(N(2\pi)-N(0))$

$\Sigma^{ij}(t) = \lim_{n\to\infty}\sum_{|q|<n}\left(1-\frac{|q|}{n}\right)(\frac{2\pi}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp^j)(s)\mathcal{F}(dp^j)(q-s)$

$$+\frac{q}{i}\mathcal{F}(N)(q)-\frac{1}{2\pi}(N(2\pi)-N(0)))\exp(iqt)$$

The volatility function $Vol(p)=\sigma^2(t)$, for $i=j$, $\Sigma^{jj}(t)=\sum_{i=1}^{d}(\sigma_i^j(t))^2=\sigma^2$

$$Vol(p)=\lim_{n\to\infty}\sum_{|q|<n}\left(1-\frac{|q|}{n}\right)(\frac{2\pi}{2n+1}\mathcal{F}(dp^j)(s)\mathcal{F}(dp^j)(q-s)$$

$$+\frac{q}{i}\mathcal{F}(N)(q)-\frac{1}{2\pi}(N(2\pi)-N(0)))\exp(iqt)$$

$$\mathcal{F}(dp^j)(s)=-\frac{s}{i}\mathcal{F}(p^j)(s)+\frac{1}{2\pi}(p^j(2\pi)-p^j(0))$$

4. APPLICATIONS OF STOCHASTIC VOLATILITY ESTIMATION FOR A SPECIFIC CASE USING FOURIER TRANSFORMATION

We let:

$$dp^j(t)=\sum_{i=1}^{d}\sigma_i^j(t)\sigma_i^k(t)+\alpha^j(t)dt+\beta^j(t)dM^j(t),\quad j,k=1,\cdots,n,$$

$$\mathcal{F}(dp^j)(q)=\mathcal{F}\left(\sum_{i=1}^{d}\sigma_i^j\sigma_i^k dB^i\right)(q)+\mathcal{F}(\alpha^j dt)(q)+\mathcal{F}(\beta^j dM^j)(q),\quad j=1,\cdots,n$$

$$dp_\sigma^j(t)=\sum_{i=1}^{d}\sigma_i^j\sigma_i^k dB^i(t),$$

$$\mathcal{F}(dp^j)(q)=\mathcal{F}(dp_\sigma^j)(q)+\mathcal{F}(\alpha^j dt)(q)+\mathcal{F}(\beta^j dM^j)(q)$$

Given that $\mathcal{F}(\beta^j dM^j)(q)=\phi_{M1}(q)$, then,

$$(\phi_\sigma+\phi_\alpha+\phi_{M1})_{*B}(\phi_\sigma+\phi_\alpha+\phi_{M1})$$

$$=(\phi_{\sigma*B}+\phi_{\alpha*B}+\phi_{M1*B})(\phi_\sigma+\phi_\alpha+\phi_{M1})=\phi_{\sigma*B}\phi_\alpha+\phi_{M*B}\phi_{M1}$$

$$(\phi_{M*B}\phi_{M1})(q)=\lim_{n\to\infty}\frac{1}{2n+1}\sum_{s=-n}^{n}\phi_{M1}(s)\phi_{M1}(q-s)$$

$$\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j)=(\phi_{\sigma*B}*\phi_\sigma)+(\phi_{M1*B}*\phi_{M_1})$$

$$=\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp_\sigma^j)\mathcal{F}(\beta^j dM^j)_{*B}\mathcal{F}(\beta^j dM^j)$$

$$=\frac{1}{2\pi}\mathcal{F}\left([dp_\sigma^i,dp_\sigma^j]\right)+\frac{1}{2\pi}\mathcal{F}([\beta^j dM^j,\beta^j dM^j])$$

$$=\frac{1}{4\pi^2}\int_0^t e^{-iqs}d[p_\sigma^i,p_\sigma^j](s)+\frac{1}{4\pi^2}\int_0^t e^{-iqs}([\beta^j dM^j,\beta^j dM^j])(s)$$

$$([\beta^j dM^j,\beta^j dM^j])(s)=\beta^{2j}(s)dN^j(s),$$

$$\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j)=\frac{1}{4\pi^2}\int_0^t e^{-iqs}d[p_\sigma^i,p_\sigma^j](s)+\frac{1}{4\pi^2}\int_0^t e^{-iqs}\beta^{2j}dN^j(s)$$

$$\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q)+\frac{1}{2\pi}\mathcal{F}(\beta^{2j}dN^j)(q)=\mathcal{F}(dp^i)_{*B}\mathcal{F}(dp^j)$$

$$(\mathcal{F}(dp^j)_{*B}(\mathcal{F}(dp^j))(q)=\frac{1}{2\pi}\mathcal{F}(\Sigma^{ij})(q)+\frac{1}{2\pi}\mathcal{F}(\beta^{2j}dN^j)(q)$$

$$\mathcal{F}(\Sigma^{ij})(q)+\mathcal{F}(\beta^{2j}dN^j)(q)=\lim_{n\to\infty}\frac{2\pi}{2n+1}\sum_{s=-n}^{n}\mathcal{F}(dp^j)(s)\mathcal{F}(dp^j)(q-s)$$

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$$\Sigma^{ij}(t) = \lim_{n\to 0} \sum_{|q|<n} \left(1-\frac{|q|}{n}\right)\left(\frac{2\pi}{2n+1}\sum_{s=-n}^{n} mathcalF(dp^j)(s)\mathcal{F}(dp^j)(q-s) - \mathcal{F}(\beta^{2j}dN^j)(q)\right)\exp(iqt$$

$$vol(p) = \lim_{n\to\infty} \sum_{|q|<n} \left(1-\frac{|q|}{n}\right)\left(\frac{2\pi}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp^j)(s)\mathcal{F}(dp^j)(q-s) - \mathcal{F}(\beta^{2j}dN^j)(q)\right)\exp(iqt)$$

$$vol\,(p) = \lim_{n\to\infty} \sum_{|q|<n} \left(1-\frac{|q|}{n}\right)\left(\frac{2\pi}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp)(s)\mathcal{F}(dp)(q-s) - \mathcal{F}(\beta^{2}dN)(q)\right)\exp(iqt$$

$$\mathcal{F}(dp)(s) = -\frac{s}{i}\mathcal{F}(p)(s) + \frac{1}{2\pi}(p(2\pi) - p(0))$$

5. NUMERICAL EXAMPLE

We let $\beta = 0.1$, then,

$$Vol(p) = \lim_{n\to\infty} \sum_{|q|<n} \left(1-\frac{|q|}{n}\right)\left(\frac{2\pi}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp)(s)\mathcal{F}(dp)(q-s) - \mathcal{F}(0.1^2 dN)(q)\right)\exp(iqt)$$

$$\mathcal{F}(dN)(q) = -\frac{q}{i}\mathcal{F}(N)(q) + \frac{1}{2\pi}(N(t) - N(0))$$

For $N\leq 1$ otherwise (0)=0,

$$\mathcal{F}(dN)(q) = -\frac{q}{i}\mathcal{F}(N)(q) + \frac{1}{2\pi}(N(t)) = -\frac{q}{2\pi i}\int_0^t N(s)e^{-iqs}ds + \frac{1}{2\pi}(N(t))$$

if $N(t) = 0$ then $\mathcal{F}(dN)(q) = 0$ but if $N(t) = 1$

$$\mathcal{F}(dN)(q) = -\frac{q}{2\pi i}\left(-\frac{1}{iq}e^{-iqs}\right)_0^t + \frac{1}{2\pi} = \frac{1}{\pi} - \frac{1}{2\pi}e^{-iqt}$$

$$Vol(p) = \lim_{n\to\infty} \sum_{|q|<n} \left(1-\frac{|q|}{n}\right)\left(\frac{2\pi}{2n+1}\sum_{s=-n}^{n} \mathcal{F}(dp)(s)\mathcal{F}(dp)(q-s) - \frac{0.01}{\pi} + \frac{0.01}{2\pi}e^{-iqt}\right)\exp(iqt)$$

$$\mathcal{F}(dp)(s) = -\frac{s}{i}\mathcal{F}(p)(s) + \frac{1}{2\pi}(p(2\pi) - p(0)).$$

Which revealed and indicates that meaning that an increase in volatility, $((p))$ will also lead to increase in the price process and vice versa given that all parameters are constant which shows that the price process is directly proportional to the volatility as the parameter $\beta(t)$ added to the jump component had greater influence on the volatility as an increase or decrease in the parameter $\beta(t)$ will affect the volatility in direct proportionality.

6. CONCLUSION

Determination of compensated poison jump price process with stochastic volatility estimation through the Fourier transformation methods were successful examine and evaluated. Here the theoretical basis of Fourier transformations through Bohr's convolution were adopted which showed an identical correlation between the Fourier transformation of the price process with a compensated Poisson jump and the volatility with estimated instantaneous volatility. The identical correlation for a special case (t) added to the jump showed that it is directly proportional to the volatility when all parameters are constant using both univariate and multivariate volatility settings as was numerically evaluated.

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